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Application of homotopy perturbation and Adomian decomposition methods for solving an inverse heat problem

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Abstract. In this paper, the homotopy perturbation method is proposed to solve an inverse problem of finding an unknown function in parabolic equation with overspecified data. Comparison is made between Adomian decomposition method and the proposed method. It is shown; Adomian decomposition method is equivalent to the homotopy perturbation method in the model problem. To show the efficiency of these methods, several test problems are presented for one-, two- and three-dimensional cases. Comparison of the applied methods with exact solutions reveals that both methods are tremendously effective.

Keywords: Homotopy perturbation method (HPM), Adomian decomposition method (ADM), inverse parabolic problem, integral overspecified data.

1. Introduction

In this article, we consider the following inverse problem. Find $u(\boldsymbol{x},t)$ and $p(t)$ satisfy:

$$u_t(\boldsymbol{x},t) = \Delta u(\boldsymbol{x},t) + p(t)u(\boldsymbol{x},t) + g(\boldsymbol{x},t); \qquad o < t < T, \qquad \boldsymbol{x} \in \Omega, \tag{1}$$

with initial condition:

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega, \tag{2}$$

and boundary conditions:

$$u(\boldsymbol{x},t) = \mathrm{h}(\boldsymbol{x},t); \quad o < t < T, \qquad \boldsymbol{x} \in \partial\Omega. \tag{3}$$

An additional boundary condition which can be the integral overspecification over the spatial domain for (1) is given in the following form:

$$\int_{\Omega} u(\boldsymbol{x},t)d\boldsymbol{x} = E(t); \qquad o < t < T, \tag{4}$$

where Δ is the Laplace operator, $\Omega = [0,1]^d$ is spatial domain of the problem for $d = 1,2,3$, $\partial\Omega$ is the boundary of Ω, $\boldsymbol{x} = (x_1, \dots, x_d)$ and g, u_0, h and E are known functions, while u and p are unknown. The integral overspecification (4) is considered for one-dimensional case in an example of certain chemical absorbing light at various frequencies in [3]. Also in the case of $d = 1$ the integral overspecification can be written in the following form:

$$\int_0^1 k(x)u(x,t)dx = E(t); \qquad o < t < T, \tag{5}$$

or

$$\int_0^{s(t)} u(x,t)dx = E(t); \qquad o < t < T, \tag{6}$$

where $k(x)$ and $s(t)$ are known functions. It is assumed that, for some constant $\rho > 0$, the kernel $k(x)$

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satisfies:

$$\int_0^1 |k(x)|dx \le \rho. \tag{7}$$

This problem appears in the mathematical modeling of many phenomena [1, 2]. Certain types of physical problems can be modeled by (1)-(4). For example if u is a temperature, then (1)-(4) can be regarded as a control problem finding the control $p(t)$. We want to identify the control function $p(t)$ that will yield a desired energy prescribed in a portion of the spatial domain.

The existence and uniqueness of the solution of this inverse problem is established under certain assumptions in [2, 4]. Also the theoretical discussion about this problem is found in [5]. In [11-13] the solution of this problem and similar higher dimensional problems are investigated. Some numerical methods are presented in [3, 13, 15] for solving this problem.

This inverse problem has many important applications. The interested readers can see [6-10]. In this paper, we propose two powerful methods to solve the discussed problem. The first is the HPM developed by He in [18, 19] and used in [20-24] among many others. The second is ADM developed by Adomian in [25, 26] and used heavily in the literature in [27-30] and the references therein. It is shown; Adomian decomposition method is equivalent to the homotopy perturbation method in the model problem. The two methods give rapidly convergent series with specific significant features for each scheme. The two methods, which accurately compute the solutions in a series form or in an exact form, are of great interest to applied sciences. The main advantage of the two methods is that it can be applied directly for all types of differential and integral equations, homogeneous or inhomogeneous. Another important advantage is that the methods are capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution. The effectiveness and the usefulness of both methods are demonstrated by finding exact solutions to the models that will be investigated. However, each method has its own characteristic and significance that will be examined.

This paper is arranged in the following manner. In Section 2, we present homotopy perturbation and Adomian decomposition methods. In Section 3, we apply the HPM and the ADM on the inverse parabolic problem with a control parameter. to present a clear overview of method, in Section 4, we implement these methods for finding the exact solution of a control parameter in one-, two- and three-dimensional parabolic equations. A conclusion is drawn in Section 5.

2. The methods

In what follows we will highlight briefly the main points of each of the two methods, where details can be found in [18-30].

2.1. Homotopy perturbation method

To illustrate the basic idea of this method [18-24], we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0; \qquad r \in \Omega. \tag{8}$$

Considering the boundary condition of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0; \qquad r \in \Gamma, \tag{9}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain $\Omega \subset \mathbb{R}^d; d = 1,2,3$. Generally speaking, the operator A can be divided into two parts which are L and N, where L is a simple part which is easy to handle and N contains the remaining parts of A.

Therefore equation (8) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{10}$$

Using the homotopy technique, we construct a homotopy as $v(r,p): \Omega \times [0,1] \longrightarrow \mathbb{R}$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0; \qquad p \in [0,1], \tag{11}$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0; \qquad p \in [0,1], \tag{12}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of equation (8). Clearly, we have:

$$H(v,0) = L(v) - L(u_0) = 0, \tag{13}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{14}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to HPM, we can first use the embedding parameter p as a "small parameter" and assume that the solution of equations (11) and (12) can be written as a power series in p:

$$v = u_0 + pu_1 + p^2u_2 + \cdots. \tag{15}$$

Setting $p = 1$ results the approximate solution of equation (8):

$$u = \lim_{p \to 1} v = u_0 + u_1 + u_2 + \cdots. \tag{16}$$

In most cases the series (16) is a convergent one which leads to the exact solution of equation (8).

One can take the closed form or truncate the series for obtaining approximate solutions. The series (16) is convergent for most cases. However, the convergence rate depends on the nonlinear operator, $A(u)$ [18].

2.2. Adomian decomposition method

Consider the functional equation:

$$A(u) - f(r) = 0; \qquad r \in \Omega, \tag{17}$$

where A is a general differential operator and $f(r)$ a known analytical function. Suppose that the differential operator A can be decomposed into three operators:

$$A = L + R + N, \tag{18}$$

where L is an invertible operator, R is a linear and N is an analytic nonlinear operator.

Now we can write the equation (17) as the following:

$$L(u) + R(u) + N(u) - f(r) = 0. \tag{19}$$

Applying the inverse operator L^{-1} to both sides of equation (19), we obtain:

$$u(r) = L^{-1}\big(f(r)\big) - L^{-1}\big(R(u)\big) - L^{-1}(N(u)). \tag{20}$$

Adomian defines the unknown function $u(r)$ by an infinite series; say [25-30]:

$$u(r) = \sum_{n=0}^{\infty} u_n(r), \tag{21}$$

where the components $u_n(r)$ are usually determined recurrently. Substituting this infinite series into equation (20) leads to:

$$\sum_{n=0}^{\infty} u_n(r) = L^{-1}\big(f(r)\big) - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n(r)\right)\right) - L^{-1}\left(N\left(\sum_{n=0}^{\infty} u_n(r)\right)\right). \tag{22}$$

Adomian also considers $N(u)$ as the summation of an infinite series of polynomials, say:

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n). \quad (23)$$

Polynomials A_n, which are called Adomian polynomials, are generated for all kinds of nonlinearity so that A_0 depends only on u_0, A_1 depends on u_0 and u_1 and in general A_n depends on $u_0, u_1, \dots, u_n$.

Adomian introduces these polynomials as:

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}; \qquad n = 0,1,2,\dots \quad (24)$$

Adomian procedure can be presented as the following:

$$u_0(r) = L^{-1}\big(f(r)\big), \quad (25)$$

$$u_{n+1}(r) = -L^{-1}\big(R(u)\big) - L^{-1}\big(A_n(u_0, u_1, \dots, u_n)\big); \qquad n = 0,1,2,\dots \quad (26)$$

3. The application of HPM and ADM in an inverse problem

In this section the application of HPM and ADM is discussed for solving the discussed problem.
Our approach begins with the utilization of the following transformations:

3.1. The employed transformation

Let:

$$r(t) = \exp\left(-\int_0^t p(s)ds\right), \qquad w(\boldsymbol{x}, t) = r(t)u(\boldsymbol{x}, t). \quad (27)$$

Therefore, from equations (1)-(3), we have the following non-local parabolic problem:

$$w_t(\boldsymbol{x}, t) = \Delta w(\boldsymbol{x}, t) + r(t)g(\boldsymbol{x}, t); \qquad o < t < T, \qquad \boldsymbol{x} \in \Omega, \quad (28)$$

$$w(\boldsymbol{x}, t) = u_0(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega, \quad (29)$$

$$w(\boldsymbol{x}, t) = r(t)\mathrm{h}(\boldsymbol{x}, t); \quad o < t < T, \qquad \boldsymbol{x} \in \partial\Omega. \quad (30)$$

Assume for any $t \in [0, T]$, $E(t) \neq 0$, then from equations (4) and (27), we have:

$$r(t) = \frac{\int_\Omega w(\boldsymbol{x}, t)d\boldsymbol{x}}{E(t)}. \quad (31)$$

Also in the cases (5) and (6), we have:

$$r(t) = \frac{\int_0^1 k(x)w(x, t)dx}{E(t)}, \quad (32)$$

$$r(t) = \frac{\int_0^{s(t)} w(x, t)dx}{E(t)}, \quad (33)$$

respectively. And finally from (27), we can obtain:

$$u(\boldsymbol{x}, t) = \frac{w(\boldsymbol{x}, t)}{r(t)}, \quad (34)$$

$$p(t) = \frac{\acute{r}(t)}{r(t)}. \quad (35)$$

3.2. Using homotopy perturbation method

In order to solve problem (28)-(31) by HPM, we choose the initial approximation, $L(w)$ and $N(w)$ as following:

$$w_0(\boldsymbol{x},t) = w(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \qquad L(w) = \frac{\partial w}{\partial t}, \qquad N(w) = -\Delta w - \frac{g(\boldsymbol{x},t)}{E(t)} \int_\Omega w(\boldsymbol{x},t) d\boldsymbol{x}. \tag{36}$$

Note that, in the cases (32) and (33), we have:

$$N(w) = -\Delta w - \frac{g(\boldsymbol{x},t)}{E(t)} \int_0^1 k(x) u(x,t) dx,$$

$$N(w) = -\Delta w - \frac{g(\boldsymbol{x},t)}{E(t)} \int_0^{s(t)} u(x,t) dx,$$

respectively. And from equation (12) construct the following homotopy:

$$v_t - w_{0t} + p\left[w_{0t} - \Delta v - \frac{g(\boldsymbol{x},t)}{E(t)} \int_\Omega v(\boldsymbol{x},t) d\boldsymbol{x}\right] = 0. \tag{37}$$

Assume the solution of equation (37) in the form:

$$v = w_0 + p w_1 + p^2 w_2 + \cdots. \tag{38}$$

Substituting (38) into equation (37) and collecting terms of the same power of p gives:

$$p^0: \qquad w_{0t} - w_{0t} = 0, \tag{39}$$

$$p^1: \qquad w_{1t} - w_{0t} - \Delta w_0 - \frac{g(\boldsymbol{x},t)}{E(t)} \int_\Omega v(\boldsymbol{x},t) d\boldsymbol{x} = 0, \tag{40}$$

$$p^n: \qquad w_{nt} - \Delta w_{n-1} - \frac{g(\boldsymbol{x},t)}{E(t)} \int_\Omega v(\boldsymbol{x},t) d\boldsymbol{x} = 0; \qquad n \geq 2. \tag{41}$$

The above partial differential equations must be supplemented by conditions ensuring a uniqueness of the solution. For equation (40) we assume the following conditions:

$$w_0(\boldsymbol{x},t) + w_1(\boldsymbol{x},t) = h(\boldsymbol{x},t); \qquad \boldsymbol{x} \in \partial\Omega,$$

while for equation (41) conditions are in the form ($n \geq 2$):

$$w_n(\boldsymbol{x},t) = 0; \qquad \boldsymbol{x} \in \partial\Omega.$$

We can start with $w_0(\boldsymbol{x},t) = u_0(\boldsymbol{x})$ and all the equations above can be easily solved, we get all the solutions. The solution of (28)-(31) can be obtained by setting $p = 1$ in equation (38):

$$w = w_0 + w_1 + w_2 + \cdots. \tag{42}$$

Having $w(\boldsymbol{x},t)$ determined, then $u(\boldsymbol{x},t)$ and $p(t)$ can be computed by using equations (34) and (35). Also, get the n-th approximation of the exact solution as $\Phi_n = u_0 + u_1 + u_2 + \cdots + u_n$.

3.3. Using Adomian decomposition method

Let:

$$L(w) = \frac{\partial w}{\partial t}, \qquad R(w) = -\Delta w - \frac{g(\boldsymbol{x},t)}{E(t)} \int_\Omega w(\boldsymbol{x},t) d\boldsymbol{x}, \qquad N(w) = 0.$$

The inverse L^{-1} is assumed as an integral operator given by:

$$L^{-1}(.) = \int_0^t (.) dt. \tag{43}$$

Applying the inverse operator L^{-1} on both sides of (28) and using the initial condition, we find:

$$w(\boldsymbol{x},t) = u_0(\boldsymbol{x}) + L^{-1}\left(\Delta w - \frac{g(\boldsymbol{x},t)}{E(t)} \int_\Omega w(\boldsymbol{x},t) d\boldsymbol{x}\right). \tag{44}$$

Substituting equation (21) into the functional equation (44) gives:

$$\sum_{n=0}^{\infty} w_n(\boldsymbol{x},t) = u_0(\boldsymbol{x}) + L^{-1}\left(\Delta \sum_{n=0}^{\infty} w_n(\boldsymbol{x},t) - \frac{g(\boldsymbol{x},t)}{E(t)} \int_{\Omega} \sum_{n=0}^{\infty} w_n(\boldsymbol{x},t)\, d\boldsymbol{x}\right). \quad (45)$$

Identifying the zero-th component $w_0(\boldsymbol{x},t)$ by $u_0(\boldsymbol{x})$, the remaining components $w_n(\boldsymbol{x},t);\ \ n \geq 1$, can be determined by using the recurrence relation:

$$w_0(\boldsymbol{x},t) = u_0(\boldsymbol{x}), \quad (46)$$

$$w_n(\boldsymbol{x},t) = L^{-1}\left(\Delta w_{n-1} - \frac{g(\boldsymbol{x},t)}{E(t)} \int_{\Omega} w_{n-1}(\boldsymbol{x},t) d\boldsymbol{x}\right); \qquad n \geq 1. \quad (47)$$

Note that, in the cases (32) and (33) we have:

$$R(w) = -\Delta w - \frac{g(x,t)}{E(t)} \int_0^1 k(x) w(x,t) dx,$$

$$R(w) = -\Delta w - \frac{g(x,t)}{E(t)} \int_0^{s(t)} w(x,t) dx,$$

respectively. A comparison between equations (39)-(41) and (46)-(47) show that Adomian decomposition method is equivalent to the homotopy perturbation method for this problem.

4. Test examples

To show the efficiency of the present methods on the semi-linear inverse parabolic partial differential equation, four examples are given. These tests are chosen from [11-13, 15] such that their analytical solutions are known. But this method can be applied on more complicated problems.

Example 1. Consider problems (1)-(3) and (5) with [15]:
$u_0(x) = x + \cos(\pi x)\,, u(0,t) = \exp(t)\,, u(1,t) = 0, E(t) = \exp\left(\frac{3}{4} - \frac{2}{\pi^2}\right), k(x) = 1 + x^2$ and $g(x,t) = \exp(t)[x + \cos(\pi) + \pi^2 \cos(\pi x)] - \exp(t)(1 + t^2)[x + \cos(\pi)]$ for which the exact solution is $u(x,t) = \exp(t)(x + \cos(\pi x))$ and $p(t) = 1 + t^2$.

Using the HPM:

We can select $w_0(x,t) = u_0(x) = x + \cos(\pi x)$ by using the given initial value and from equations (40) and (41), we obtain:

$$w_1(x,t) = -\frac{t^3}{3}(x + \cos(\pi x)),$$

$$w_2(x,t) = \frac{t^6}{18}(x + \cos(\pi x)),$$

$$w_3(x,t) = -\frac{t^9}{162}(x + \cos(\pi x)),$$

and generally, we obtain:

$$w_5(x,t) = \frac{(-1)^n t^{3n}}{3^n n!}(x + \cos(\pi x)).$$

Thus from (42), we have:

$$w(x,t) = w_0 + w_1 + w_2 + \cdots + w_n + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k t^{3k}}{3^k k!}(x + \cos(\pi x)) = \exp\left(-\frac{t^3}{3}\right)(x + \cos(\pi x)),$$

and form (32),we can write:

$$r(t) = \frac{\int_0^1 (1 + x^2) \exp\left(-\frac{t^3}{3}\right)(x + \cos(\pi x)) dx}{E(t)} = \exp\left(-\frac{t^3}{3} - t\right).$$

Therefore, using (34) and (35), we obtain:

$$u(x,t) = \exp(t)(x + \cos(\pi x)),$$

$$p(t) = 1 + t^2,$$

which is the exact solution of the problem.
Using the ADM:
By using Adomian decomposition method, the same results will be obtained as follows:
From equations (46) and (47), we obtain:

$$w_0(x,t) = x + \cos(\pi x),$$
$$w_1(x,t) = -\frac{t^3}{3}(x + \cos(\pi x)),$$
$$w_2(x,t) = \frac{t^6}{18}(x + \cos(\pi x)),$$
$$w_3(x,t) = -\frac{t^9}{162}(x + \cos(\pi x)),$$

and generally, we obtain:

$$w_n(x,t) = \frac{(-1)^n t^{3n}}{3^n n!}(x + \cos(\pi x)).$$

In view of the above equations, the solution $w(x,t)$ is readily obtained in a series form by:

$$w(x,t) = w_0 + w_1 + w_2 + \cdots + w_n + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k t^{3k}}{3^k k!}(x + \cos(\pi x)) = \exp\left(-\frac{t^3}{3}\right)(x + \cos(\pi x)),$$

also from (32), we can obtain:

$$r(t) = \frac{\int_0^1 (1 + x^2) \exp\left(-\frac{t^3}{3}\right)(x + \cos(\pi x))dx}{E(t)} = \exp\left(-\frac{t^3}{3} - t\right).$$

Thus, using (34) and (35), we obtain:

$$u(x,t) = \exp(t)(x + \cos(\pi x)),$$
$$p(t) = 1 + t^2.$$

Author of [15] used the several explicit and implicit finite difference methods to solve this problem and achieved the approximation solution at mesh point only while the HPM and the ADM provide the solution in a closed form. The results obtained for u at $t = 1$, computed using the finite difference methods in [15] and our methods are listed in Table 1.

Table 1. Absolute error $|u(x,1) - \Phi_5(x,1)|$ and comparison with some well-known finite difference methods from Example 1

x	FTCS[15]	BTCS[15]	Crank-Nicolson[15]	Saulyev I[15]	Saulyev II[15]	HPM and ADM
0.2	7.7×10^{-3}	7.7×10^{-3}	6.9×10^{-3}	9.6×10^{-3}	9.9×10^{-3}	1.0×10^{-9}
0.4	8.5×10^{-3}	3.7×10^{-3}	6.5×10^{-3}	9.9×10^{-3}	9.8×10^{-3}	1.0×10^{-9}
0.6	8.2×10^{-3}	3.3×10^{-3}	6.7×10^{-3}	9.8×10^{-3}	9.9×10^{-3}	3.0×10^{-10}
0.8	7.9×10^{-3}	3.3×10^{-3}	6.3×10^{-3}	9.8×10^{-3}	9.0×10^{-3}	1.0×10^{-9}

Example 2. In this example, we consider (1)-(3) and (6) as follows [12]:
$u_0(x) = x + \cos(\pi x)$, $u(0,t) = \exp(t)$, $u(1,t) = 0$, $E(t) = \exp\left(\frac{\sin\left(0.5(1+\sqrt{t})\right)}{\pi} + \frac{(1+\sqrt{t})^2}{8}\right)$, $s(t) = 0.5\left(1 + \sqrt{t}\right)$ and $g(x,t) = (\pi^2 + 2t)\exp(t)\cos(\pi) + \exp(t)xt$
for which the exact solution is $u(x,t) = \exp(t)(x + \cos(\pi x))$ and $p(t) = 1 - 2t$.
Using the HPM:

We can select $w_0(x,t) = u_0(x) = x + \cos(\pi x)$ by using the given initial value and from equations (40) and (41), we obtain:

$$w_1(x,t) = t^2(x + \cos(\pi x)),$$
$$w_2(x,t) = \frac{t^4}{2}(x + \cos(\pi x)),$$

$$w_3(x,t) = -\frac{t^6}{6}(x+\cos(\pi x)),$$

and generally, we obtain:

$$w_n(x,t) = \frac{t^{2n}}{n!}(x+\cos(\pi x)).$$

Thus from (42), we have:

$$w(x,t) = w_0 + w_1 + w_2 + \cdots + w_n + \cdots = \sum_{k=0}^{\infty} \frac{t^{2k}}{k!}(x+\cos(\pi x)) = \exp(t^2)\,(x+\cos(\pi x)),$$

and form (33),we can write:

$$r(t) = \frac{\int_0^{s(t)} \exp(t^2)\,(x+\cos(\pi x))dx}{E(t)} = \exp(t^2 - t).$$

Therefore, using (34) and (35), we obtain:

$$u(x,t) = \exp(t)(x+\cos(\pi x)), \qquad \mathrm{p(t) = 1 - 2t},$$

which is the exact solution of the problem. Using the Adomian decomposition method, the same results will be obtained. Author of [12] used the numerical technique to solve this problem and achieved the approximation solution at mesh point only while the HPM and the ADM provide the solution in a closed form. The results obtained for u, computed using the numerical techniques in [12] and our methods are listed in Table 2.

Table 2. Absolute error $|u(x,0.5) - \Phi_5(x,0.5)|$ and comparison with some well-known finite difference methods from Example 2.

x	FTCS[12]	5-point FTCS [12]	BTCS [12]	Crank-Nicolson[12]	HPM and ADM
0.2	3.1×10^{-3}	3.8×10^{-3}	4.6×10^{-3}	4.1×10^{-3}	0
0.4	2.8×10^{-3}	3.9×10^{-3}	4.3×10^{-3}	3.7×10^{-3}	0
0.6	2.4×10^{-3}	3.5×10^{-3}	4.1×10^{-3}	3.2×10^{-3}	1.0×10^{-10}
0.8	2.7×10^{-3}	3.2×10^{-3}	4.3×10^{-3}	3.5×10^{-3}	1.9×10^{-9}

Example 3. We consider the problem (1)-(4) as follows [11]:

$u_0(x,y) = \sin\left(\frac{\pi}{4}(x+2y)\right), u(0,y,t) = \exp(t)\sin\left(\frac{\pi y}{2}\right), u(1,y,t) = \exp(t)\sin\left(\frac{\pi}{4}(1+2y)\right)$

$u(x,0,t) = \exp(t) sin\left(\frac{\pi}{4}x\right), u(x,1,t) = \exp(t) sin(\frac{\pi}{4}(x+2)),$

$E(t) = \frac{8}{\pi^2}\exp(t)(\sin(0.15\pi) + \sin(0.2\pi) - \sin(0.35))$ and

$g(x,y,t) = (\frac{5\pi^2}{16} - 5\mathrm{t})\exp(t) sin\left(\frac{\pi}{4}(x+2)\right)$ for which the exact solution is:

$u(x,t) = \exp(t) sin(\frac{\pi}{4}(x+2y)), \qquad p(t) = 1 + 5\mathrm{t}.$

Using the HPM:

We can select $w_0(x,y,t) = u_0(x,y) = \sin\left(\frac{\pi}{4}(x+2y)\right)$ by using the given initial value and from equations (40) and (41), we obtain:

$$w_1(x,y,t) = -\frac{5t^2}{2}\sin\left(\frac{\pi}{4}(x+2y)\right),$$

$$w_2(x,y,t) = \frac{25t^4}{4}\sin\left(\frac{\pi}{4}(x+2y)\right),$$

$$w_3(x,y,t) = -\frac{125t^6}{48}\sin\left(\frac{\pi}{4}(x+2y)\right),$$

and generally, we obtain:

$$w_n(x,y,t) = \frac{(-1)^n 5^n t^{2n}}{2^n n!}\sin\left(\frac{\pi}{4}(x+2y)\right).$$

Thus from (42), we have:

$$w(x,y,t) = w_0 + \cdots + w_n + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k 5^k t^{2k}}{2^n n!} \sin\left(\frac{\pi}{4}(x+2y)\right) = \exp\left(\frac{-5t^2}{2}\right) \sin\left(\frac{\pi}{4}(x+2y)\right),$$

and form (31),we can write:

$$r(t) = \frac{\int_0^1 \int_0^{\mathbf{1}} \exp\left(\frac{-5t^2}{2}\right) \sin\left(\frac{\pi}{4}(x+2y)\right) dxdy}{E(t)} = \exp(-5t^2 - t).$$

Therefore, using (34) and (35), the exact solution of the problem results. Using Adomian decomposition method, the same results will be obtained. Author of [11] used the three different finite difference schemes to solve this problem and achieved the approximation solution at mesh point only while the HPM and the ADM provide the solution in a closed form.

Example 4. We consider (1)-(4) as follows [13]:

$u_0(x,y,z) = \sin\left(\frac{\pi}{4}(x+2y+3z)\right), u(0,y,z,t) = \exp(t)\sin\left(\frac{\pi}{4}(2y+3z)\right),$

$u(1,y,z,t) = \exp(t)\sin\left(\frac{\pi}{4}(1+2y+3z)\right), u(x,0,z,t) = \exp(t)\sin\left(\frac{\pi}{4}(x+3z)\right)$

$u(x,1,z,t) = \exp(t)\sin\left(\frac{\pi}{4}(x+2+3z)\right), u(x,y,0,t) = \exp(t)\sin\left(\frac{\pi}{4}(x+2y)\right)$

$u(x,y,1,t) = \exp(t)\sin\left(\frac{\pi}{4}(x+2y+3)\right), g(x,y,z,t) = (\frac{7\pi^2}{8} - 10t)\exp(t)\sin\left(\frac{\pi}{4}(x+2y+3z)\right)$

for which the exact solution is $u(x,y,z,t) = \exp(t)\sin\left(\frac{\pi}{4}(x+2y+3z)\right)$ and p(t)=1+10t.

Using the HPM:

We can select $w_0(x,y,z,t) = u_0(x,y,z) = \sin\left(\frac{\pi}{4}(x+2y+3z)\right)$ by using the given initial value and from equations (40) and (41), we obtain:

$$w_1(x,y,z,t) = -5t^2 \sin\left(\frac{\pi}{4}(x+2y+3z)\right),$$

$$w_2(x,y,z,t) = -\frac{25t^4}{2}\sin\left(\frac{\pi}{4}(x+2y+3z)\right),$$

$$w_3(x,y,z,t) = -\frac{125t^6}{48}\sin\left(\frac{\pi}{4}(x+2y+3z)\right),$$

and generally, we obtain:

$$w_n(x,y,z,t) = \frac{-(5t^2)^n}{n!}\sin\left(\frac{\pi}{4}(x+2y+3z)\right).$$

Thus from (42), we have:

$$w(x,y,z,t) = w_0 + \cdots = \sum_{k=0}^{\infty} \frac{-(5t^2)^k}{k!}\sin\left(\frac{\pi}{4}(x+2y+3z)\right) = \exp(-5t^2)\sin\left(\frac{\pi}{4}(x+2y+3z)\right),$$

and form (31),we can write:

$$r(t) = \frac{\int_0^1 \int_0^1 \int_0^1 \exp(-5t^2)\sin\left(\frac{\pi}{4}(x+2y+3z)\right) xdy\, dz}{E(t)} = \exp(-5t^2 - t).$$

Therefore, using (34) and (35), the exact solution of the problem results. Using the Adomian decomposition method, the same results will be obtained. In [13] a generalization of the well-known, explicit Euler finite difference technique is used to compute the solution and achieved the approximation solution at mesh point only while HPM and ADM provide the solution in a closed form.

5. Conclusion

In this paper, we propose two powerful methods to obtain the approximation analytical solution of the inverse heat problem. The first is the homotopy perturbation method and the second is Adomian decomposition method. It is shown; Adomian decomposition method is equivalent to the homotopy perturbation method in the model problem. The two methods give rapidly convergent series with specific significant features for each scheme. The main advantage of the two methods is that the methods are capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution. Implementation of these methods is easy and straightforward. Also, the HPM and the ADM solve the problem without any discretization of the variables, therefore is free from rounding off errors in computational process. The analytical approximation to the solutions is reliable, and confirms the power and ability of these methods as easy devices for computing the solution of partial differential equations.

6. References

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