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On the Characterization of Nonuniform Wavelet Sets on Positive Half Line

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Abstract. In this paper, we study the characterization of nonuniform wavelet sets on positive half line and we also prove the existence of nonuniform wavelet sets associated with the dilation N and the translation set $\Lambda^+_{r,N} = \left\{0, \frac{r}{N}\right\} + \mathbb{Z}^+, N > 1$ (an integer) and r is an odd integer with $1 \le r \le 2N-1$ such that r and N are relatively prime and $\mathbb{Z}^+$ is the set of non-negative integers.

Keywords: Nonuniform Wavelets and nonuniform wavelet set

1. Introduction

The concepts of wavelet and multiresolution analysis has been extended to many different setups. One can replace the dilation factor 2 by an integer $M \ge 2$ and one needs to construct $N-1$ wavelets to generate the whole space $L^2(\mathbb{R})$. In general, in higher dimensions, it can be replaced by a dilation matrix A, in which case the number of wavelets required is $|\det A| - 1$. But in all these cases, the translation set is always a group. In the two papers [5, 6], Gabardo and Nashed considered a generalization of Mallat's [13] celebrated theory of MRA, in which the translation set acting on the scaling function associated with the MRA to generate the subspace V_0 is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. More precisely, this set is of the form $\left\{0, \frac{r}{N}\right\} + 2\mathbb{Z}$, where $N \ge 1$ is an integer, $1 \le r \le 2N-1$, r is an odd integer relatively prime to N. They call this a nonuniform multiresolution analysis (NUMRA) and is based on the theory of spectral pairs. Farkov [3] has given general construction of compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$. Farkov et al. [4] gave an algorithm for biorthogonal wavelets related to Walsh functions on positive half line. Shah [15], studied the construction of p-wavelet packets associated with the multiresolution analysis defined by Farkov [3], for $L^2(\mathbb{R}^+)$. Meenakshi et al. [14] studied NUMRA on positive half line. Recently, Shah and Abdullah [16], have constructed nonuniform multiresolution analysis on local fields of positive characteristic and proved the necessary and sufficient condition for the existence of associated wavelets.

In the present paper, we study characterization of nonuniform wavelet sets on positive half line and we also prove the existence of nonuniform wavelet sets associated with the dilation N and the translation set $\Lambda^+_{r,N} = \left\{0, \frac{r}{N}\right\} + \mathbb{Z}^+, N > 1$ is an integer and r is an odd integer with $1 \le r \le 2N-1$ such that r and N are relatively prime. This paper is organized as follows. In Sec. 2, we present a brief review of generalized Walsh functions and polynomials, the Walsh-Fourier transform. In Sec. 3, we study characterization for nonuniform wavelet sets and proves their existence.

2. Notations and preliminaries

Throughout we shall denote $|A|$ and $\chi_A(\xi)$, respectively Lebesgue measure and characteristic function of A. If $N \ge 1$ is an integer, we define

$$\Gamma^+_N = \{mN + j : m \in \mathbb{Z}^+, \quad j = 0, 1, \dots, N-1\}$$

and if r is any odd integer with $1 \le r \le 2N-1$ such that r and N are relatively prime, the set $\Lambda^+_{r,N} = \left\{0, \frac{r}{N}\right\} + \mathbb{Z}^+$.

2.1. Walsh-Fourier Analysis

Let p be a fixed natural number greater than 1. As usual, let $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{Z}^+ = \{0, 1, \dots\}$. Denote by $[x]$ the integer part of x. For $x \in \mathbb{R}^+$ and for any positive integer j, we set

$$x_j = [p^j x] (\mathrm{mod}\, p), \qquad x_{-j} = [p^{1-j} x] (\mathrm{mod}\, p), \tag{2.1}$$

where $x_j,\ x_{-j} \in \{0, 1, \dots, p-1\}$.

Consider the addition defined on $\mathbb{R}^+$ as follows:

$$x \oplus y = \sum_{j<0} \xi_j p^{-j-1} + \sum_{j>0} \xi_j p^{-j} \tag{2.2}$$

with

$$\xi_j = x_j + y_j (\mathrm{mod}\, p), \qquad j \in \mathbb{Z} \setminus \{0\}, \tag{2.3}$$

where $\xi_j \in \{0, 1, \dots, p-1\}$ and $x_j,\ y_j$ are calculated by (2.1). Moreover, we write $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

For $x \in [0,1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1, & x \in \left[0, \frac{1}{p}\right) \\ \epsilon_p^j, & x \in [jp^{-1}, (j+1)p^{-1}),\ j = 1, 2, \dots, p-1, \end{cases} \tag{2.4}$$

where $\epsilon_p = \exp\left(\frac{2\pi i}{p}\right)$. The extension of the function r_0 to $\mathbb{R}^+$ is defined by the equality $r_0(x+1) = r_0(x),\ x \in \mathbb{R}^+$. Then the generalized Walsh functions $\{\omega_m(x)\}_{m \in \mathbb{Z}^+}$ are defined by

$$\omega_0(x) = 1, \qquad \omega_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j},$$

where $m = \sum_{j=0}^{k} \mu_j p^j,\ \mu_j \in \{0, 1, 2, \dots, p-1\}, \mu_k \neq 0$.

For $x,\ \omega \in \mathbb{R}^+$, let

$$\chi(x, \omega) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j \omega_{-j} + x_{-j} \omega_j)\right), \tag{2.5}$$

where x_j and ω_j are calculated by (2.1).

We observe that

$$\chi\left(x, \frac{m}{p^{n-1}}\right) = \chi\left(\frac{x}{p^{n-1}}, m\right) = \omega_m\left(\frac{x}{p^{n-1}}\right) \qquad \forall x \in [0, p^{n-1}), \qquad m \in \mathbb{Z}^+.$$

The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}^+)$ is defined by

$$\tilde{f}(\omega) = \int_{\mathbb{R}^+} f(x)\,\overline{\chi(x,\omega)}\,dx, \tag{2.6}$$

where $\chi(x,\omega)$ is given by (2.5).

If $f \in L^2(\mathbb{R}^+)$ and

$$J_a f(\omega) = \int_0^a f(x)\overline{\chi(x,\omega)}\,dx \qquad (a < 0), \tag{2.7}$$

then $\hat{f}$ is defined as limit of $J_a f$ in $L^2(\mathbb{R}^+)$ as $a \longrightarrow \infty$.

The properties of Walsh-Fourier transform are quite similar to the classical Fourier transform. It is known that systems $\{\chi(\alpha,.)\}_{\alpha=0}^{\infty}$ and $\{\chi(.,\alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2(0,1)$. Let us denote by $\{\omega\}$ the fractional part of ω. For $l \in \mathbb{Z}^+$, we have $\chi(l,\omega) = \chi(l,\{\omega\})$.

If $x, y, \omega \in \mathbb{R}^+$ and $x \oplus y$ is *p*-adic irrational, then

$$\chi(x \oplus y,\omega) = \chi(x,\omega)\,\chi(y,\omega), \quad \chi(x \oplus y,\omega) = \chi(x,\omega)\,\chi(y,\omega), \tag{2.8}$$

Definition 2.2. Let $N \geq 1$ be an integer and *r* be an odd integer with $1 \leq r \leq 2N-1$. A collection of functions $\Psi = \{\psi^k : k = 1,2,\dots,K\} \subset L^2(\mathbb{R}^+)$ is called a set of wavelets associated with the dilation *N* and the translation set $\Lambda_{\mathrm{r,N}}^{+}$ if the family $\{\psi_{j,\lambda}^{k} : k = 1,2,\dots,K,\ j \in \mathbb{Z},\ \lambda \in \Lambda_{\mathrm{r,N}}^{+}\}$ forms a complete orthonormal system for $L^2(\mathbb{R}^+)$, where $\psi_{j,\lambda}^{k}(x) = N^{\frac{j}{2}}\psi^k(N^j x \ominus \lambda)$.

3. Main Results

Lemma 3.1. Let $N \geq 1$ be an integer and *r* be an odd integer with $1 \leq r \leq 2N-1$. Let $\varphi \in L^2(\mathbb{R}^+)$ with $\|\varphi\|_{L^2(\mathbb{R}^+)} = 1$. Then

(i) For a given odd *r*, the collection $\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_{\mathrm{r,N}}^{+}}$ is an orthonormal system in $L^2(\mathbb{R}^+)$ if and only if

$$\sum_{p\in\mathbb{Z}^+} |\tilde{\varphi}(\xi + p)|^2 = 1 \qquad for\ a.e.\ \xi \in \mathbb{R}^+ \tag{3.1}$$

and

$$\sum_{p\in\mathbb{Z}^+} \overline{\chi\left(\frac{r}{N},p\right)}\,|\tilde{\varphi}(\xi + p)|^2 = 0 \qquad for\ a.e.\ \xi \in \mathbb{R}^+. \tag{3.2}$$

(ii) The collection $\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_{\mathrm{r,N}}^{+}}$ is an orthonormal system for every odd integer *r* with $1 \leq r \leq 2N-1$ if and only if

$$\sum_{\gamma\in\Gamma_N^+} |\tilde{\varphi}(\xi \ominus \gamma)|^2 = 1 \qquad for\ a.e.\ \xi \in \mathbb{R}^+. \tag{3.3}$$

Proof. The proof of (i) is given in [14]. We will only prove (ii). By (i), the collection $\{\varphi(x \ominus \lambda)\}_{\lambda\in\Lambda_{\mathrm{r,N}}^{+}}$ is an orthonormal system for every odd integer *r* with $1 \leq r \leq 2N-1$. Define,

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$$c_j(\xi) = \sum_{q\in\mathbb{Z}^+} |\tilde{\varphi}(\xi \ominus (qN+j))|^2, \quad 0 \le j \le N, \qquad for\ a.e.\ \xi \in \mathbb{R}^+$$

and $c_0(\xi) = c_N(\xi)$. Suppose that (3.1) and (3.2) hold for every odd r with $1 \le r \le 2N-1$. By (3.1), we have

$$\sum_{j=0}^{N-1} c_j(\xi) = 1 \tag{3.4}$$

By (3.2), we have

$$0 = \sum_{p\in\mathbb{Z}^+} \overline{\chi\left(\frac{r}{N}, p\right)} |\tilde{\varphi}(\xi+p)|^2$$

$$= \sum_{q\in\mathbb{Z}^+} \sum_{j=0}^{N-1} \overline{\chi\left(\frac{r}{N}, qN+j\right)} |\tilde{\varphi}(\xi \ominus (qN+j))|^2$$

$$= \sum_{j=0}^{N-1} \overline{\chi\left(\frac{r}{N}, j\right)} \left[\sum_{q\in\mathbb{Z}^+} |\tilde{\varphi}(\xi \ominus (qN+j))|^2\right] = \sum_{j=0}^{N-1} \overline{\chi\left(\frac{r}{N}, j\right)}\, c_j(\xi). \tag{3.5}$$

From (3.4), we have

$$\sum_{j=0}^{N-1} \sum_{q\in\mathbb{Z}^+} |\tilde{\varphi}(\xi \ominus (qN+j))|^2 = 1 \qquad for\ a.e.\ \xi \in \mathbb{R}^+,$$

i.e.

$$\sum_{\gamma\in\Gamma_N^+} |\tilde{\varphi}(\xi \ominus \gamma)|^2 = 1 \qquad for\ a.e.\ \xi \in \mathbb{R}^+.$$

Conversely, suppose that (3.3) holds, for a. e. $\xi \in \mathbb{R}^+$. Clearly (3.1) and (3.2) hold for each odd r with $1 \le r \le 2N-1$ by using (3.4) and (3.5). This completes the proof.

The following theorem follows as a special case of Calogero's characterization of wavelets on general lattices with expansive matrix dilations ([1, Theorem 3.1]). This result can also be obtained from various more general results that have appeared recently in the literature characterizing certain normalized tight frame system (see [8, 9, 17]).

Theorem 3.2. Let $1 \le r \le 2N-1$ be a fixed odd integer. A collection $\Psi = \{\psi^k : k = 1, 2, \dots, K\} \subset L^2(\mathbb{R}^+)$ with $\|\psi^k\|_{L^2(\mathbb{R}^+)} = 1,\ k = 1, 2, \dots, K,$ is a set of wavelets associated with the dilation N and the translation set $\Lambda_{r,N}^+$ if and only if for a.e. $\xi \in \mathbb{R}^+$;

$$\sum_{k=1}^{K} \sum_{l\in\mathbb{Z}} \left|\tilde{\psi}^k(N^l\xi)\right|^2 = 1 \tag{3.6}$$

and for any $q \in \mathbb{Z}^+ \setminus 2N\,\mathbb{Z}^+$

$$t_q(\xi) = \sum_{k=1}^{K} \sum_{l=1}^{\infty} \tilde{\psi}^k(N^l\xi)\overline{\tilde{\psi}^k(N^l(\xi+q))} + \left(1 + \chi\left(\frac{r}{N}, q\right)\right) \sum_{k=1}^{K} \tilde{\psi}^k(\xi)\,\overline{\tilde{\psi}^k(\xi+q)} = 0 \tag{3.7}$$

Definition 3.3. Let $N \geq 1$ be an integer and r be an odd integer with $1 \leq r \leq 2N-1$. Let $W \subset \mathbb{R}^+$ be a measurable set with $|W| = 1$. We call W a wavelet set associated with the dilation N and translation set $\Lambda^+_{r,N}$ if $\mathcal{F}^{-1}\left(\chi_W(\xi)\right)$ is an orthonormal wavelet in the sense of Definition 2.2 (with $K = 1$), where $\mathcal{F}^{-1}$ denotes the inverse Walsh-Fourier transform.

Lemma 3.4. Let $W \subset \mathbb{R}^+$ be a measurable set with finite measure and $\psi(x) \in L^2(\mathbb{R}^+)$ with $|\tilde{\psi}(\xi)| = \chi_W(\xi)$. If ψ satisfies (3.1) and (3.2) for a fixed odd integer r with $1 \leq r \leq 2N-1$, then it also satisfies (3.7) (with $K = 1$ and $\psi^1 = \psi$) for the same r.

Proof. Fix any $q \in \mathbb{Z}^+ \setminus 2N\,\mathbb{Z}^+$. If $\tilde{\psi}\left(N^l\xi\right) \neq 0$ for a fixed $l \geq 0$, then

$$\left|\tilde{\psi}\left(N^l\xi\right)\right| = \chi_W\left(N^l\xi\right) = 1, \qquad a.e.\ \xi \in \mathbb{R}^+.$$

From Eqn.(3.1), we have

$$\sum_{p \in \mathbb{Z}^+ \setminus \{0\}} \left|\tilde{\psi}\left(N^l\xi + p\right)\right| = 1.$$

There exists an integer $p^{(l)}(\xi) \neq 0$ such that

$$\left|\tilde{\psi}\left(N^l\xi + p^{(l)}(\xi)\right)\right| = 1$$

and

$$\tilde{\psi}\left(N^l\xi + p\right) = 0, \quad p \in \mathbb{Z}^+ \setminus \left\{0, p^{(l)}(\xi)\right\} \tag{3.8}$$

From Eqn.(3.2) and above equalities, we have

$$\overline{\chi\left(\frac{r}{N}, p^{(l)}(\xi)\right)} = -1.$$

So $p^{(l)}(\xi)\frac{r}{N} \in 2\mathbb{Z}^+ + 1$ and $p^{(l)}(\xi) \notin 2N\mathbb{Z}^+$.

When $l \geq 1$, from Eqn.(3.8), we have

$$\tilde{\psi}\left(N^l(\xi + q)\right) = \tilde{\psi}\left(N^l\xi + N^l q\right) = 0,$$

since $p^{(l)}(\xi) \neq N^l q$. Therefore,

$$\sum_{l=1}^{\infty} \tilde{\psi}\left(N^l\xi\right) \overline{\tilde{\psi}(N^l(\xi + q))} = 0 \quad for\ a.e.\ \xi \in \mathbb{R}^+. \tag{3.9}$$

If $\tilde{\psi}(\xi) \neq 0$ then, whether or not $q = p^{(0)}(\xi)$, the previous argument with $l = 0$ shows that

$$\tilde{\psi}(\xi + q)\left(1 + \chi\left(\frac{r}{N}, q\right)\right) = 0.$$

Eqns.(3.9) and (3.10), proves the equality (3.7) with $K = 1$.

Lemma 3.5. Let $W \subset \mathbb{R}^+$ be a measurable set with finite measure. Then, the following two statements are equivalent

(A) For a.e. $\xi \in \mathbb{R}^+$, we have

$$\sum_{p \in \mathbb{Z}^+} \chi_W(\xi + p) = 1 \quad and \quad \sum_{p \in \mathbb{Z}^+} \chi\left(\frac{r}{N}, p\right) \chi_W(\xi + p) = 0,$$

where r is a fixed odd integer coprime to N with $1 \leq r \leq 2N - 1$.

(B) For a.e. $\xi \in \mathbb{R}^+$, we have

$$\sum_{\gamma \in \Gamma_N^+} \chi_W(\xi \ominus \gamma) = 1.$$

Proof. Lemma 3.1 with $\tilde{\varphi} = \chi_W$ shows that (B) ⇒(A). Conversely, assume that (A) holds. For a.e. $\xi \in \mathbb{R}^+$, there exists a positive integer $p_1 = p_1(\xi)$ such that

$$\xi + p_1 \in W \quad and \quad \xi + p \notin W, \; p \in \mathbb{Z}^+ \setminus \{p_1\}$$

by the first equality in (A). Then $\chi\left(\frac{r}{N}, p_1\right) = 0$ by the second equality in (A). Then $\xi \in W - p_1 = W + mN + j$. This means that $W - p_1$ must be a component of $\bigcup_{\gamma \in \Gamma_N^+} (W \oplus \gamma)$. since ξ is an arbitrary element of $\mathbb{R}^+$. Therefore $\bigcup_{\gamma \in \Gamma_N^+} (W \oplus \gamma) = \mathbb{R}^+$ where the union is disjoint, which is equivalent to (B).

The following theorem provides a characterization for nonuniform wavelet sets and proves their existence.

Theorem 3.6. Let $W \subset \mathbb{R}^+$ be a measurable set with $|W| = 1$. Then, the following three statements are equivalent

(A) $\mathcal{F}^{-1}(\chi_W(\xi))$ is a wavelet associated with the dilation N and the translation set $\Lambda_{r,N}^+$ for one particular odd r prime to N with $1 \leq r \leq 2N - 1$.

(B) (i) $\bigcup_{l \in \mathbb{Z}} N^l W = \mathbb{R}^+$ and (ii) $\bigcup_{\gamma \in \Gamma_N^+} (W \oplus \gamma) = \mathbb{R}^+$, where both unions are disjoint almost everywhere.

(C) For every odd integer r with $1 \leq r \leq 2N - 1$, $\mathcal{F}^{-1}(\chi_W(\xi))$ is a wavelet associated with the dilation N and the translation set $\Lambda_{r,N}^+$.

Moreover, a measurable set W satisfying the three previous equivalent statements always exists.

Proof. From Eqn.(3.6) with $K = 1$; we have

$$\sum_{l \in \mathbb{Z}} \left|\tilde{\psi}^k(N^l \xi)\right|^2 = 1, \qquad for\ a.e.\ \xi \in \mathbb{R}^+,$$

by replacing $\tilde{\psi}(\xi)$ by $\chi_W(\xi)$, we have

$$\sum_{l \in \mathbb{Z}} \chi_W(N^l \xi) = 1, \qquad i.e. \qquad \bigcup_{l \in \mathbb{Z}} N^l W = \mathbb{R}^+.$$

In Eqn.(3.3), by replacing φ by ψ, we have

$$\sum_{\gamma\in\Gamma_N^+}\left|\tilde{\psi}(\xi\ominus\gamma)\right|^2=1 \qquad for\ a.e.\ \xi\in\mathbb{R}^+,$$

above is equivalent to

$$\sum_{\gamma\in\Gamma_N^+}\chi_W(\xi\ominus\gamma)=1 \quad ie. \quad \bigcup_{\gamma\in\Gamma_N^+}(W\oplus\gamma)=\mathbb{R}^+.$$

If (A) holds, part (i) of (B) is a consequence of Theorem 3.2, while part (ii) of (B) follows from Lemma 3.1 and Lemma 3.5. On the other hand, if (B) holds, (C) follows from Lemma 3.1, Lemma 3.4 and Theorem 3.2. Since (C) obviously implies (A), this proves our claim.

We now prove the existence of such a set W. Consider the set

$$E=\left[0,\frac{N}{4}\right)\cup\left[\frac{1}{4},\frac{3}{4}\right)\cup\left(\frac{3N}{4},N\right).$$

Note that E satisfies the following condition

$$\sum_{m\in\mathbb{Z}^+}\sum_{j=0}^{N-1}\chi_E(\xi\ominus(Nm+j))=1. \tag{3.11}$$

Let

$$F=\left[0,\frac{1}{4}\right)\cup\left[\frac{3N}{4},N\right)$$

and F satisfies the following condition

$$\sum_{l\in\mathbb{Z}^+}\chi_F(N^l\xi)=1. \tag{3.12}$$

Since $\frac{1}{2}$ is an interior point of E and F is bounded away from $\frac{1}{2}$ and has nonempty interior. By using the result of Dai et al. [2], we can construct a measurable set W which is both $N\mathbb{Z}^+$-translation congruent to E and N-dilation congruent to F. Equivalently W satisfies

$$\sum_{m\in\mathbb{Z}^+}\chi_W(\xi\ominus Nm)=\sum_{m\in\mathbb{Z}^+}\chi_E(\xi\ominus Nm) \tag{3.13}$$

and

$$\sum_{l\in\mathbb{Z}^+}\chi_W(N^l\xi)=\sum_{l\in\mathbb{Z}^+}\chi_F(N^l\xi). \tag{3.14}$$

It follows immediately from(3.11), (3.12), (3.13) and (3.14) that the conditions in (B) are all satisfied by the set W, and this completes the proof..

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