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Finite Element Method for solving Linear Volterra Integro-Differential Equations of the second kind

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Abstract. In this paper, we present a method for numerical solution of linear Volterra integro - differential equations with boundary conditions. First, we obtain variational form of the problem, and then, finite element method and basis functions will be used. Also, the error analysis of the method is considered. Furthermore, the efficiency of the proposed method will be considered through numerical examples.

Keywords: Linear Volterra Integro-Differential Equations, Finite element method, Error estimation.

1. Introduction

Many authors have studied finite element methods for integral equations. See, Atkinson[2] , Ikebe [7], Nedelec [12], Sloan [16], and Wendland [19]. Adaptive finite element methods for integral equations have been considered more recently. See,[13,19].

Integro-differential equations have been discussed in many applied fields, such as biological, physical and engineering problems. They are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution. There are several methods for solving integro-differential equations, Yanik and Fairweather in [20], used finite element methods for solving integro-differential equation of parabolic type and obtained an $O(h^{r+1}+(\Delta t)^2)$ order estimate for L^2 norm of the error.

In [9], Leroux and Thomee analyzed a Galerkin approximation in space with the Euler method in time for a semilinear integro-differential equations of parabolic type with non smooth data. The stability of Ritz-Volterra projections and error estimates for finite element methods for a class of integro-differential equations of parabolic type is studied by Lin and Zhang [10]. Sloan and Thomée, used time discretization of an integro-differential equation of parabolic type [17]. Brunner applied a collocation-type method to Volterra-Hammerstein integral equation as well as integro-differential equations, [3]. Volk used projection method to solve linear integro-differential equations, [18]. High order nonlinear Volterra Fredholm integro-differential equations has been solved in [11] by using Taylor polynomial. Sabri-Nadjafi [15] proposed He's variational iteration method for two systems of Volterra integro-differential equations.

In this paper, we use Lagrange polynomials with Finite element method to obtain an approximate solution of the problem. To illustrate the basic approach, we consider the following volterra integro-differential equation

$$-u''+b(x)u'(x)+c(x)u(x)=f(x)+\int_a^x K(x,t)u(t)dt u(a)=0, \quad u(b)=0,\ \Omega=[a,b] \tag{1}$$

We assume that $K(x,t)$ and $f(x)$ are continuous functions respect to their arguments, and $b(x)$ and $c(x)$ are nonnegative functions and belong to $C^1(\Omega)$. First, for using finite element method, by suitable linear transform, we convert the essential boundary condition to homogeneous one, and then we define

$$V=H_0^1(\Omega)=\{v\in H^1(\Omega), \quad v(a)=v(b)=0\}$$

where V is a Sobolev space together with following norm:

$$\|u\|_V^2=\|u\|_{L^2(\Omega)}^2+\|u'\|_{L^2(\Omega)}^2 .$$

For obtaining varational form, we let $B:V\times V\to R$ and $L:V\to R$ be bilinear form and linear functional, respectively.

The varational form of the problem is given as follows

$$B(u,v) = L(v), \quad \forall v \in V, \tag{2}$$

where

$$B(u,v) = \int_\Omega u'(x)v'(x)dx + \int_\Omega b(x)u'(x)v(x)dx + \int_\Omega c(x)u(x)v(x)dx -$$
$$\int_\Omega v(x)(\int_a^b K(x,t)u(t)dt)dx L(v) = \int_\Omega f(x)v(x)dx \tag{3}$$

where $v(x) \in V$ is an arbitary function.

Lemma 1.1 Let B be bilinear form defined by (3). If $M_1 \le c(x) \le M_2$ and $P_1 \le b(x) \le P_2$, then B is continuous.

Proof. For B, we can write,

$$|B(u,v)| = |\int_\Omega u'(x)v'(x)dx + \int_\Omega b(x)u'(x)v(x)dx + \int_\Omega c(x)u(x)v(x)dx - \int_\Omega v(x)(\int_a^x K(x,t)u(t)dt)dx|$$

Using the Cauchy-Schwarz inequality and Sobolev norm, we have

$$|B(u,v)| \le \|u\|_{H^1}\|v\|_{H^1} + P_2\|u\|_{H^1}\|v\|_{H^1} + M_2\|u\|_{H^1}\|v\|_{H^1} + KR\|u\|_{H^1}\|v\|_{H^1} =$$
$$(1+P_2+M_2+KR)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} = C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

which $K = \max_{\substack{a\le x\le b \\ a\le t\le x}} |K(x,t)|$, $R = \|1\|^2_{L^2(\Omega)}$ and $C = 1+P_2+M_2+KR$. So B is continous.

In addition of the hypothesis of lemma 1.1, suppose $0 \le b'(x) \le T_2$. Now we consider the V-ellipticity of B. For this purpose we write

$$\int_\Omega v'(x)v'(x)dx + \int_\Omega c(x)v(x)v(x)dx \ge \int_\Omega v'^2(x)dx \ge \frac{1}{1+c}\|v\|^2_{H^1}, \tag{4}$$

and

$$\int_\Omega b(x)v'(x)v(x)dx = \frac{-1}{2}\int_a^b b'(x)(v(x))^2dx \ge \frac{-T_2}{2}\int_a^b (v(x))^2dx \ge \frac{-T_2}{2}\|v\|^2_{H^1}, \tag{5}$$

also

$$-\int_\Omega v(x)(\int_a^x K(x,t)v(t)dt)dx \ge -|\int_\Omega v(x)(\int_a^x K(x,t)v(t)dt)dx| \ge -KR\|v\|^2_{L^2} \ge -KR\|v\|^2_{H^1}. \tag{6}$$

By ((4)), ((5)), ((6)), we have

$$B(v,v) \ge (\frac{1}{1+c} - \frac{T_2}{2} - KR)\|v\|^2_{H^1}, \tag{7}$$

or

$$B(v,v) \ge \alpha\|v\|^2_{H^1}, \tag{8}$$

where $\alpha = (\frac{1}{1+c} - \frac{T_2}{2} - KR)$, c is poincare's constant. So, the following lemma can be expressed.

Lemma 1.2 If $\alpha > 0$, B is V-elliptic.

By using Lax-Milgram theorem and lemmas 1.1, 1.2, the problem ((1)) has a unique solution.

2. Finite element method

Now, we explain how to solve the problem with finite element method. Since V is a infinite dimensional space, we choose a subspace of V with finite dimension and call it V_h. So the problem is converted to find $u_h \in V_h$ so that $B(u_h, v_h) = L(v_h), \quad \forall v_h \in V_h$. We consider a set of basis continuous piecewise polynomials functions of degree at most m such as $\{\phi_i\}_{i=1}^n$, which $V_h = span\{\phi_1, \phi_2, ..., \phi_n\}$. Let $\{\Omega_e\}_{e=1}^M$ be a regular partition of $\Omega = [a,b]$. We choose $m+1$ nodes in each subinterval and denote nodes by $\{x_1, x_2, ..., x_N\}$. Corresponding to each node, we construct a basis function, such that satisfies the following properties:

$i)\phi_i(x_j) = \delta_{ij}, \qquad i, j = 1, 2, ..., N$

$ii)\phi_i|_{\Omega_e} = \psi_i^{(e)} \qquad \psi_i^{(e)}(x_j) = \delta_{ij}, \qquad i, j = 1, 2, ..., N$

where $\psi_i^{(e)}$ are called local functions. we can write $u_h(x)$ and $v_h(x)$ as a linear combinations of the basis functions of V_h, so we have

$$u_h(x) = \sum_{i=1}^n a_i \phi_i(x) \qquad v_h(x) = \sum_{j=1}^n b_j \phi_j(x) \tag{9}$$

Hence, by substituting ((9)) in variational formulation of the problem, we have

$$\sum_{j=1}^n b_j \{\sum_{i=1}^n a_i \{\int_\Omega \phi_i'(x)\phi_j'(x)dx + \int_\Omega b(x)\phi_{i'}(x)\phi_j(x)dx + \int_\Omega c(x)\phi_i(x)\phi_j(x)dx$$
$$-\int_\Omega \phi_j(x)(\int_a^x K(x,t)\phi_i(t)dt)dx\} - \int_\Omega f(x)\phi_j(x)dx\} = 0 \tag{10}$$

Since, the b_j 's are arbitrary, we have

$$\sum_{i=1}^n a_i \{\int_\Omega \phi_{i'}(x)\phi_{j'}(x)dx + \int_\Omega b(x)\phi_{i'}(x)\phi_j(x)dx + \int_\Omega c(x)\phi_i(x)\phi_j(x)dx$$
$$-\int_\Omega \phi_j(x)(\int_a^x K(x,t)\phi_i(t)dt)dx - \int_\Omega f(x)\phi_j(x)dx\} = 0 \tag{11}$$

Now, we define

$$C_{i,j} = \int_\Omega \phi_{i'}(x)\phi_{j'}(x)dx + \int_\Omega b(x)\phi_{i'}(x)\phi_j(x)dx + \int_\Omega c(x)\phi_i(x)\phi_j(x)dx$$
$$-\int_\Omega \phi_j(x)(\int_a^x K(x,t)\phi_i(t)dt)dx \qquad i, j = 1, 2, ..., n. \tag{12}$$

and

$$F_j = \int_\Omega f(x)\phi_j(x)dx \qquad j = 1, 2, ..., n. \tag{13}$$

In this case, the following system is obtained

$$\sum_{i=1}^n C_{ij} a_i = F_j \qquad j = 1, 2, ..., n. \tag{14}$$

We assume, $a = [a_1, a_2, ..., a_n]^T$, $F = [F_1, F_2, ..., F_n]^T$ and $C = (C_{ij})$, then system ((14)) can be written as $C^T a = F$. Let $C_{ij}^{(e)}$ and $F_j^{(e)}$ are restriction of C_{ij} and F_j respectively. So, we have

$$C = \sum_{e=1}^M C^{(e)}, \qquad C^{(e)} = (C_{ij}^{(e)}) \qquad i, j = 1, 2, ..., n.$$
$$F = \sum_{e=1}^M F^{(e)}, \qquad F^{(e)} = (F_j^{(e)}) \qquad j = 1, 2, ..., n. \tag{15}$$

Now, by solving the system $C^T a = F$, the coefficients a_i is obtained, and with these coefficients, we can obtain the approximate solution.

3. Error Analysis

Suppose u is the exact solution of the problem and u_h be its approximate solution, then we have

$$B(u, v_h) = l(v_h) \qquad \forall v_h \in V_h, \tag{16}$$

and also we have

$$B(u_h, v_h) = l(v_h) \qquad \forall v_h \in V_h \tag{17}$$

If $e = u - u_h$, then

$$B(e, v_h) = 0 \quad \forall v_h \in V_h. \tag{18}$$

Definition 3.1 Let V be a Hilbert space, and suppose B be a symmetric and V -elliptic bilinear form. We define an inner product as follows

$$(.,.) : V \times V \to R$$
$$(u, v)_B = B(u, v)$$

which is called the inner product energy. Also we define energy norm as follows

$$\| u \|_E^2 = (u, u)_B$$

By Schwarz inequality, we have the following relation between energy norm and inner product,

$$| B(v, w) | \leq \| v \|_E \| w \|_E \qquad \forall v, w \in V. \tag{19}$$

So, from ((18)) we obtain

$$(e, v_h)_B = B(e, v_h) = 0.$$

Therefore e is orthogonal to each v_h .

$$(u - u_h, v_h) = 0 \tag{20}$$

According to [4] , we have the following theorem and lemma.

Theorem 3.2 $\| u - u_h \|_E = min\{\| u - v_h \|_E ; \quad v_h \in V_h\}$.

Lemma 3.3 (Cea's Lemma)

Suppose V is a Hilbert space, and B is a continuous bilinear form and V -elliptic, and l is a continuous linear functional on V . Then, there is a constant c independent of h such that

$$\| u - u_h \|_V \leq c \inf_{v_h \in V_h} \| u - v_h \|_V .$$

Definition 3.4 (projection operator)

Projection operators are defined as follows:

$$\Pi : V \to V_h$$
$$\Pi u = \tilde{u}_h = \sum_{i=1}^{n} \tilde{a}_i \phi_i(x)$$

In other words, each member of V , correspond to its interpolated function by projection operator. Since for each particular $\tilde{v}_h$ in V_h, we have

$$\inf \| u - v_h \|_V \leq \| u - \tilde{v}_h \|_V$$

for finding an upper bound for $u - u_h$, we can take $\tilde{v}_h$ equal to $\tilde{u}_h$. Then

$$\| u - u_h \|_V \leq c \| u - \tilde{u}_h \|_V .$$

Therefore it is sufficient to get an upper bound for the interpolation error. If the bilinear form is symmetric, then from theorem 3.2, we have

$$\|u-u_h\|_E=\min_{v_h\in V_h}\|u-v_h\|_E$$

then

$$\alpha\|u-u_h\|_V^2\le B(u-u_h,u-u_h)=\|u-u_h\|_E^2\Rightarrow\|u-u_h\|_V\le\frac{1}{\sqrt{\alpha}}\|u-u_h\|_E, \tag{21}$$

by continuity of B, we have

$$\|u-v_h\|_E^2=B(u-v_h,u-v_h)\le C\|u-v_h\|_V\|u-v_h\|_V\Rightarrow\|u-v_h\|_E\le\sqrt{C}\|u-v_h\|_V \tag{22}$$

and from ((21)) and ((22))

$$\|u-u_h\|_V\le\sqrt{\frac{C}{\alpha}}\min_{v_h\in V_h}\|u-v_h\|_V . \tag{23}$$

The above inequality is a special case of lemma 3.3. [4]

Let the basis functions be piecewise quadratic polynomial, we define

$$E(x)=u(x)-\tilde{u}_h(x)\qquad \text{(interpolation error)}$$

First, we examine the error on Ω_e, we have

$$E(x_1^{(e)})=E(x_2^{(e)})=E(x_3^{(e)})=0$$

so, according to Rolle's theorem there exists $\xi_1\in(x_1^{(e)},x_2^{(e)})$ and $\xi_2\in(x_2^{(e)},x_3^{(e)})$, such that $E'(\xi_1)=0$ and $E'(\xi_2)=0$, and so there exists $\eta\in(x_1^{(e)},x_3^{(e)})$, such that $E''(\eta)=0$ and note that $E''(x)=u''(x)-u_h''(x)$, then

$$E''(x)=\int_\eta^x E'''(t)dt.$$

Since the polynomial interpolation is piecewise quadratic polynomial,

$$|E''(x)|=\left|\int_\eta^x u'''(t)dt\right|\le\int_\eta^x|u'''(t)|dt\le\int_{x_1^{(e)}}^{x_3^{(e)}}|u'''(t)|dt$$

and by using the Cauchy - Schwarz inequality, we have

$$|E''(x)|\le\|1\|_{L^2}\|u'''\|_{L^2}$$

or

$$|E''(x)|\le h^{\frac{1}{2}}|u|_{H^3(\Omega_e)}$$

then

$$\int_{\Omega_e}|E''(x)|^2dx\le h|u|_{H^3(\Omega_e)}^2\int_{\Omega_e}dx\Rightarrow\|E''\|_{L^2(\Omega_e)}^2=|E|_{H^2(\Omega_e)}^2\le h^2|u|_{H^3(\Omega_e)}^2$$

Now, we can obtain an upper bound on Ω, as follows:

$$\|E''\|_{L^2(\Omega)}^2=|E|_{H^2(\Omega)}^2=\int_\Omega|E''(x)|^2dx=\sum_{e=1}^M\int_{\Omega_e}|E''(x)|^2dx=\sum_{e=1}^M|E|_{H^2(\Omega_e)}^2\le$$

$$h^2\sum_{e=1}^M|u|_{H^3(\Omega_e)}^2=h^2\sum_{e=1}^M\int_{\Omega_e}(u'''(x))^2dx=h^2\int_\Omega|u'''(x)|^2dx=h^2|u|_{H^3(\Omega)}^2$$

thus

$$|E(x)|_{H^2(\Omega)}\le h|u|_{H^3(\Omega)} \tag{24}$$

Similarly, we can write

$$|E|^2_{H^1(\Omega)} \le h^2 |E|^2_{H^2(\Omega)} \tag{25}$$

And so, we can obtain

$$\|E\|_{L^2(\Omega)} \le h|E|_{H^1(\Omega)} \tag{26}$$

From ((24)), ((25)) and (26), we have

$$\|E\|^2_{L^2(\Omega)} \le h^2h^2h^2 |u|^2_{H^3(\Omega)} = h^6 |u|^2_{H^3(\Omega)} \tag{27}$$

By using Sobolev norm, we have

$$\|E\|^2_{H^1(\Omega)} = \|E\|^2_{L^2(\Omega)} + \|E'\|^2_{L^2(\Omega)} \le h^6 |u|^2_{H^3(\Omega)} + |E|^2_{H^1(\Omega)}$$

$$\le h^6 |u|^2_{H^3(\Omega)} + h^4 |u|^2_{H^3(\Omega)} \le 2h^4 |u|^2_{H^3(\Omega)} \rightarrow \|E\|_{H^1(\Omega)} \le \sqrt{2}h^2 |u|_{H^3(\Omega)} \tag{28}$$

Upper bound for the interpolation error

Since the variational form has a unique solution, therefore $|u|_{H^2(\Omega)}$ is a constant number.

However, according to Lemma 3.3

$$\|u - u_h\|_V \le \frac{C}{\alpha} \|u - \tilde{u}_h\|_V$$

where C is the continuity constant and α is the V-ellipticity constant. Then

$$\|u - u_h\|_{H^1(\Omega)} \le C_1 \sqrt{2} h^2 |u|_{H^3(\Omega)}, \qquad C_1 := \frac{C}{\alpha}.$$

By the above error bounded, one can see that the order of method is $O(h^2)$. As, we observe, since $|u|_{H^3(\Omega)}$ is constant, the norm of error tends to zero as $h \to 0$, and the convergence of the method is demonstrated.

4. Numerical Examples

Example 4.1 Consider the following Volterra Integro-Differential Equation:

$$-u''(x) + 4u(x) = -f(x) - \int_0^x (xt) u(t) dt, \qquad u(0) = u(1) = 0$$

where

$$f(x) = \frac{x^3}{2} cosh(1) - \frac{x^2}{2} sinh(2x-1) + \frac{x}{4} cosh(2x-1) - \frac{x}{4} cosh(1) + 4cosh(1)$$

and $t, x \in [0,1]$, with the exact solution $u(x) = cosh(2x-1) - cosh(1)$.

For $M = 20$ and using polynomials of degree 2, exact and approximate values at some points are given in Table 4.1, and approximation error is shown in Figure 4.1.

Table 1: Comparison of exact and approximate solutions in some points for example 4.1

x	0.1	0.3	0.5	0.7	0.9
$\tilde{u}_h(x)$	–0.20564574	–0.46200834	–0.54308068	–0.46200832	–0.20564572
$u(x)$	–0.20564568	–0.46200826	–0.54308063	–0.46200826	–0.20564568

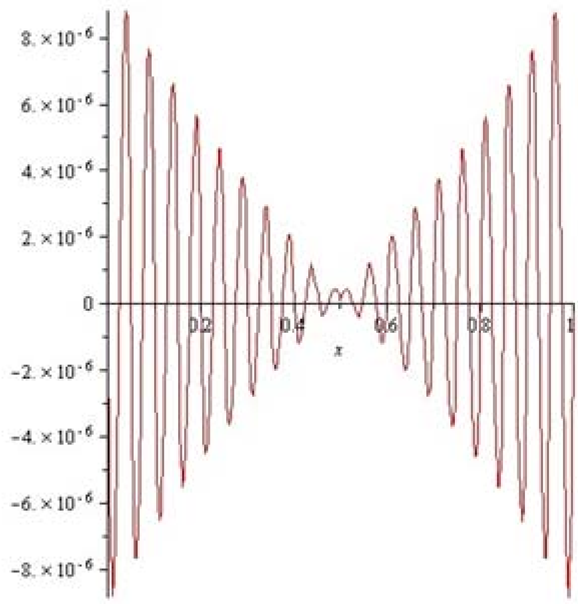

Figure 1: graph of error example 4.1

As Figure 4.1 shows the maximum error is less than 9×10^{-6}.

Example 4.2 Consider the following Volterra Integro-Differential Equation:

$$-y''(x)-3y(x)=f(x)-\int_0^x sin(x+t)y(t)dt \qquad y(0)=y(1)=0,$$

where

$$f(x)=2-3x+3x^2+(x^2-x-2)cos(2x)-(2x-1)sin(2x)-sin(x)+2cos(x)$$

with exact solution $y(x)=x^2-x$. For $M=20$ and $m=2$ graph of error is shown in Figure 4.2.
As it can be seen the maximum error is less than $\frac{3}{2}\times10^{-6}$.

Example 4.3 Let us consider the following linear Volterra Integro-Differential Equation

$$-y''(x)+\frac{\pi^2}{cos(\frac{x}{\pi})}y'-\frac{1}{\pi^2}y-\int_0^x(xt+1)y(t)dt=\pi(1+x^2)cos(\frac{x}{\pi})-x\pi^2 sin(\frac{x}{\pi})$$

with boundary condition $y(0)=0$, $y(1)=sin(\frac{1}{\pi})a$ and the exact solution $y(x)=sin(\frac{x}{\pi})$. At first, we convert the boundary condition at $x=1$ such that to be homogeneous. For $M=30$ and $m=2$ graph of error is shown in Figure 4.3.

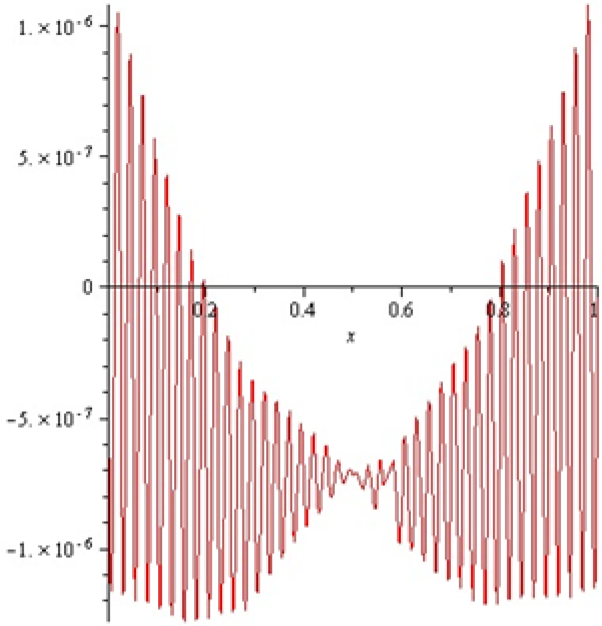

Figure 2: graph of error example 4.2

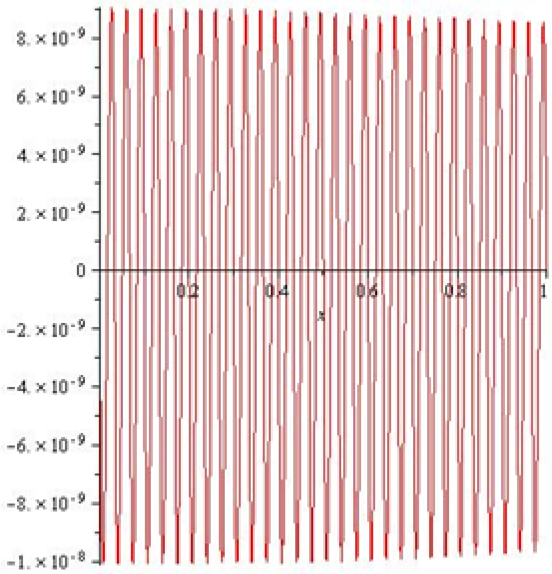

Figure 3: graph of error example 4.3

As Figure 4.3 shows the maximum error is less than 1×10^{-8}.

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