

# The use of radial basis functions for the solution of a partial differential equation with an unknown time-dependent coefficient

F. Parzlivand and A. M. Shahrezaee

Department of Mathematics, Alzahra University, Vanak, Post Code 19834, Tehran, Iran  
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**Abstract.** In this paper, a numerical technique is presented for the solution of a parabolic partial differential equation with a time-dependent coefficient subject to an extra measurement. For solving the discussed inverse problem, at first we transform it into a nonlinear direct problem and then use the proposed method. This method is a combination of collocation method and radial basis functions. The radial basis functions (RBFs) method is an efficient meshfree technique for the numerical solution of partial differential equations. The main advantage of numerical methods which use radial basis functions over traditional techniques is the meshless property of these methods. The accuracy of the method is tested in terms of maximum and RMS errors. Illustrative examples are included to demonstrate the validity and applicability of the technique.

**Keywords:** Radial basis functions, Inverse parabolic problems, Scattered data, Interpolation problem.

## 1. Introduction

Inverse problems are the problems that consist of finding an unknown property of an object, or a medium, from the observation of a response of this object, or medium, to a probing signal. Thus, the theory of inverse problems yields a theoretical basis for remote sensing and non-destructive evaluation. For example, if an acoustic plane wave is scattered by an obstacle, and one observes the scattered field far from the obstacle, or in some exterior region, then the inverse problem is to find the shape and material properties of the obstacle. Such problems are important in identification of flying objects (airplanes missiles, etc.), objects immersed in water (submarines, paces of fish, etc.) and in many other situations. In geophysics one sends an acoustic wave from the surface of the earth and collects the scattered field on the surface for various positions of the source of the field for a fixed frequency, or for several frequencies. The inverse problem is to find the subsurface inhomogeneities. In technology one measures the eigenfrequencies of a piece of a material, and the inverse problem is to find a defect in this material, for example, a hole in a metal. In geophysics the inhomogeneity can be an oil deposit, a cave, a mine. In medicine it may be a tumor or some abnormality in a human body. If one is able to find inhomogeneities in a medium by processing the scattered field on the surface, then one does not have to drill a hole in a medium. This, in turn, avoids expensive and destructive evaluation. The practical advantages of remote sensing are what make the inverse problems important [6].

The parameter identification in a parabolic differential equation from the overspecified data plays an important role in engineering and physics [1-3]. This technique has been widely used to determine the unknown properties of a region by measuring only data on its boundary or a specified location in the domain. These unknown properties such as the conductivity medium are important to the physical process but usually cannot be measured directly or very expensive to be measured [4, 5].

In this paper we shall consider an inverse problem of finding an unknown parameter  $q(t)$  in a parabolic partial differential equation. The classical example is that one needs to find the temperature distribution  $u(x, t)$  as well as the thermal coefficient  $q(t)$  simultaneously that satisfy:

$$u_t = u_{xx} + q(t)u_x + f(x, t); \quad 0 < x < 1, \quad 0 < t < T, \quad (1)$$

with the initial-boundary conditions:

$$u(x, 0) = u_0(x); \quad 0 \leq x \leq 1, \quad (2)$$

$$u(0, t) = g_0(t); \quad 0 \leq t \leq T, \quad (3)$$

$$u(1, t) = g_1(t); \quad 0 \leq t \leq T, \quad (4)$$

and subject to an extra measurement:

$$\int_0^1 u(x,t)dx = E(t); \quad 0 \leq t \leq T, \quad (5)$$

where  $T > 0$  is constant and  $f$ ,  $u_0$ ,  $g_0$ ,  $g_1$  and  $E$  are known functions.

The existence and uniqueness of this inverse problem is discussed in [7, 8] and to interpret the integral equation (5), the reader can refer to [9, 10].

There is a fundamental difference between the direct and the inverse problems. In all cases, the inverse problem is ill-posed or improperly posed in the sense of Hadamard, while the direct problem is well-posed. A mathematical model for a physical problem is called as well-posed in the sense that it has the following three properties:

- There exists a solution of the problem (existence).
- There is at most one solution of the problem (uniqueness).
- The solution depends continuously on the data (stability).

Thus an important task is to formulate the problem properly and to find the conditions that ensure its well posedness. If the solution of the given problem exists and is unique but it does not depend continuously on the data, then in general the computed solution has nothing to do with the true solution. The ill-posedness may be a main difficulty for the inverse problems. Since it is hard to avoid some errors in the observation  $E(t)$  which is obtained from experiments, a small perturbation in  $E(t)$  may result in a big change in  $q(t)$  which may make the obtained results meaningless [11, 12].

In this paper, we solve this problem by using radial basis functions (RBFs) as a truly meshless method. In a meshless (meshfree) method a set of scattered nodes is used instead of meshing the domain of the problem. The use of radial basis functions as a meshless method for numerical solution of partial differential equations is based on the collocation method. Because of the collocation technique, this method does not need to evaluate any integral. The main advantage of numerical methods, which use radial basis functions over traditional techniques, is the meshless property of these methods. The radial basis functions method is used actively for solving partial differential equations. For example see [14-19]. Also some applications of this approach in solving inverse problems can be found in [20-23].

The organization of this article is as follows. In Section 2, we describe radial basis functions and its properties. In Section 3, the presented technique is used to approximate the solution of the inverse problem (1)-(5). In Section 4, an error analysis is presented. In Section 5, we give some computational results of numerical experiments with RBFs method to support our theoretical discussion. The conclusions are discussed in Section 6.

## 2. Radial basis function approximation

The problem of interpolating functions of  $d$  real variables ( $d > 1$ ) occurs naturally in many areas of applied mathematics and the sciences. Radial basis functions method can provide interpolants to function values given at irregularly positioned points for any value of  $d$ . Further, these interpolants are often excellent approximations to the underlying function, even when the number of interpolation points is small. Although polynomials (e.g., Chebyshev and Legendre) are very powerful tools for interpolating a set of points in one-dimensional domains, the use of these functions is not efficient in higher-dimensional or irregular domains. In the use of these functions, the points in the domain of the problem should be chosen in a special form, which is very limiting when the interpolation of a scattered set of points is needed. The main advantage of RBFs is that it requires neither domain nor surface discretization, so the method is independent of the dimension of the problem. The method is meshless and is not complicated. Some meshless schemes are the diffuse element method [24], the partition of unity method [25], the element-free Galerkin method [26], the reproducing kernel particle method [27], the finite point method [28], the meshless local Petrov-Galerkin method [29] and the general finite difference method [30].

Let  $\square^+ = \{x \in \square, x \geq 0\}$ ,  $\|\cdot\|_2$  denotes the Euclidean norm and  $\varphi: \square^+ \rightarrow \square$  be a continuous function

with  $\varphi(0) \geq 0$ . A radial basis function on  $\square^d$  is a function of the form:

$$\phi_i(x) = \varphi(\|x - x_i\|_2),$$

which depended only on the distance between  $x \in \square^d$  and a fixed point  $x_i \in \square^d$ . So that the radial basis function  $\phi_i$  is radially symmetric about the center  $x_i$ .

Let  $x_1, x_2, \dots, x_N \in \Omega \subset \square^d$  be a given set of scattered data. A radial basis function interpolation problem may be described as, given data  $f_i = f(x_i); i = 1, 2, \dots, N$ , the interpolation RBFs approximation is:

$$S_f(x) = \sum_{i=1}^N \lambda_i \phi_i(x), \tag{6}$$

where  $\lambda_i, i = 1, 2, \dots, N$  are chosen so that  $S_f(x_j) = f_j, j = 1, 2, \dots, N$ , that is the interpolation conditions provide the linear system:

$$A\lambda = f, \tag{7}$$

where for  $i, j \in \{1, 2, \dots, N\}$ ,  $A_{ij} = \phi_i(x_j)$ ,  $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_N]^T$  and  $f = [f_1, f_2, \dots, f_N]^T$ . Let  $r$  be the Euclidean distance between a fixed point  $x_i \in \square^d$  and  $x \in \square^d$ , i.e.  $\|x - x_i\|_2$ . For the RBFs that we have considered in Table 1, the interpolation matrix can be shown to be invertible for distinct interpolation points by C. Micchelli [35] and H. Wendland[36].

A generalized interpolation problem also may be considered. The generalized interpolation problem is:

$$S_f(x) = \sum_{i=1}^N \lambda_i \phi_i(x) + \sum_{k=0}^M \alpha_k p_k(x), \tag{8}$$

in which a finite number of  $d$ -variate polynomials of at most order  $M$  are added to the RBFs basis. The polynomials  $p_k(x)$  are the polynomials spanning  $\Pi_M$ , that is they are the polynomials of degree at most  $M$ . The extra equation(s) needed to complete the generalized interpolation problem are chosen to be:

$$\sum_{j=0}^N \lambda_j p_k(x_j) = 0; \quad k = 0, 1, \dots, M. \tag{9}$$

Interpolation problem (8) must be considered when using RBFs, such as the cubics  $\phi(r) = r^3$ , as the basic interpolation problem (6) does not lead to a guaranteed invertible interpolation matrix [37]. Also, the generalized interpolation problem may lead to an approximation with some desirable properties that an approximation from the standard interpolation problem may lack, such as a degree of polynomial accuracy [38]. The standard radial basis functions are categorized into two major classes [39, 40].

Name of Radial Basis Function	Definition
Multiquadric(MQ)	$\varphi(r) = \sqrt{c^2 + r^2}$
InverseQuadratic(IQ)	$\varphi(r) = \sqrt{c^2 + r^2}$
Inverse Multiquadric(IMQ)	$\varphi(r) = 1/\sqrt{c^2 + r^2}$
Gaussian(GA)	$\varphi(r) = e^{-c^2 r^2}$
Thin Plate Splines(TPS)	$\varphi(r) = r^2 \text{Log}(r)$

Table1. Some well-known functions that generate RBFs.

**Class 1.** Infinitely smooth RBFs.

These basis functions are infinitely differentiable and involve a parameter, called shape factor (such as multiquadric (MQ), inverse multiquadric (IMQ) and Gaussian) which needs to be selected so that the required accuracy of the solution is attained.

**Class 2.** Infinitely smooth (except at centers) RBFs.

These basis functions are not infinitely differentiable. These basis functions are shape parameter free and have comparatively less accuracy than basis functions discussed in Class 1. Examples are thin plate splines.

Despite research done by many scientists to develop algorithms for selecting the values of  $c$  which produce the most accurate interpolation (e.g. see [31, 32]), the optimal choice of shape parameter is still an open question. For example, Franke [33] evaluated about 30 interpolation schemes in two dimensions and found that the most accurate two schemes were MQ and TPS. He suggested the shape parameter  $c^2 = 1.25D / \sqrt{N}$  in MQ basis. Where  $D$  is the diameter of the minimal circle enclosing all data points and  $N$  is the number of data points. Hardy [34] recommended  $c^2 = 0.815d$  where  $d = (1/N) \sum_{i=1}^N d_i$  and  $d_i$  is the distance between the  $i$ th data point and its nearest neighbor.

Despite the fact that  $A$  can be shown to be invertible for all  $\phi$  of the interest, the linear system (7) may often be very ill-conditioned, therefore a small perturbation in initial data may produce a large amount of perturbation in the solution and it may be impossible to solve accurately using standard floating point arithmetic. Thus, we have to use more precision arithmetics than standard floating point arithmetic in our computational algorithm. The conditioning of  $A$  is measured by the condition number defined as:

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2.$$

The condition number of  $A$  is influenced by the number of centers, the minimum separation distance of the centers and shape parameter. The shape parameter affects both the accuracy of the approximation and the conditioning of the interpolation matrix. In general, for a fixed number of centers  $N$ , smaller shape parameters produce the more accurate approximations, but also are associated with a poorly conditioned  $A$ . The condition number also grows with  $N$  for fixed values of the shape parameter  $c$ .

### 3. The application of RBFs in inverse parabolic problem

In this section, the radial basis functions method is used for solving the problem (1)-(5). To use the radial basis functions for solving the discussed problem, at first we use the following transformation.

#### 3.1. The employed transformation

If one differentiates with respect to the variable  $t$  in the equation (5) then one obtains:

$$q(t) = \frac{E'(t) - [u_x(1,t) - u_x(0,t)] - \int_0^1 f(x,t) dx}{g_1(t) - g_0(t)}, \tag{10}$$

provided that, for any  $t \in [0, T]$ ,  $g_1(t) \neq g_0(t)$  and  $\int_0^1 f(x,t) dx$  is exists. Consequently, inverse problem

(1)-(5) is equivalent to the following non-local parabolic problem:

$$u_t = u_{xx} + \frac{E'(t) - [u_x(1,t) - u_x(0,t)] - \int_0^1 f(x,t) dx}{g_1(t) - g_0(t)} u_x + f(x,t); \quad 0 < x < 1, \quad 0 < t < T, \tag{11}$$

$$u(x, 0) = u_0(x); \quad 0 \leq x \leq 1, \tag{12}$$

$$u(0, t) = g_0(t); \quad 0 \leq t \leq T, \tag{13}$$

$$u(1, t) = g_1(t); \quad 0 \leq t \leq T, \tag{14}$$

Therefore, for solving the inverse problem (1)-(5), we shall investigate the direct problem (11)-(14).

#### 3.2. Application

Let  $\Omega = \{\chi_k = (x_i, t_j), 0 \leq x_i \leq 1, 0 \leq t_j \leq T, i, j \in \{0, 1, \dots, n\}, k = 1, 2, \dots, N = (n+1)^2\}$ , be a set of scattered nodes, then the solution of the problem (11)-(14) is considered as follows:

$$\tilde{u}(\chi) = \sum_{m=1}^N \lambda_m \phi_m(\chi), \tag{15}$$

where  $\chi = (x, t)$ ,  $\phi_m(\chi) = \phi(\|\chi - \chi_m\|_2)$  and  $\lambda_m, m = 1, 2, \dots, N$  are unknown constants that must be found.

The collocation technique is used for finding unknown  $\lambda_m, m = 1, 2, \dots, N$ . Let  $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$  where

$$\begin{aligned} \Omega_1 &= \{\chi_k = (x_i, t_j), 0 \leq x_i \leq 1, t_j = 0, \quad i = 0, 1, \dots, n, k = 1, 2, \dots, n+1\}, \\ \Omega_2 &= \{\chi_k = (x_i, t_j), x_i = 0, 0 < t_j \leq T, \quad j = 1, 2, \dots, n, k = 1, 2, \dots, n\}, \\ \Omega_3 &= \{\chi_k = (x_i, t_j), x_i = 1, 0 < t_j \leq T, \quad j = 1, 2, \dots, n, k = 1, 2, \dots, n\}, \\ \Omega_4 &= \{\chi_k = (x_i, t_j), 0 < x_i < 1, 0 < t_j \leq T, \quad i = 1, 2, \dots, n-1, j = 1, 2, \dots, n, k = 1, 2, \dots, n(n-1)\}. \end{aligned}$$

Also we assume  $\Omega_i \neq \emptyset$  for  $1 \leq i \leq 4$ . Now approximated (11)-(14) using (15).

Using (15) in (11), we obtain:

$$\begin{aligned} G(t) \sum_{m=1}^N \lambda_m \frac{\partial}{\partial t} \phi_m(\chi) = & G(t) \sum_{m=1}^N \lambda_m \frac{\partial^2}{\partial x^2} \phi_m(\chi) + E'(t) \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(\chi) - \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(1, t) \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(\chi) \\ & + \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(0, t) \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(\chi) - F(t) \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(\chi) + G(t) f(\chi), \end{aligned} \tag{16}$$

where  $G(t) = g_1(t) - g_0(t)$  and  $F(t) = \int_0^1 f(x, t) dx$ . And using (15) in (12)-(14) yields:

$$\sum_{m=1}^N \lambda_m \phi_m(x, 0) = u_0(x), \tag{17}$$

$$\sum_{m=1}^N \lambda_m \phi_m(0, t) = g_0(t), \tag{18}$$

$$\sum_{m=1}^N \lambda_m \phi_m(1, t) = g_1(t). \tag{19}$$

We collocate (16) in  $n(n-1)$  points  $\chi_k = (x_i, t_j) \in \Omega_4$ , we get:

$$\begin{aligned} G(t_j) \sum_{m=1}^N \lambda_m \frac{\partial}{\partial t} \phi_m(\chi_k) = & G(t_j) \sum_{m=1}^N \lambda_m \frac{\partial^2}{\partial x^2} \phi_m(\chi_k) + E'(t_j) \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(\chi_k) \\ & - \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(1, t_j) \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(\chi_k) + \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(0, t_j) \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(\chi_k) \\ & - F(t_j) \sum_{m=1}^N \lambda_m \frac{\partial}{\partial x} \phi_m(\chi_k) + G(t_j) f(\chi_k). \end{aligned} \tag{20}$$

Now, collocation (17)-(19) in  $n+1$  points  $\chi_k = (x_i, 0) \in \Omega_1, i = 0, 1, \dots, n$ ,  $n-1$  points  $\chi_k = (0, t_j) \in \Omega_2$  and  $\chi_k = (1, t_j) \in \Omega_3, j = 1, 2, \dots, n$ , yields:

$$\sum_{m=1}^N \lambda_m \phi_m(x_i, 0) = u_0(x_i), \tag{21}$$

$$\sum_{m=1}^N \lambda_m \phi_m(0, t_j) = g_0(t_j), \tag{22}$$

$$\sum_{m=1}^N \lambda_m \phi_m(1, t_j) = g_1(t_j). \tag{23}$$

Equations (20)-(23) give a system of  $N$  nonlinear algebraic equations with the  $N$  unknown coefficients  $\lambda_m$ . Solving this nonlinear system, the approximate solution of the transformed problem (11)-(14) is obtained. And finding the approximate value of  $q(t)$  is:

$$q(t) = \frac{E'(t) - [\tilde{u}_x(1,t) - \tilde{u}_x(0,t)] - \int_0^1 f(x,t) dx}{g_1(t) - g_0(t)}, \tag{24}$$

### 4. Convergence analysis and error bound

This section covers the error analysis of the proposed method. Also the sufficient conditions are presented to guarantee the convergence of RBFs, when applied to solve the differential equations.

#### 4.1. Approximation error

Here, we are concerned with the error of the approximation of a given two-variate function by its expansion in terms of Gaussian radial basis functions.

Let  $X = L^2(\Omega)$  where  $\Omega = [0,1] \times [0,T]$  (also note that the solution  $u \in C^{4,2}(\Omega) \subset L^2(\Omega)$ ), the inner product in this space is defined by:

$$\langle f(x,t), g(x,t) \rangle = \int_0^T \int_0^1 f(x,t) \overline{g(x,t)} dx dt,$$

and the norm is as follows:

$$\|f(x,t)\|_2 = \langle f(x,t), f(x,t) \rangle^{\frac{1}{2}} = \left( \int_0^T \int_0^1 |f(x,t)|^2 dx dt \right)^{\frac{1}{2}}.$$

And suppose that  $\{\phi_1(x,t), \phi_2(x,t), \dots, \phi_N(x,t)\} \subset X$  be the set of Gaussian radial basis functions and  $Y = span\{\phi_1(x,t), \phi_2(x,t), \dots, \phi_N(x,t)\}$  and  $f$  be an arbitrary element in  $X$ . Since  $Y$  is a finite dimensional vector space,  $f$  has the unique best approximation out of  $Y$  such as  $f_0 \in Y$  that is [41]:

$$\forall g \in Y, \quad \|f - f_0\|_2 \leq \|f - g\|_2.$$

Since  $f_0 \in Y$ , there exists the unique coefficients  $\lambda_1, \lambda_2, \dots, \lambda_N$  such that:

$$f(x,y) \square f_0(x,y) = \sum_{m=1}^N \lambda_m \phi_m(x,y).$$

**Theorem 1.** Let  $(X, \|\cdot\|_2)$  be a Hilbert space and  $Y$  be a closed subspace of  $X$  such that  $\dim Y < \infty$  and  $\{y_1, y_2, \dots, y_n\}$  is any basis for  $Y$ . Let  $f$  be an arbitrary element in  $X$  and  $f_0$  be the unique best approximation to  $f$  out of  $Y$  then [41]:

$$\|f - f_0\|_2 = \frac{G(f, y_1, y_2, \dots, y_n)}{G(y_1, y_2, \dots, y_n)},$$

where

$$G(y_1, y_2, \dots, y_n) = \begin{bmatrix} \langle y_1, y_1 \rangle & \langle y_1, y_2 \rangle & \dots & \langle y_1, y_n \rangle \\ \langle y_2, y_1 \rangle & \langle y_2, y_2 \rangle & \dots & \langle y_2, y_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_n, y_1 \rangle & \langle y_n, y_2 \rangle & \dots & \langle y_n, y_n \rangle \end{bmatrix},$$

### 4.2. Convergence analysis

We have the following theorem about the convergence of RBFs interpolation:

**Theorem 2.** Assume  $\{x_i\}_{i=1}^N$  are  $N$  nodes in  $\Omega \subset \mathbb{R}^d$  which is convex, let:

$$h = \max_{x \in \Omega} \min_{1 \leq i \leq N} \|x - x_i\|_2,$$

when  $\phi(\eta) < c(1 + |\eta|)^{-2l+d}$  for any  $y(x)$  satisfies  $\int (y(\eta))^2 / \phi(\eta) d\eta < \infty$ , we have:

$$\|y_N^\alpha - y^\alpha\|_\infty < ch^{1-\alpha},$$

where  $\phi(x)$  is RBFs and the constant  $c$  depends on the RBFs,  $d$  is space dimension,  $l$  and  $\alpha$  are nonnegative integer.

**Proof.** A complete proof is given by authors [19, 20].

It can be seen that not only RBFs itself but also its any order derivative has a good convergence.

### 5. Numerical examples

In this section, three examples are used to illustrate performance of the RBFs method. In all examples, we use the Gaussian (GA) RBFs. In the process of computation, all the symbolic and numerical computations are performed by using Maple.

We tested the accuracy and stability of the method presented in this paper by performing the mentioned method for different values of  $N$ . To study the convergence behaviour of the RBFs method, we apply the following laws:

- (1) The max error is described using:

$$L_\infty(u) = \max_{1 \leq k \leq N} |u(\chi_k) - \tilde{u}(\chi_k)|,$$

- (2) The root mean square (RMS) is described using:

$$RMS(u) = \sqrt{\frac{1}{N} \sum_{k=1}^N |u(\chi_k) - \tilde{u}(\chi_k)|^2},$$

where  $u$  is the exact value and  $\tilde{u}$  is the RBFs approximation.

#### 5.1. Example 1

Solve problem (1)-(5) with the input data:

$$u_0(x) = -(x - 0.5)x, g_0(t) = 2t^2 - 2t, g_1(t) = 2t^2 - 2t - 0.5, E(t) = 2t^2 - 2t - \frac{1}{12},$$

$f(x, t) = 4t(0.5 + 2x)$  and  $T = 1$  for which the true solution is  $u(x, t) = -(x - 0.5)x + 2t^2 - 2t$  and  $q(t) = 4t$ . Table 2 shows  $L_\infty$  and RMS error values of the method presented in the previous section for various values of  $n$  and  $c$  with  $\delta = 40$  (the number of floating point arithmetics) and the set of collocation points  $x_i = (i - 1) / (n - 1)$  and  $t_j = (j - 1) / (n - 1)$ . In addition, the graphs of the error functions  $|u(x, t) - \tilde{u}(x, t)|$  and  $|q(t) - \tilde{q}(t)|$  for  $n = 8$  are plotted in Figure 1. It can be obtained from Table 2 and Figure 1 that the accuracy increases with the increase of the number of collocation points and the decrease of the value of  $c$ .

<i>n</i>	<i>c</i>	$L_\infty(u)$	$RMS(u)$	$L_\infty(q)$	$RMS(q)$
<b>2</b>	0.5	2.462E-01	9.504E-02	3.950E-01	2.679E-01
	0.1	1.236E-02	4.707E-03	2.737E-01	1.381E-01
	0.08	7.045E-03	3.023E-03	1.333E-01	5.125E-02
	0.06	4.482E-03	1.705E-03	7.525E-02	5.009E-02
	0.04	1.996E-03	7.596E-04	3.353E-02	2.231E-02
<b>4</b>	0.5	8.615E-03	4.455E-03	1.876E-01	1.153E-01
	0.1	1.897E-05	7.583E-06	3.182E-04	1.953E-04
	0.08	7.777E-06	3.108E-06	1.304E-04	8.007E-05
	0.06	2.462E-06	9.842E-07	4.130E-05	2.535E-05
	0.04	4.866E-07	1.945E-07	8.163E-06	5.009E-06
<b>6</b>	0.5	1.763E-04	7.124E-05	2.461E-03	1.789E-03
	0.1	1.178E-08	4.757E-09	1.642E-07	1.195E-07
	0.08	3.092E-09	1.247E-09	4.307E-08	3.136E-08
	0.06	5.506E-10	2.222E-10	7.670E-09	5.584E-09
	0.04	4.836E-11	1.951E-11	6.735E-10	4.904E-10
<b>8</b>	0.5	1.925E-06	7.839E-07	3.266E-05	1.955E-05
	0.1	5.164E-12	1.210E-12	8.709E-11	5.214E-11
	0.08	8.671E-12	3.526E-13	1.472E-11	8.754E-12
	0.06	8.277E-14	3.532E-14	1.474E-12	8.769E-13
	0.04	3.390E-15	1.378E-15	5.756E-14	3.422E-14

Table1.  $L_\infty$  and RMS errors for  $u$  and  $q$  by using GA-RBFs for Example 1.

**5.2. Example 2**

As the second example, consider (1)-(5) with:

$$u_0(x) = -(x - 0.5)x, g_0(t) = 2 \sin(t) - 2t, g_1(t) = 2 \sin(t) - 2t - 0.5, E(t) = 2 \sin(t) - 2t - \frac{1}{12},$$

$f(x, t) = 2 \sin(t)(0.5 + 2x)$  and  $T = 1$  for which the true solution is  $u(x, t) = -(x - 0.5)x + 2 \sin(t) - 2t$  and  $q(t) = 2 \cos(x)$ . Table 3 shows  $L_\infty$  and RMS error values of the method presented in the previous section for various values of  $n$  and  $c$  with  $\delta = 40$  (the number of floating point arithmetics) and the set of collocation points  $x_i = (i - 1) / (n - 1)$  and  $t_j = (j - 1) / (n - 1)$ . In addition, the graphs of the error functions  $|u(x, t) - \tilde{u}(x, t)|$  and  $|q(t) - \tilde{q}(t)|$  for  $n = 8$  are plotted in Figure 2.



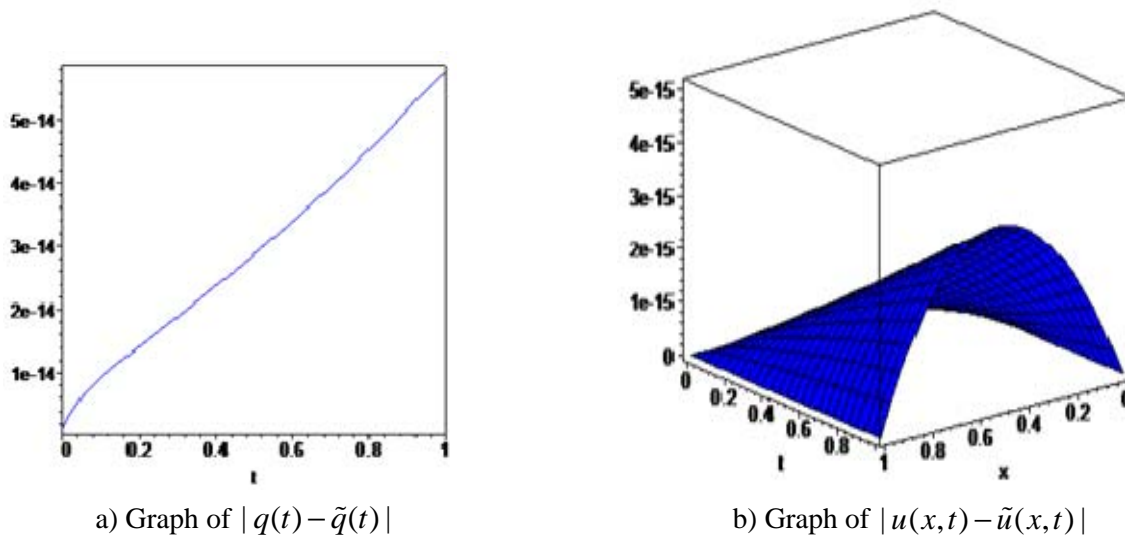


Figure 1. Graph of absolute error by using GA-RBF for Example 1 with  $c = 0.04$ .

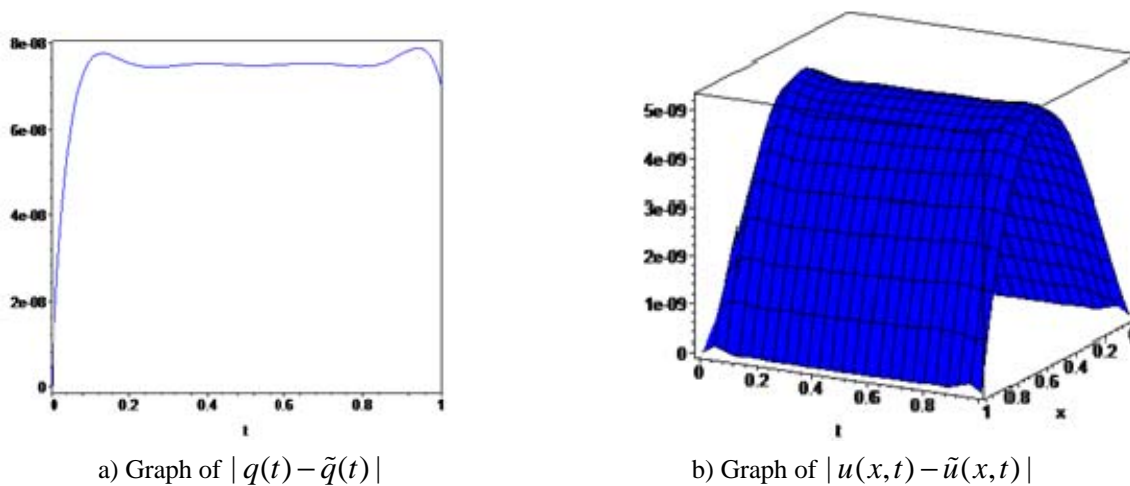


Figure 2. Graph of absolute error by using GA-RBF for Example 2 with  $c = 0.04$ .

### 5.3. Example 3

We consider the inverse problem (1)-(5) with:

$$u_0(x) = -(x-0.5)x \quad , \quad g_0(t) = 2t^4 - 2t \quad , \quad g_1(t) = 2t^4 - 2t - 0.5 \quad , \quad E(t) = 2t^4 - 2t - \frac{1}{12} \quad ,$$

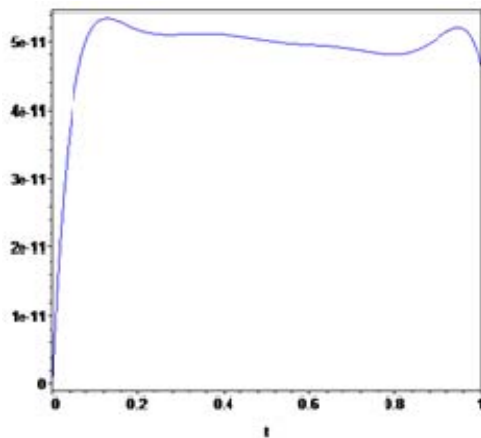
$f(x,t) = 4t^3(1.5+2x)$  and  $T=1$  for which the true solution is  $u(x,t) = -(x-0.5)x + 2t^4 - 2t$  and  $q(t) = 4t^3$ . Table 4 shows  $L_\infty$  and RMS error values of the method presented in the previous section for various values of  $n$  and  $c$  with  $\delta = 40$  (the number of floating point arithmetics) and the set of collocation points  $x_i = (i-1)/(n-1)$  and  $t_j = (j-1)/(n-1)$ . In addition, the graphs of the error functions  $|u(x,t) - \tilde{u}(x,t)|$  and  $|q(t) - \tilde{q}(t)|$  for  $n = 8$  are plotted in Figure 3.

## 6. Conclusion

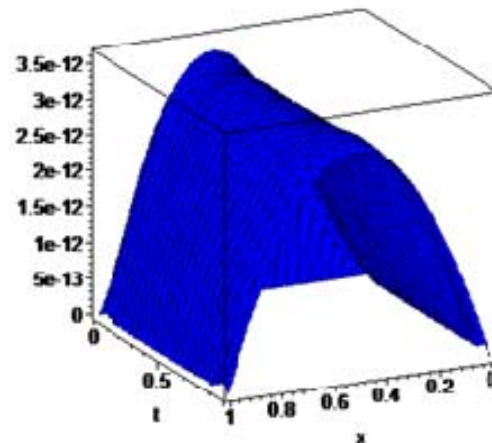
Radial basis functions were used for solving an inverse parabolic equation with an unknown time-dependent coefficient subject to an extra measurement. The meshless property of RBFs methods is the most important advantage of these methods over the traditional mesh-dependent techniques such as finite difference, finite element and boundary element methods. Using the RBFs methods a closed form of the solution is provided. Also, the meshfree nature of these techniques allows solving problems with nonregular geometry. The numerical tests obtained by using the radial basis functions described in this paper give acceptable results.

$n$	$c$	$L_\infty(u)$	$RMS(u)$	$L_\infty(q)$	$RMS(q)$
2	0.5	1.862E-01	1.638E-01	7.362E-01	6.248E-01
	0.1	1.132E-01	9.471E-02	6.458E-01	5.805E-01
	0.08	9.672E-01	7.927E-02	5.749E-01	5.112E-01
	0.06	8.451E-02	5.834E-02	5.167E-01	4.738E-01
	0.04	8.640E-02	3.708E-02	4.342E-01	4.010E-01
4	0.5	4.410E-03	2.640E-03	7.802E-02	6.893E-02
	0.1	7.234E-04	4.607E-04	1.096E-02	9.313E-02
	0.08	7.859E-04	4.389E-04	1.289E-02	1.100E-02
	0.06	8.352E-04	4.651E-04	1.263E-02	1.106E-02
	0.04	8.709E-04	4.845E-04	1.317E-02	2.207E-02
6	0.5	1.277E-04	4.483E-05	2.122E-03	1.138E-03
	0.1	1.849E-06	1.114E-06	2.778E-05	2.449E-05
	0.08	2.158E-06	1.344E-06	3.345E-05	2.956E-05
	0.06	2.461E-06	1.535E-06	3.819E-05	3.337E-05
	0.04	2.777E-06	1.679E-06	4.174E-05	3.693E-05
8	0.5	1.341E-06	5.313E-07	2.442E-05	1.328E-05
	0.1	2.464E-09	1.550E-09	3.677E-08	3.349E-08
	0.08	3.015E-09	2.200E-09	5.223E-08	7.750E-08
	0.06	4.417E-09	2.794E-09	6.632E-08	6.035E-08
	0.04	5.165E-09	3.267E-09	7.755E-08	3.057E-08

Table 2.  $L_\infty$  and RMS errors for  $u$  and  $q$  by using GA-RBFs for Example 2.



a) Graph of  $|q(t) - \tilde{q}(t)|$



b) Graph of  $|u(x,t) - \tilde{u}(x,t)|$

Figure 3. Graph of absolute error by using GA-RBF for Example 3 with  $c = 0.04$ .

$n$	$c$	$L_\infty(u)$	$RMS(u)$	$L_\infty(q)$	$RMS(q)$
2	0.5	6.745E-01	5.138E-01	8.362E-01	7.130E-01
	0.1	6.132E-01	4.738E-01	7.739E-01	6.871E-01
	0.08	5.625E-01	3.884E-01	7.109E-01	6.792E-01
	0.06	5.613E-01	3.507E-01	6.816E-01	6.648E-01

	0.04	5.556E-01	3.341E-01	6.645E-01	6.214E-01
<b>4</b>	0.5	6.657E-02	3.652E-02	9.753E-01	8.661E-01
	0.1	2.542E-03	1.429E-03	4.090E-02	3.336E-02
	0.08	1.626E-03	9.142E-04	2.620E-02	2.134E-02
	0.06	9.154E-04	5.140E-04	1.147E-02	1.200E-02
	0.04	4.066E-04	2.284E-04	6.559E-03	5.332E-03
<b>6</b>	0.5	4.708E-03	2.831E-03	7.499E-02	6.598E-02
	0.1	1.866E-04	1.081E-04	1.787E-03	2.441E-03
	0.08	8.817E-05	6.904E-05	9.547E-04	1.557E-03
	0.06	6.756E-05	3.874E-05	4.480E-04	8.729E-04
	0.04	3.011E-05	1.717E-05	4.480E-04	3.866E-04
<b>8</b>	0.5	9.778E-06	8.312E-06	1.975E-04	1.185E-04
	0.1	8.830E-10	5.212E-10	1.303E-08	1.115E-08
	0.08	2.316E-10	1.365E-10	3.417E-09	3.013E-09
	0.06	4.124E-11	2.430E-11	6.085E-10	5.361E-10
	0.04	3.242E-12	2.133E-12	5.343E-11	2.878E-12

Table 3.  $L_\infty$  and RMS errors for  $u$  and  $q$  by using GA-RBFs for Example 3.

## 7. References

- [1]. J. R. Cannon, Y. Lin and S. Wang, Determination of source parameter in a parabolic equations, *Meccanica* **27** (2) (1992) 85-94.
- [2]. G. Fatullayev, N. Gasilov and I.Yusubov, Simultaneous determination of unknown coefficients in a parabolic equation, *Applicable Analysis* **87** (10-11) (2008) 1167-1177.
- [3]. M. I. Ivanchov, Inverse problems for the heat conduction equation with non-local boundary condition, *Ukrainian Mathematical Journal* **45** (8) (1993) 1186-1192.
- [4]. M. Dehghan, Parameter determination in a partial differential equation from the overspecified data, *Math. Comput. Model.* **41** (2005) 197-213.
- [5]. M. Dehghan, An inverse problem of finding a source parameter in a semilinear parabolic equation, *Appl. Math. Modelling* **25** (2001) 743-754.
- [6]. G. Ramm, *Mathematical and Analytical Techniques with Application to Engineering*, Springer , NewYork, 2005.
- [7]. J. R. Cannon and H. M. Yin, On a class of non-linear parabolic equations with non-linear trace type functionals, *Inverse Problems* **7** (1) (1991) 149-161.
- [8]. J. R. Cannon and S. P. Esteva, Determination of the coefficient of  $u_x$  in a linear parabolic equation, *Inverse Problems* **10** (3) (1994) 521-531.
- [9]. M. Dehghan, Parameter determination in a partial differential equation from the overspecified data, *Math. Comput. Model.* **41** (2-3) (2005) 196-213.
- [10]. H. Azari, Numerical procedures for the determination of an unknown coefficient in parabolic differential equations, *Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorit.* **9** (4) (2002) 555-573.
- [11]. Tikhonov and V. Arsenin, *Solutions of Ill-posed Problems*, Geology Press, Beijing, 1979.
- [12]. J. J. Liu, *Regularization Method and Application for the Ill-posed Problem*, Science Press, Beijing, 2005.
- [13]. G. R. Liu and Y. T. Gu, Boundary meshfree methods based on the boundary point interpolation methods, *Eng. Anal. Bound Elements* **28** (2004) 475-487.
- [14]. S. Chantrasirivan, Cartesian grid method using radial basis functions for solving Poisson, Helmholtz, and diffusion-convection equations, *Eng. Anal. Bound Elements* **28** (2004) 1417-1425.

- [15]. D. Brown, L. Ling, E. Kansa and J. Levesley, On approximate cardinal preconditioning methods for solving PDEs with radial basis functions, *Eng. Anal. Bound Elements* **29** (2005) 343-353.
- [16]. W. Chen and M. Tanaka, A meshless, integration-free and boundary only RBF technique, *Comput. Math. Appl.* **4** (3) (2002) 379-391.
- [17]. E. Kansa and Y. C. Hon, Circumventing the ill-conditioning problem with multiquadric radial basis functions: applications to elliptic partial differential equations, *Comput. Math. Appl.* **39** (2000) 123-137.
- [18]. S. A. Sarra, Adaptive radial basis function method for time dependent partial differential equations, *Appl. Numer. Math.* **54** (2005) 79-94.
- [19]. C. Shu, H. Ding, and K. S. Yeo, Solution of partial differential equations by a global radial basis function-based differential quadrature method, *Eng. Anal. Boundary Elements*, **28** (2004) 1217-1226.
- [20]. J. Li, Application of radial basis meshless methods to direct and inverse biharmonic boundary value problems, *Commun. Numer. Methods Eng.* **21** (2005) 169-182.
- [21]. J. Li, A radial basis meshless method for solving inverse boundary value problem, *Commun. Numer. Methods Eng.* **20** (2004) 51-61.
- [22]. H. Azari, F. Parzlivand and S. Zhang, A Mesh free numerical method for the solution of an inverse heat problem, *Conference Applications of Mathematics in honor of the 60th birthday of Michal Krizek. Institute of Mathematics AS CR, Prague* 2012.
- [23]. H. Azari and F. Parzlivand, Determination of the coefficient in the advection diffusion equation using collocation and radial basis function, *Inter. J. Inf. Syat. Sci.* **1** (1) (2009) 1-11.
- [24]. B. Nayroles, G. Touzot and P. Villon, Generalizing the finite element method: diffuse approximation and diffuse elements, *Comput. Mech.* **10** (1992) 307-318.
- [25]. Babuska and J. Melenk, The partition of unity method, *Int. J. Numer. Meth. Engrg.* **40** (1997) 727-758.
- [26]. T. Belytschko, Y. Y. Lu and L. Gu, Element-free Galerkin methods, *Int. J. Numer. Meth. Engrg.* **37** (1994) 229-256.
- [27]. W. Liu, S. Jun and Y. Zhang, Reproducing kernel particle methods, *Int. J. Numer. Meth. Fluids* **20** (1995) 1081-1106.
- [28]. E. Onate, S. Idelsohn, O. C. Zienkiewicz and R. L. Taylor, A finite point method in computational mechanics, Application to convective transport and fluid flow, *Int. J. Numer. Meth. Engrg.* **39** (1996) 3839-3866.
- [29]. S. N. Atluri and T. Zhu, New meshless local Petrov-Galerkin (MLPG) approach in computational mechanics, *Comput. Mech.* **22** (2) (1998) 117-127.
- [30]. T. Liszka, An interpolation method for an irregular net of nodes, *Int. J. Numer. Meth. Engrg.* **20** (1984) 1599-1612.
- [31]. S. Rippa, An algorithm for selecting a good value for the parameter  $\rho$  in radial basis function interpolation, *Adv. Comput. Math.* **11** (2-3) (1999) 193-210.
- [32]. R. E. Carlson and T. A. Foley, Interpolation of track data with radial basis methods, *Comput. Math. Appl.* **24** (12) (1992) 27-34.
- [33]. Z. M. Wu, Radial basis function scattered data interpolation and the meshless method of numerical solution of PDEs, *J. Eng. Math.* **19** (2002) 1-12.
- [34]. Z. M. Wu and R. Schaback, Local error estimates for radial basis function interpolation of scattered data, *IMA J. Numer. Anal.* **13** (1993) 13-27.
- [35]. C. Micchelli, Interpolation of scattered data: distance matrices and conditionally positive definite functions, *Constr. Approx.* **2** (1) (1986) 11-22.
- [36]. H. Wendland, Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree, *Adv. Comput. Math.* **4** (4) (1995) 389-396.
- [37]. M. Powell, The theory of radial basis function approximation in 1990. In W. Light, editor, *Advances in Numerical Analysis, Vol. II: Wavelets, Subdivision Algorithms and Radial Functions*. 1990.
- [38]. R. L. Hardy. Theory and applications of the multiquadric-biharmonic method, *Comput. Math. Appl.* **19** (8/9) (1990) 163-208.
- [39]. J. Khattak, S. I. A. Tirmizi and S. U. Islam, Application of meshfree collocation method to a class of non-linear partial differential equations, *Eng. Anal. Bound. Elem.* **33** (2009) 661-667.
- [40]. M. Dehghan and A. Shokri, A meshless method for numerical solution of the one-dimensional wave equation with an integral condition using radial basis functions, *Numer. Algorithms* **52** (2009) 461-477.
- [41]. E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley and Sons Press, New York, 1978.