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Analytical Treatment of Abel Integral Equations by Optimal Homotopy Analysis Transform Method

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Abstract. In this paper, a user friendly algorithm based on the optimal homotopy analysis transform method (OHATM) is proposed to solve a system of generalized Abel's integral equations. The classical theory of elasticity of material is modeled by the system of Abel integral equations. It is observed that the approximate solutions converge to the exact solutions. Illustrative numerical examples are given to demonstrate the efficiency and simplicity of the proposed method in solving such types of systems of Abel's integral equations. Finally, several numerical examples are given to illustrate the accuracy and stability of this method. Comparison of the approximate solution with the exact solutions we show that the proposed method is very efficient and computationally attractive.

Keywords: integeral equation, Abel integral equation, optimal homotopy analysis transform method, Laplace transform.

1. Introduction

An integral equation is defined as equations in which the unknown function $y(x)$ to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. Abel's equation is one of the integral equations derived directly from a concrete problem of physics, without passing through a differential equation. This integral equation occurs in the mathematical modeling of several models in physics, astrophysics, solid mechanics and applied sciences. The great mathematician Niels Abel, gave the initiative of integral equations in 1823 in his study of mathematical physics [1-4]. In 1924, generalized Abel's integral equation on a finite segment was studied by Zeilon [5]. The different types of Abel integral equation in physics have been solved by Pandey et al. [6], Kumar and Singh [7], Kumar et al. [8], Dixit et al. [9], Yousefi [10], Khan and Gondal [11], Li and Zhao [12] by applying various kinds of analytical and numerical methods.

The main aim of this article is to present analytical and approximate solution of integral equations by using new mathematical tool like optimal homotopy analysis transform method. The proposed method is coupling of the homotopy analysis method HAM and Laplace transform method. The HAM, first proposed in 1992 by Liao, has been successfully applied to solve many problems in physics and science [13-18]. In recent years many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform [19-27]. A typical form of an integral equation in $y(x)$ is of the form:

$$\mathbf{y(x)=f(x)+\lambda\int_{\alpha(x)}^{\beta(x)}K(x,t)y(t)dt}, \tag{1}$$

where $K(x,t)$ is called the kernel of the integral equation (1), and $\alpha(x)$ and $\beta(x)$ are the limits of integration. It can be easily observed that the unknown function $y(x)$ appears under the integral sign. It is to be noted here that both the kernel $K(x,t)$ and the function $f(x)$ in equation (1) are given functions; and λ is a constant parameter. The prime objective of this text is to determine the unknown function $y(x)$ that will satisfy equation (1) using a number of solution techniques. We shall devote considerable efforts in exploring these methods to find solutions of the unknown function

2. Bounded extended Cesàro operators

In order to elucidate the solution procedure of the optimal homotopy analysis transform method, we consider the following integral equations of second kind:

$$\mathbf{y(x) = f(x) + \int_0^x K(x,t)y(t)dt, 0 \le x \le 1} \tag{2}$$

Now operating the Laplce transform on both sides in Eq. (2), we get

$$\mathbf{L[y(x)] = L[f(x)] + L\left\{\int_0^x K(x,t)y(t)dt\right\}} \tag{3}$$

We define the nonlinear operator

$$\mathbf{N[\phi(x;q)] = L[\phi(x;q)] - L[f(x)] - L\left\{\int_0^x K(x,t)\phi(x;q)dt\right\}} \tag{4}$$

where $q \in [0,1]$ be an embedding parameter and $\boldsymbol{\phi(x;q)}$ is the real function of x and q. By means of generalizing the traditional homotopy methods, the great mathematician Liao [13-14] construct the zero order deformation equation

$$\mathbf{(1-q)L[\phi(x;q) - y_0(x)] = \hbar q H(x) N[\phi(x;q)]}, \tag{5}$$

where is a nonzero auxiliary parameter, $\mathbf{H}(x) \neq 0$ an auxiliary function, $\mathbf{y_0(x)}$ is an initial guess of $\mathbf{y(x)}$ and $\boldsymbol{\phi(x;q)}$ is an unknown function. It is important that one has great freedom to choose auxiliary thing in OHATM. Obviously, when $q=0$ and $q=1$, it holds

$$\boldsymbol{\phi(x;0) = y_0(x), \phi(x;1) = y(x)}, \tag{6}$$

Respectively. Thus, as q increases from 0 to 1, the solution varies from the initial guess to the solution. Expanding $\phi(x;q)$ in Taylor's series with respect to q, we have

$$\boldsymbol{\phi(x;q) = y_0(x,t) + \sum_{m=1}^{\infty} q^m y_m(x)}, \tag{7}$$

where

$$\mathbf{y}_m(\mathbf{x}) = \frac{1}{m!}\frac{\partial^m \boldsymbol{\phi}(\mathbf{x;q})}{\partial q^m}\Big|_{q=0} \tag{8}$$

If the auxiliary linear operator, the initial guess , the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen , the series (7) converges at $q=1$, we have

$$\mathbf{y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x)}, \tag{9}$$

which must be one of the solutions of the original integral equations. Define the vectors

$$\overleftrightarrow{y}_n = \{\mathbf{y}_0(x), \mathbf{y}_1(x), \ldots \mathbf{y}_n(x)\}. \tag{10}$$

Differentiating equation (6) m -times with respect to the embedding parameter q, then setting $q=0$ and finally dividing them by $m!$, we obtain the m^{th} -order deformation equation.

$$\Im[\mathbf{y}_m(\mathbf{x}) - \chi_m \mathbf{y}_{m-1}(\mathbf{x})] = \hbar \mathbf{q}\mathbf{H}(\mathbf{x})\mathbf{R}_m(\vec{\mathbf{y}}_{m-1}, x) \tag{11}$$

where

$$\mathbf{R}_m(\vec{\mathbf{y}}_{m-1}, x) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(x;q)}{\partial q^{m-1}}|_{\mathbf{q=0}} \tag{12}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases} \tag{13}$$

In this way, it is easily to obtain $\mathbf{y}_m(\mathbf{x})$ for $m \ge 1$, at m^{th} order , we have

$$\mathbf{y}(\mathbf{x}) = \sum_{\mathbf{m=0}}^{M} \mathbf{y}_m(\mathbf{x}), \tag{14}$$

when M $\to \infty$ we get an accurate approximation of the original Eq. (2).

Mohamed S. Mohamed et. al [28-29] applied the homotopy analysis method to nonlinear ODE's and suggested the so called optimization method to find out the optimal convergence control parameters by minimum of the square residual error integrated in the whole region having physical meaning. Their approach is based on the square residual error. Let $\Delta(h)$ denote the square residual error of the governing equation (2) and express as

$$\Delta(h) = \int_{\Omega} (N[\tilde{u}_n(t)])^2 d\Omega, \tag{15}$$

where

$$\tilde{u}_m(t) = u_0(t) + \sum_{k=1}^{m} u_k(t) \tag{16}$$

the optimal value of h is given by a nonlinear algebraic equation as:

$$\frac{d\Delta(h)}{dh} = 0. \tag{17}$$

3. Numerical results

In this section, we discuss the implementation of our proposed algorithm and investigate its accuracy by applying the homotopy analysis transform method. The simplicity and accuracy of the proposed method is illustrated through the following numerical examples by computing the absolute error,

$$E_i(x) = |u_i(x) - \tilde{u}_{im}(x)|, \qquad 1 \le i \le n, \tag{18}$$

where $u_i(x)$ is the exact solution and $\tilde{u}_{im}(x)$ is the approximate solution of the problem. To demonstrate the effectiveness of the HATM algorithm discussed above, several examples of variation problems will be studied in this section. Here all the results are calculated by using the symbolic calculus software Mathematica 9.

Example 1. Consider the Abel integral equation [30]:

$$y(x) = x + \tfrac{4}{3}x^{\frac{1}{2}} - \int_0^x \frac{y(t)}{(x-t)^{\frac{1}{2}}} dt, \qquad 0 \le x \le 1, \tag{19}$$

with the initial condition,

$$y(x,0) = x + \tfrac{4}{3}x^{\frac{3}{2}}), \tag{20}$$

with the exact solution

$$y(x) = x. \quad (21)$$

To solve equation (19) by means of the homotopy analysis transform method we consider the following linear

$$\Im[\phi(x;q)] = L[\phi(x;q)],$$

with the property that

$$\Im[c] = 0,\ c \text{ is constants},$$

Taking Laplace transform of equation (19) both of sides subject to the initial condition, we get

$$L[y(x)] - L\left[\left(x + \tfrac{4}{3}x^{\frac{3}{2}}\right)\right] + \sqrt{\tfrac{\pi}{s}}\, L[y(x)] = 0. \quad (22)$$

We now define the nonlinear operator as:

$$L[y(x)] - L\left(x + \tfrac{4}{3}x^{\frac{3}{2}}\right) + \sqrt{\frac{\pi}{s}}\, L[y(x)] = 0. \quad (23)$$

and then the mth-order deformation equation is given by

$$L\left[y_m(x) - \chi_m y_{m-1}(x)\right] = hH(x)\,\mathbf{R}_m(\overleftrightarrow{\mathbf{y}}_{m-1})]. \quad (24)$$

Taking inverse Laplace transform of Eq. (24), we get

$$\mathbf{y}_m(\mathbf{x},\mathbf{t}) = \boldsymbol{\chi}_m \mathbf{y}_{m-1} + \hbar \mathbf{L}^{-1}[\mathbf{H}(\mathbf{x})\mathbf{R}_m(\overleftrightarrow{\mathbf{y}}_{m-1})], \quad (25)$$

where

$$\mathbf{R}_m(\overleftrightarrow{\mathbf{y}}_{m-1}) = \mathbf{L}[\mathbf{y}_{\mathbf{m-1}}] - \mathbf{L}\left[x + \frac{4}{3}x^{\frac{3}{2}}\right](1-\chi_m) + \sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\mathbf{y}_{m-1}]), \quad (26)$$

with assumption $\mathbf{H}(\mathbf{x}) = \mathbf{1}$.

Let us take the initial approximation as

$$y_0(x) = x + \frac{4}{3}x^{\frac{3}{2}}, \quad (27)$$

the other components are given by

$$\mathbf{y}_1(x) = -\frac{4hx^{\frac{5}{2}}}{3} - \frac{1}{2}hx^2\pi, \quad (28)$$

$$\mathbf{y}_2(x) = -\frac{4h(1+h)x^{\frac{3}{2}}}{3} + \frac{8}{15}h^2x^{\frac{5}{2}}\pi + \frac{1}{2}x^2(-h(1+h)\pi + h^2\pi), \quad (29)$$

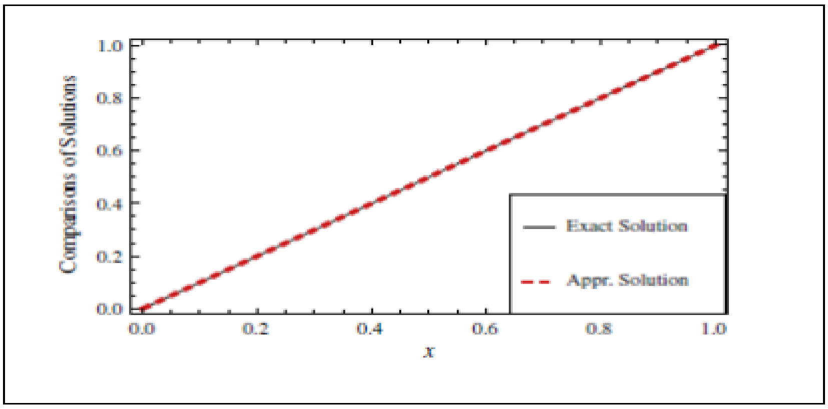
:

:

Hence the solution of the Eq. (19) is given as

$$\mathbf{y}(\mathbf{x}) = \mathbf{y}_0(x) + \sum_{\mathbf{m=1}}^{\infty}\mathbf{y}_m(\mathbf{x}) = \sum_{\mathbf{i=0}}^{n}\mathbf{y}_i(\mathbf{x}) + \mathbf{O}(\mathbf{x}^{1+\frac{n}{2}}) \to x \text{ as } \mathrm{n} \to \infty \text{ and } h = -1. \quad (30)$$

the homotopy analysis transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter h if we select $h = -1$, then the above result is in complete agreement with [30].



Figs. 1: The comparison between the exact solution and the approximate solution of the Abel integral equation (19)

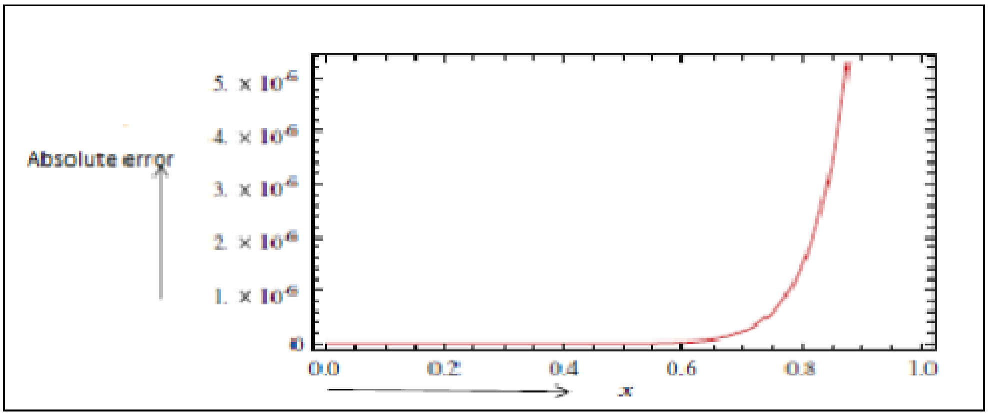


Fig.2 The absolute error between the exact solution and approximate solution of Abel integeral equation (19) at $h_{\text{optimal}} = -1.05$.

Example 2. Consider the following system of Abel's integral equations [31],

$$\mathbf{u(x)} + \int_0^{\mathbf{x}} \frac{\mathbf{v(t)}}{\sqrt{\mathbf{x-t}}}\mathbf{dt} = x + \frac{\pi}{2}x, \quad 0 \le x \le 1,$$

$$\mathbf{v(x)} + \frac{\mathbf{1}}{\mathbf{2}}\int_0^{\mathbf{x}} \frac{\mathbf{u(t)+v(t)}}{\sqrt{\mathbf{x-t}}}\mathbf{dt} = \sqrt{\mathbf{x}} + \frac{\mathbf{2}}{3}x^{\frac{3}{2}} + \frac{\pi}{4}x, \tag{31}$$

with the initial condition

$$\mathbf{u(x,0) = x} + \frac{\pi}{\mathbf{2}}\mathbf{x},$$

$$\mathbf{v(x,0)} = \sqrt{\mathbf{x}} + \frac{\mathbf{2}}{3}x^{\frac{3}{2}} + \frac{\pi}{4}x \tag{32}$$

and the exact solution

$$\mathbf{u(x) = x},$$

$$\mathbf{v(x)} = \sqrt{\mathbf{x}}, \tag{33}$$

where

$$\Im[\phi(x;q)] = L[\phi(x;q)], \tag{34}$$

with the property that

$$\Im[c] = 0,\ c \text{ is constants},$$

which implies that

$$\Im^{-1}(\bullet) = \int_0^t (\bullet)dt.$$

Taking Laplace transform of equation (31) both of sides subject to the initial condition, we get

$$\begin{aligned} &\mathbf{L[u(x)] - L[x + \frac{\pi}{2}x] + \sqrt{\frac{\pi}{s}}(L[v(x)]) = 0}, \\ &\mathbf{L[v(x)] - L[\sqrt{x} + \frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}x] + \frac{1}{2}\sqrt{\frac{\pi}{s}}(L[u(x)] + L[v(x)]) = 0} \end{aligned} \tag{35}$$

We now define the nonlinear operator as

$$\begin{aligned} &\mathbf{N[\phi_1(x;q), \phi_2(x;q)] = L[\phi_1(x;q)] - L[x + \frac{\pi}{2}x] + \sqrt{\frac{\pi}{s}}L[\phi_2(x;q)] = 0}, \\ &\mathbf{N[\phi_1(x;q), \phi_2(x;q)] = L[\phi_2(x;q)] - L[\sqrt{x} + \frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}x] + \frac{1}{2}\sqrt{\frac{\pi}{s}}L[\phi_1(x;q) + \phi_2(x;q)] = 0} \end{aligned} \tag{36}$$

and then the m^{th} - order deformation equation is given by

$$\begin{aligned} &\mathbf{L[u}_m\mathbf{(x)} - \chi_m \mathbf{u}_{m-1}\mathbf{(x)]} = \hbar_1 \mathbf{H}_1\mathbf{(x)R}_{1m}(\overleftrightarrow{\mathbf{u}}_{m-1}), \\ &\mathbf{L[v}_m\mathbf{(x)} - \chi_m \mathbf{v}_{m-1}\mathbf{(x)]} = \hbar_2 \mathbf{H}_2\mathbf{(x)R}_{2m}(\overleftrightarrow{\mathbf{v}}_{m-1}) \end{aligned} \tag{37}$$

Taking inverse Laplace transform of Eq. (37), we get

$$\begin{aligned} &\mathbf{u}_m\mathbf{(x)} = \chi_m \mathbf{u}_{m-1} + \hbar_1 \mathbf{L}^{-1}[\mathbf{H_1(x)R_{1m}}(\overleftrightarrow{\mathbf{u}}_{\mathbf{m-1}})], \\ &\mathbf{v}_m\mathbf{(x)} = \chi_m \mathbf{v}_{m-1} + \hbar_2 \mathbf{L}^{-1}[\mathbf{H_1(x)R_{2m}}(\overleftrightarrow{\mathbf{v}}_{\mathbf{m-1}})], \end{aligned} \tag{38}$$

where

$$\begin{aligned} &\mathbf{R}_{1m}(\overleftrightarrow{\mathbf{u}}_{m-1}) = \mathbf{L[u_{m-1}] - L[x + \frac{\pi}{2}x]}(1-\chi_m) + \sqrt{\frac{\pi}{s}}(\mathbf{L[v}_{m-1}]), \\ &\mathbf{R}_{2m}(\overleftrightarrow{\mathbf{v}}_{m-1}) = \mathbf{L[v_{m-1}] - L[\sqrt{x} + \frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}x]}(1-\chi_m) + \frac{1}{2}\sqrt{\frac{\pi}{s}}(\mathbf{L[u_{m-1}] + L[v}_{m-1}]), \end{aligned} \tag{39}$$

with assumption $\mathbf{H_1(x) = H_2(x) = 1}$.

Let us take the initial approximation as

$$\begin{aligned} &\mathbf{u}_0(x) = \mathbf{x} + \frac{\pi}{2}\mathbf{x}, \\ &\mathbf{v}_0(x) = \sqrt{\mathbf{x}} + \frac{2}{3}\mathbf{x}^{\frac{3}{2}} + \frac{\pi}{4}\mathbf{x}, \end{aligned} \tag{40}$$

the other components are given by

$$\mathbf{u}_1(x) = \frac{h\pi\mathbf{x}}{2} + \frac{1}{3}h\pi\mathbf{x}^{\frac{3}{2}} + \frac{1}{4}h\pi\mathbf{x}^2,$$
$$\mathbf{v}_1(x) = \frac{h\pi\mathbf{x}}{4} + \frac{1}{6}h(4+3\pi)\mathbf{x}^{\frac{3}{2}} + \frac{1}{8}h\pi\mathbf{x}^2, \quad (41)$$

$$\mathbf{u}_2(x) = \frac{1}{2}h(1+h)\pi\mathbf{x} + \frac{1}{3}h(1+2h)\pi\mathbf{x}^{\frac{3}{2}} + \frac{1}{16}h\pi(\mathbf{4}+h(8+3\pi))\mathbf{x}^2 + \frac{2}{15}h^2\pi\mathbf{x}^{\frac{5}{2}},$$
$$\mathbf{v}_2(x) = \frac{1}{4}h(1+h)\pi\mathbf{x} + \frac{1}{6}h(4+3\pi+h(4+6\pi))\mathbf{x}^{\frac{3}{2}} + \frac{1}{32}h\pi(4+h(8+5\pi))\mathbf{x}^2 + \frac{1}{5}h^2\pi\mathbf{x}^{\frac{5}{2}}, \quad (42)$$

:

:

Proceeding in this manner, the rest of the components $y_n(x)$ for $n \geq 5$ can be completely obtained and the series solutions are thus entirely determined. The solution of the problem is given as

$$\mathbf{y(x)} = \mathbf{y}_0(x) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{y}_m(\mathbf{x}), \quad (43)$$

however, mostly, the results given by the Laplace decomposition method and homotopy analysis transform method converge to the corresponding numerical solutions in a rather small region. But, different from those two methods, the homotopy analysis transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary h if we select $h = -1$, then

$$\mathbf{u(x)} = \mathbf{u}_0(x) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{u}_m(\mathbf{x}) = \sum_{\mathbf{i=0}}^{n} \mathbf{y}_i(\mathbf{x}) + \mathbf{O}(\mathbf{x}^{\frac{3}{2}+\frac{n}{2}}) \to x \text{ as n} \to \infty \text{ and } h = -1$$
$$\mathbf{v(x)} = \mathbf{v}_0(x) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{v}_m(\mathbf{x}) = \sum_{\mathbf{i=0}}^{n} \mathbf{v}_i(\mathbf{x}) + \mathbf{O}(\mathbf{x}^{\frac{3}{2}+\frac{n}{2}}) \to \sqrt{\mathbf{x}} \text{ as n} \to \infty \text{ and } h = -1 \quad (44)$$

The above result is in complete agreement with [31],

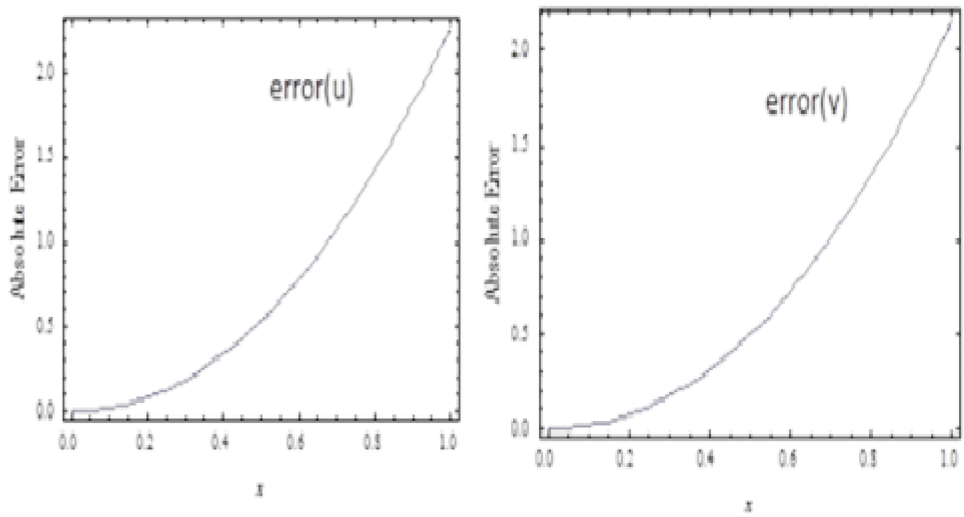


Fig.3 The absolute error between the exact solution and approximate solution of Abel integeral equation (31) at $h_{\text{optimal}} = -0.98$.

The values of h−curve derived from Figs. 2-3.	
$u(x)$	$-1.15 \le h \le -0.4$
$v(x)$	$-1.11 \le h \le -0.4$

Table 1: The values of h.

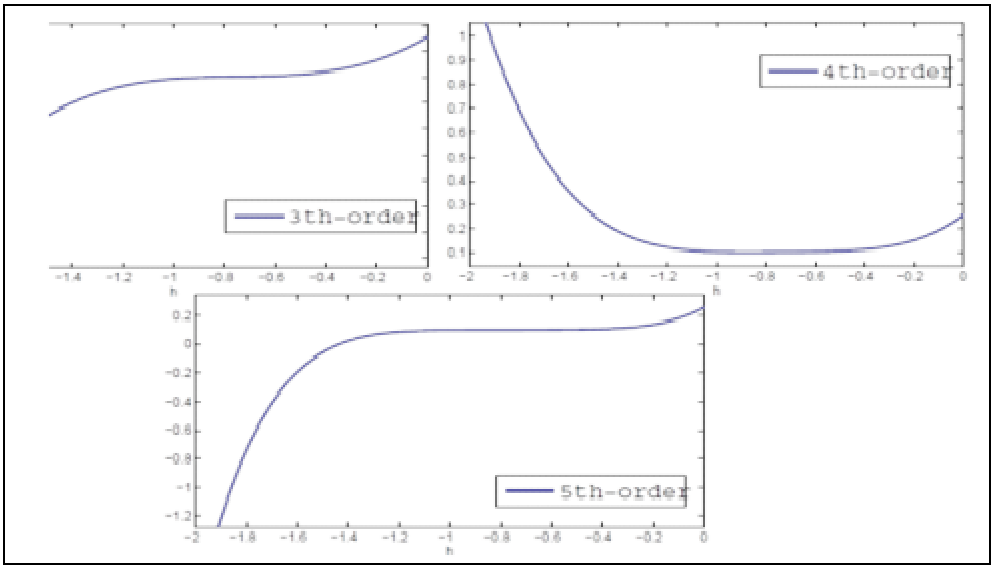


Fig. 4 - The HATM approximate solutions $u(x)$ with $x = 0.1$ lead to the h -curves

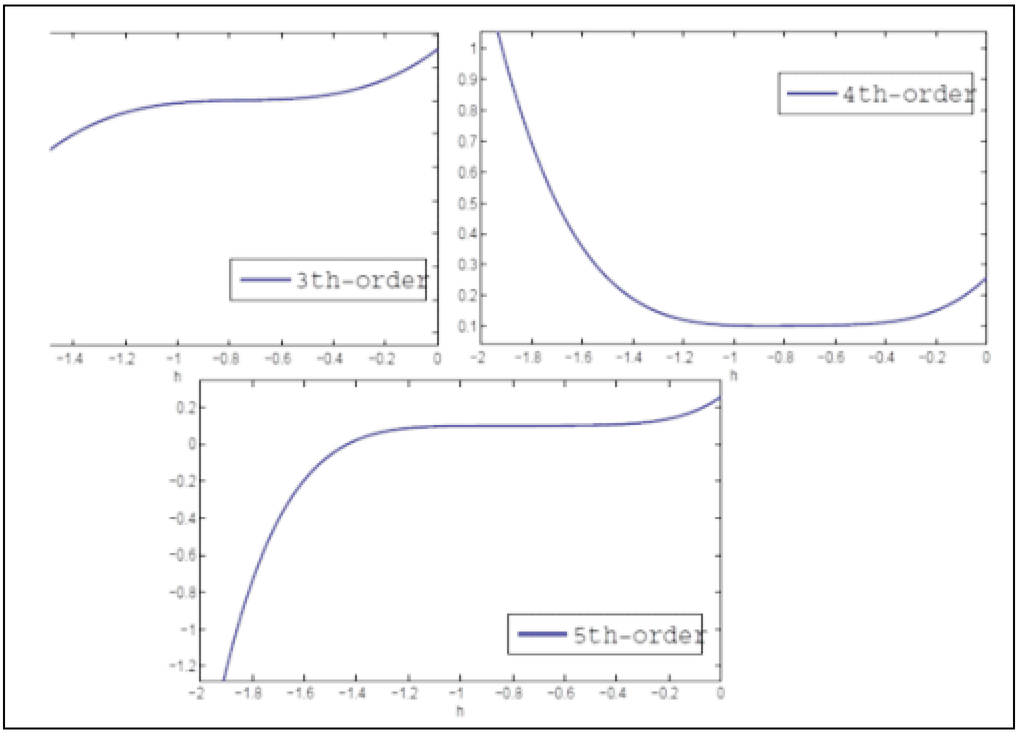


Fig. 5 - The HATM approximate solutions $v(x)$ with $x = 0.1$ lead to the h -curves. The graphical comperison between the exact solution and the approximate solution obtained by the HATM at $h_{\text{optimal}} = -0.98$. It can be seen that the solution obtained by the present method nearly identical to the exact solution. The above result is in complete agreement with [31].

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Also, from Figs. 1 to 5 shows the graphical comparison between the exact solution and the approximate solution obtained by the OHATM. It can be seen that the solution obtained by the present method nearly identical to the exact solution. The above result is in complete agreement with [30] and [31].

4. Conclusions

The main aim of this work is to provide the system of Abel integral equations of the second kind has been studied by the optimal Homotopy analysis transform method OHATM. The OHATM is more suitable than other analytic methods. OHATM is coupling of homotopy analysis and Laplace transform method. The new modification is a powerful tool to search for solutions of Abel's integral equation. An excellent agreement is achieved. The proposed method is employed without using linearization, discretization or transformation. It may be concluded that the OHATM is very powerful and efficient in finding the analytical solutions for a wide class of differential and integral equation. The approximate solution of this system is calculated in this form of series which its components are computed by applying a recursive relation. Results indicate that the solution obtained by this method converges rapidly to an exact solution. The graphs are plotted confirm the results. In comparison with the methods in Ref. [30] and [31], the numerical results show that the HATM is more accurate.

5. References

[1]. A.M. Wazwaz, Linear and Nonlinear Integral equations methods and applications, Springer, Heidelberg, Dordrecht, London, New York, 2011.

[2]. A.M. Wazwaz, A first course in integral equations, World Scientific, New Jersey, 1997.

[3]. R. Gorenflo, S. Vessella, Abel Integral Equations, Springer, Berlin, 1991.

[4]. A.M. Wazwaz, M.S. Mehanna, The combined Laplace Adomian method for handling singular integral equation of heat transfer, Int. J. Nonlinear Sci. 10 (2010) 248-252.

[5]. N. Zeilon, Sur quelques points de la theorie de l'equationintegraled'Abel, Arkiv. Mat. Astr. Fysik. 18 (1924) 1-19.

[6]. R.K. Pandey, Om P. Singh, V.K. Singh, Efficient algorithms to solve singular integral equations of Abel type, Comput. Math. Appl. 57 (2009) 664--676.

[7]. S. Kumar, Om P. Singh, Numerical inversion of Abel integral equation using homotopy perturbation method, Z. Naturforsch. 65a (2010) 677-682.

[8]. S. Kumar, Om P. Singh, S. Dixit, Homotopy perturbation method for solving system of generalized Abel's integral equations, Appl. Appl. Math. 6 (2009) 268-283.

[9]. S. Dixit, Om P. Singh, S. Kumar, A stable numerical inversion of generalized Abel's integral equation, Appl. Numer. Math. 62 (2012) 567-579.

[10]. S.A. Yousefi, Numerical solution of Abel's integral equation by using Legendre wavelets, Appl. Math. Comput. 175 (2006) 574- 580.

[11]. M. Khan, M.A. Gondal, A reliable treatment of Abel's second kind singular integral equations, Appl. Math. Lett. 25 (11) (2012) 1666-1670.

[12]. M. Li, W. Zhao, Solving Abel's type integral equation with Mikusinski's operator of fractional order, Adv. Math. Phys. (2013). Article ID 806984, 4 pp.

[13]. S. J. Liao. On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147(2004): 499-513.

[14]. S. J. Liao. Comparison between the homotopy analysis method and homotopy perturbation method, Appl. Math. Comput. 169(2005): 1186-1194.

[15]. K. M. Hemida, K. A. Gepreel and M. S. Mohamed. Analytical approximate solution to the time- space nonlinear partial fractional differential equations, International journal of pure and applied

mathematics, 78(2)(2012):233-244.

[16]. K. Hemida, M. S. Mohamed. Numerical simulation of the generalized Huxley equation by homotopy analysis method, Journal of applied functional analysis, 5(4)(2010)344-350.

[17]. H. A. Ghany and M. S. Mohamed. White noise functional solutions for the wick-type stochastic fractional Kdv-Burgers-Kuramoto equations, Journal of the Chinese Journal of Physics, 50(4)(2012):619-626.

[18]. K. A. Gepreel and M. S. Mohamed. Analytical approximate solution for nonlinear space-time fractional Klein Gordon equation, Chinese physics B, 22(1)(2013)010201-6.

[19]. S. A. Khuri. A Laplace decomposition algorithm applied to a class of nonlinear di¤erential equations, Journal of Applied Mathematics 1 (2001): 141-155.

[20]. E. Yusufoglu. Numerical solution of Du¢ ng equation by the Laplace decomposition algorithm, Applied Mathematics and Computation 177 (2006): 572-580.

[21]. K. Yasir. An e¤ective modication of the Laplace decomposition method for nonlinear equations, International Journal of Nonlinear Sciences and Numerical Simulation 10 (2009): 1373--1376.

[22]. Y. Khan, N. Faraz, S. Kumar, A. A.Yildirim. A coupling method of homotopy method and Laplace transform for fractional modells. UPB Sci Bull Ser A Appl Math Phys 74(1)(2012):57-68.

[23]. J. Singh, D. Kumar, S. Kumar. New homotopy analysis transform algorithm to solve Volterra integral equation. Ain Shams Eng J 2013, doi: .org/10.1016/j.asej.2013.07.004.

[24]. M. M. Khader, S. Kumar, S. Abbasbandy. New homotopy analysis transform method for solving the discontinued problems arising in nanotechnology. Chin Phys B 22(11)(2013):110201.

[25]. S. Arife, S. K. Vanani, F. Soleymani. The Laplace homotopy analysis method for solving a general fractional difusion equation arising in nano-hydrodynamics. J Comput Theor Nanosci 10(2012):1-4.

[26]. M. Khan, MA. Gondal, I. Hussain, SK. Vanani, A new comparative study between homotopy analysis transform method and homotopy perturbation transform method on a semi-infinite domain. Math Comput Modell 2012;55:1143-50.

[27]. Z. M. Odibat. Differential transform method for solving Volterra integral equation with seperable kernel. Math Comput Modell 2008;48:1144-1149.

[28]. S. M. Abo-Dahab, M. S. Mohamed and T. A. Nofal, A One Step Optimal Homotopy Analysis Method for propagation of harmonic waves in nonlinear generalized magneto-thermoelasticity with two relaxation times under inuence of rotation, Journal of in Abstract and Applied Analysis, (2013) 1-14.

[29]. K. A. Gepreel, and Mohamed S. Mohamed, An optimal homotopy analysis method nonlinear fractional differential equation, Journal of Advanced Research Dynamical and Control Systems, 6(2014), 1-10.

[30]. S. Kumar, A. Kumar, D. Kumar, J. Singh and A. Singh. Analytical solution of Abel integral equation arising in astrophysics via Laplace transform, Journal of the Egyptian Mathematical Society, 2014.

[31]. Jafarian, P. Ghaderi, K. Alireza, D. aleanu, Analytical treatment of system of Abel integral equations by homotopy analysis method, 66(3)(2014) 603-611.