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Identification of the unknown diffusion coefficient in a parabolic equation using HPM

F. Parzlivand and A. M. Shahrezaee

Department of Mathematics, Alzahra University, Vanak, Post Code 19834, Tehran, Iran
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Abstract. In this paper, the homotopy perturbation method (HPM) is proposed to solve an inverse problem of finding an unknown function in parabolic equation with an extra measurement. For solving the discussed inverse problem, at first we transform it's into a nonlinear direct problem then uses the proposed method. Also an error analysis is presented for the method and prior and posterior error bounds of the approximate solution are estimated. Application of the HPM to this problem shows the rapid convergence of the sequence constructed by this method to the exact solution.

Keywords: Homotopy perturbation method (HPM), inverse parabolic problem, extra measurement.

1. Introduction

The homotopy perturbation method was first proposed by the Chinese mathematician Ji-Huan He [1-5]. The method has been used by many authors to handle a wide variety of scientific and engineering applications to solve various functional equations. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions. One of the most remarkable features of the HPM is that usually just few perturbation terms are sufficient for obtaining a reasonably accurate solution. Considerable research works have been conducted recently in applying this method to a class of linear and nonlinear equations. This method was further developed and improved by He, and applied to nonlinear oscillators with discontinuities [5], nonlinear wave equations [6], boundary value problems [7], limit cycle and bifurcation of nonlinear problems [8], nonlinear Schrodinger equations [9], nonlinear equations arising in heat transfer [10] quadratic Ricatti differential equation [11] Klein-Gordon equation [12] and Blasius equation [13].

In this paper, we propose HPM to solve the inverse problem of finding the function $u(x,t)$ and the unknown positive diffusion coefficient $\mathrm{a}(t)$ in the parabolic initial-boundary value problem as following:

$$u_t = a(\mathrm{t})u_{xx} + p(t)u + g(x,t); \qquad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

subject to the initial and boundary conditions:

$$u(x,0) = u_0(x); \qquad 0 \le x \le 1, \tag{2}$$

$$u(0,\mathrm{t}) = g_0(\mathrm{t}); \qquad 0 \le t \le T, \tag{3}$$

$$u(1,\mathrm{t}) = g_1(\mathrm{t}); \qquad 0 \le t \le T, \tag{4}$$

where $T > 0$ and g, p, u_0, g_0 and g_1 are known functions.

An additional boundary condition which can be the additional specification at a point in the spatial domain (temperature additional specification), is given in the following form:

$$u(\mathrm{x}^*,\mathrm{t}) = E(\mathrm{t}); \qquad 0 \le t \le T, \tag{5}$$

where E is known function and $x^* \in (0,1)$ is constant. Employing the condition (5), a recovery of the function $\mathrm{a}(t)$ together with the solution $u(x,t)$ can be made possible.

Therefore in this study, we solve the inverse problem (1)-(5).

Certain types of physical problems can be modeled by (1)-(5). As is said in [18], one application is in the determination of the unknown properties in a region by measuring only data on the boundary and particular attention has been focused to coefficients that present physical meaning quantities. For example, the conductivity of a medium.

The existence and uniqueness of the solution of this problem and more applications are discussed in

[14-17]. However, the theory of the numerical solution of this problem is far from satisfactory. In [18], a backward Euler finite difference scheme was discussed. Authors of [19] proved the determination of a time-dependent conductivity is possible for an arbitrary domain in $\mathbb{R}^n$ in a well-posed manner. In [20], this problem was studied from a different point of view. The authors first transformed a large class of parabolic inverse problems into a non-classical parabolic equation whose coefficients consist of trace type functional on the solution and its derivatives subject to some initial and boundary conditions. For the resulted non-classical problem, they introduced a variational form by defining a new function, then both continuous and discrete Galerkin procedures are employed to the non-classical problem. Author of [21] used the several explicit and implicit finite difference methods to solve this problem. In [22], an efficient pseudospectral Legendre method is developed to solve problem (1)-(5). Also, a method is proposed in [23] to solve this problem which is based on a semi-analytical approach. In [24], the numerical solution is also considered by use of Chebyshev cardinal functions. This problem is solved by a high-order compact finite difference method in [25].

2. Homotopy perturbation method

To illustrate the basic idea of this method [1-13], we consider the following nonlinear differential equation:

$$\mathrm{A(u)} - \mathrm{f(r)} = 0; \qquad r \in \Omega. \tag{6}$$

Considering the boundary condition of:

$$\mathrm{B}(\mathrm{u}, \frac{\partial u}{\partial n}) = 0; \qquad r \in \Gamma, \tag{7}$$

where A is a general differential operator, B a boundary operator, $f(\mathrm{r})$ a known analytical function and Γ is the boundary of the domain $\Omega \subset \mathbb{R}^d$; $d = 1, 2, 3$. Generally speaking, the operator A can be divided into two parts which are L and N, where L is a simple part which is easy to handle and N contains the remaining parts of A. Therefore equation (6) can be rewritten as follows:

$$\mathrm{L(u)} + \mathrm{N(u)} - \mathrm{f(r)} = 0. \tag{8}$$

Using the homotopy technique, we construct a homotopy as $v(\mathrm{r},\mathrm{p}) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies:

$$H(\mathrm{v},\mathrm{p}) = (1-\mathrm{p})[\mathrm{L(v)} - \mathrm{L(u_0)}] + \mathrm{p}[\mathrm{A(v)} - \mathrm{f(r)}] = 0; \qquad p \in [0,1], \tag{9}$$

or

$$H(\mathrm{v},\mathrm{p}) = \mathrm{L(v)} - \mathrm{L(u_0)} + \mathrm{pL(u_0)} + \mathrm{p}[\mathrm{N(v)} - \mathrm{f(r)}] = 0; \qquad p \in [0,1], \tag{10}$$

where p is an embedding parameter and u_0 is an initial approximation of equation (6). Clearly, we have:

$$H(\mathrm{v},0) = \mathrm{L(v)} - \mathrm{L(u_0)} = 0, \tag{11}$$

$$H(\mathrm{v},1) = A(\mathrm{v}) - f(\mathrm{r}) = 0. \tag{12}$$

The changing process of p from zero to unity is just that of $v(\mathrm{r},\mathrm{p})$ from $u_0(\mathrm{r})$ to $u(\mathrm{r})$. In topology, this is called deformation, and $\mathrm{L(v)} - \mathrm{L(u_0)}$ and $A(\mathrm{v}) - f(\mathrm{r})$ are called homotopy. According to HPM, we can first use the embedding parameter p as a "small parameter" and assume that the solution of equations (9) and (10) can be written as a power series in p:

$$v = u_0 + pu_1 + p^2 u_2 + \ldots. \tag{13}$$

Setting $p = 1$ results the approximate solution of equation (6):

$$u = \lim_{p \to 1} v = u_0 + u_1 + u_2 + \ldots. \tag{14}$$

For nonlinear term N in equation (8), we can write:

$$N(\mathrm{v}) = \mathrm{N(u_0)} + \mathrm{pN(u_0,u_1)} + \cdots = \sum_{n=0}^{\infty} p^n N(\mathrm{u_0}, \mathrm{u_1}, \cdots, \mathrm{u}_n), \tag{15}$$

where $N(\mathrm{u_0}, \mathrm{u_1}, \cdots, \mathrm{u}_n)$ is called He's polynomial [26] defined by:

$$N(u_0, u_1, \cdots, u_n) = \frac{1}{n!} \frac{d^n}{dp^n} N(\sum_{k=0}^{\infty} p^k u_k)_{p=0}. \tag{16}$$

3. Convergence analysis and error bound of HPM

In this section, the sufficient conditions are presented to guarantee the convergence of HPM when applied to solve the differential equations. Also, an error analysis is presented for the method and prior and posterior error bounds of the approximate solution are estimated.

3.1. Convergence analysis

Theorem 1. Let $(X, \|.\|_1)$ and $(Y, \|.\|_2)$ be Banach space and $T : X \to Y$ is a nonlinear mapping and suppose that:

$$\|T[v] - T[\tilde{v}]\|_2 \le \gamma \|v - \tilde{v}\|_1; \qquad v, \tilde{v} \in X,$$

for some constant $\gamma < 1$. Then, T has a unique fixed point u, that is $T[u] = u$.

The sequence generated by the homotopy perturbation method will be regarded as [27]:

$$V_n = T[V_{n-1}], \qquad V_{n-1} = \sum_{i=0}^{n-1} u_i, \qquad n = 1, 2, 3, \ldots,$$

and suppose that $V_0 = u_0 \in B_r(u)$ where $B_r(u) = \{u^* \in X \mid \|u - u^*\|_1 < r\}$, then we have the following statements:

(i) $\|V_n - u\|_1 \le \gamma^n \|u_0 - u\|_1$,

(ii) $V_n \in B_r(u)$,

(iii) $\lim_{n\to\infty} V_n = u$.

Proof. See [27].

According to the above theorem, a sufficient condition for the convergence of the variational iteration method is strictly contraction of T.

3.1. Approximation error

In the following theorem we introduce an estimation of the error of the approximate solution of problem (6) and prior and posterior error bounds of the approximate solution are estimated. The prior error bound can be used at the beginning of a calculation for estimating the number of steps necessary to obtain a given accuracy and the posterior error bound can be used at intermediate stages or at the end of a calculation.

Theorem 2. Under the conditions of Theorem 1, the prior and the posterior estimates can obtain as:

$$\|u - V_n\|_1 \le \frac{\gamma^n}{1-\gamma} \|V_1 - u_0\|_1, \tag{17}$$

$$\|u - V_n\|_1 \le \frac{\gamma}{1-\gamma} \|V_n - V_{n-1}\|_1, \tag{18}$$

Also, if suppose that $T[0] = 0$ then the error of the approximate solution V_n to problem (6) can be obtained as follows:

$$\|u - V_n\|_1 \le \frac{1+\gamma}{1-\gamma^n} \|u_0\|_1. \tag{19}$$

Proof:

$$\|V_{n+1}-V_n\|_1 \le \|T[V_n]-T[V_{n-1}]\|_2 \le \gamma\|V_n-V_{n-1}\|_1$$
$$\le \gamma^2\|V_{n-1}-V_{n-2}\|_1$$
$$\vdots$$
$$\le \gamma^n\|V_1-V_0\|_1 = \gamma^n\|V_1-u_0\|_1 .$$

Therefore for any $m>n$, we have:

$$\|V_m-V_n\|_1 \le \|V_m-V_{m-1}\|_1+\|V_{m-1}-V_{m-2}\|_1+\ldots+\|V_{n+1}-V_n\|_1$$
$$\le (\gamma^{m-1}+\gamma^{m-2}+\ldots+\gamma^n)\|V_1-u_0\|_1 = \gamma^n\frac{1-\gamma^{m-n}}{1-\gamma}\|V_1-u_0\|_1 .$$

Since $0<\gamma<1$, in the numerator, we have, $1-\gamma^{m-n}<1$. Consequently,

$$\|V_m-V_n\|_1 \le \frac{\gamma^n}{1-\gamma}\|V_1-u_0\|_1; \qquad \mathrm{m}>\mathrm{n}. \tag{20}$$

The first statement follows from (20) by using Theorem 1, as $m\to\infty$. We derive (8). Taking $n=1$ and writing v_0 for u_0 and v_1 for V_1, we have from (17):

$$\|V_1-u_1\|_1 \le \frac{\gamma}{1-\gamma}\|v_1-v_0\|_1 .$$

Setting $\mathrm{v}_0=V_{n-1}$, we have $\mathrm{v}_1=A[\mathrm{v}_0]=V_n$ and obtain (9).

Also, we have:

$$\|V_1-u_0\|_1 \le \|A[\mathrm{V}_0]-V_0\|_1 \le \|A[\mathrm{V}_0]\|_2+\|\mathrm{V}_0\|_1 .$$

We can write:

$$\|A[\mathrm{V}_0]\|_2 = \|\mathrm{A}[\mathrm{V}_0]-\mathrm{A}[0]\|_2 \le \gamma\|\mathrm{V}_0\|_1 .$$

Therefore:

$$\|V_1-u_0\|_1 \le 1+\gamma\|u_0\|_1 . \tag{21}$$

From Theorem 1, as $m\to\infty$, then $V_m\to u$ and by using (20)-(21), inequality (19) can be obtained.

4. The application of HPM in an inverse problem

In this section the application of the HPM is discussed for solving the discussed problem. To use the HPM for solving the problem (1)-(5), at first we use the following transformation.

4.1. The employed transformation

By differentiation with respect to the variable t in the equation (5) and using the resulting equation, one obtains:

$$a(\mathrm{t})=\frac{E'(\mathrm{t})-\mathrm{p}(\mathrm{t})-\mathrm{g}(\mathrm{x}^*,\mathrm{t})}{u_{xx}(\mathrm{x}^*,\mathrm{t})}, \tag{22}$$

provided that $u_{xx}(\mathrm{x}^*,\mathrm{t})\neq 0$ for any $t\in[0,\mathrm{T}]$. Assuming $F(\mathrm{t})=E'(\mathrm{t})-\mathrm{p}(\mathrm{t})-\mathrm{g}(\mathrm{x}^*,\mathrm{t})$, the inverse parabolic problem (1)-(5) is equivalent to the following nonlinear parabolic equation:

$$u_t=\frac{\mathrm{F}(\mathrm{t})}{u_{xx}(\mathrm{x}^*,\mathrm{t})}u_{xx}+p(t)u+g(x,t); \qquad 0<x<1, \quad 0<t<T, \tag{23}$$

$$u(x,0)=u_0(x); \qquad 0\le x\le 1, \tag{24}$$

$$u(0,\mathrm{t})=g_0(\mathrm{t}); \qquad 0\le t\le T, \tag{25}$$

$$u(1,\mathrm{t})=g_1(\mathrm{t}); \qquad 0\le t\le T, \tag{26}$$

Therefore, for solving the inverse problem (1)-(5), we shall investigate the nonlinear direct problem (23)-(26).

4.2. HPM solutions

In order to solve equations (23)-(26) by HPM, we choose the initial approximation, $L(\mathrm{u})$ and $\mathrm{N(u)}$ as following:

$$u_0(\mathrm{x,t}) = u(\mathrm{x},0) = u_0(\mathrm{x}), \qquad L(\mathrm{u}) = \frac{\partial u}{\partial t}, \qquad N(\mathrm{u}) = -\frac{\mathrm{F(t)}}{u_{xx}(\mathrm{x}^*,\mathrm{t})}u_{xx} - p(t)u. \tag{27}$$

And from equations (10) and (15) construct the following homotopy:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p\frac{\partial u_0}{\partial t} + p[\sum_{n=0}^{\infty} p^n N(\mathrm{u}_0,\mathrm{u}_1,....,\mathrm{u}_n) - \mathrm{g(x,t)}] = 0. \tag{28}$$

Assume the solution of equation (28) in the form:

$$v = u_0 + pu_1 + p^2u_2 + \dots. \tag{29}$$

Substituting (29) into equation (28) and collecting terms of the same power of p gives:

$$p^0: \qquad \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \tag{30}$$

$$p^1: \qquad \frac{\partial u_1}{\partial t} + \frac{\partial u_0}{\partial t} + N(\mathrm{u}_0) - \mathrm{g(x,t)} = 0, \tag{31}$$

$$p^n: \qquad \frac{\partial u_n}{\partial t} + N(\mathrm{u}_0,\mathrm{u}_1,...,\mathrm{u}_n) = 0, \qquad n \ge 2. \tag{32}$$

The above partial differential equations must be supplemented by conditions ensuring a uniqueness of the solution. For equation (31) we assume the following conditions:

$$u_0(0,\mathrm{t}) + u_1(0,\mathrm{t}) = \mathrm{g}_0(\mathrm{t}), \tag{33}$$

$$u_0(1,\mathrm{t}) + u_1(1,\mathrm{t}) = \mathrm{g}_1(\mathrm{t}), \tag{34}$$

while for equation (32) conditions are in the form ($n \ge 2$):

$$u_n(0,\mathrm{t}) + u_n(1,\mathrm{t}) = 0. \tag{35}$$

We can start with $u_0(\mathrm{x,t}) = u_0(\mathrm{x})$ and all the equations above can be easily solved, we get all the solutions. The solution of (23)-(26) can be obtained by setting $p = 1$ in equation (29):

$$u = u_0 + u_1 + u_2 + \dots. \tag{36}$$

5. Test examples

In this section, the theoretical considerations introduced in the previous sections will be illustrated with some examples. For all of the three examples, the true solutions are available.

Example 1. We solve the problem (1)-(5) with $T = 1$, $p(\mathrm{t}) = 0$, $g(\mathrm{x,t}) = 0$ and with the following initial, boundary and extra measurement conditions [21, 24, 25]:

$$u_0(\mathrm{x}) = \exp(\frac{x}{2}),\ g_0(\mathrm{t}) = \frac{1+2t^3}{1+t^3} + \sin(\frac{t}{2}),\ g_1(\mathrm{t}) = \sqrt{e}(\frac{1+2t^3}{1+t^3} + \sin(\frac{t}{2})),$$

$\mathrm{E(t)} = 1.13315(\frac{1+2t^3}{1+t^3} + \sin(\frac{t}{2}))$ and $x^* = 0.25$ for which the true solution is

$$\mathrm{u(x,t)} = \exp(\frac{x}{2})(\frac{1+2t^3}{1+t^3} + \sin(\frac{t}{2})) \text{ and } a(\mathrm{t}) = \frac{2[6\,\mathrm{t}^2 + (1+\mathrm{t}^3)^2\cos(\frac{t}{2})]}{(1+\mathrm{t}^3)[1+2\,\mathrm{t}^3 + (1+\mathrm{t}^3)\sin(\frac{t}{2})]}.$$

It is easy to see that $\mathrm{u}_{xx}(\mathrm{x,t}) = \frac{1}{4}\exp(\frac{x}{2})(\frac{1+2t^3}{1+t^3} + \sin(\frac{t}{2})) \ne 0$ for any $x > 0$ and $t \ge 0$. As the initial approximation u_0 the function that satisfies the initial condition is taken:

$$u_0(\mathrm{x},\mathrm{t}) = u_0(\mathrm{x}) = \exp(\frac{x}{2}).$$

From equations (31), (33) and (34), we have:

$$\frac{\partial u_1}{\partial t} + N(\mathrm{u}_0) = 0,$$

$$\mathrm{u}_1(0,\mathrm{t}) = \frac{1+2t^3}{1+t^3} + \sin(\frac{t}{2}) - 1,$$

$$u_1(1,\mathrm{t}) = \sqrt{e}(\frac{1+2t^3}{1+t^3} + \sin(\frac{t}{2})) - \exp(\frac{x}{2}).$$

By using the above equations and from equation (16), we obtain:

$$u_1(\mathrm{x},\mathrm{t}) = \exp(\frac{x}{2})(\frac{1+2t^3}{1+t^3} + \sin(\frac{t}{2})) - \exp(\frac{x}{2}).$$

In next step, the functions $u_n(\mathrm{x},\mathrm{t});\ \mathrm{n} \geq 2$ are determined recursively by solving the equation (32) with boundary conditions (35). Finally we obtain:

$$u_n(\mathrm{x},\mathrm{t}) = 0; \qquad \mathrm{n} \geq 2.$$

Thus from (36), we have:

$$u(\mathrm{x},\mathrm{t}) = \sum_{n=0}^{\infty} u_n(\mathrm{x},\mathrm{t}) = \exp(\frac{x}{2})(\frac{1+2t^3}{1+t^3} + \sin(\frac{t}{2})),$$

which is the exact solution of the problem. Knowing the exact $u(\mathrm{x},\mathrm{t})$, from equation (22), we can easily determine the function describing the diffusion coefficient:

$$a(\mathrm{t}) = \frac{2[6\,\mathrm{t}^2 + (1+\mathrm{t}^3)^2\cos(\frac{t}{2})]}{(1+\mathrm{t}^3)[1+2\,\mathrm{t}^3 + (1+\mathrm{t}^3)\sin(\frac{t}{2})]}$$

which is equal to the exact $a(\mathrm{t})$ of this example. Authors of [21, 24, 25] used the numerical methods to solve this problem and achieved the approximation solution at mesh point only while the HPM provide the solution in a closed form.

Example 2. As the second example, consider (1)-(5) with [25]:

$u_0(\mathrm{x}) = \cos(\mathrm{x})$, $g_0(\mathrm{t}) = \exp(\mathrm{t})$, $g_1(\mathrm{t}) = \exp(\mathrm{t})\cos(1)$, $p(\mathrm{t}) = 0$, $g(\mathrm{x},\mathrm{t}) = (3+\cos(\mathrm{x}))\exp(\mathrm{t})\cos(\mathrm{x})$

$\mathrm{E}(\mathrm{t}) = \exp(\mathrm{t})\cos(\frac{4}{9})$, $T = 1$, and $x^* = \frac{4}{9}$. The true solution is $\mathrm{u}(\mathrm{x},\mathrm{t}) = \exp(\mathrm{t})\cos(\mathrm{x})$ and

$a(\mathrm{t}) = 2 + \cos(\mathrm{t})$. We can see that $\mathrm{u}_{xx}(\mathrm{x}^*,\mathrm{t}) \neq 0$.

As the initial approximation u_0 we assume the function satisfying the initial condition:

$$u_0(\mathrm{x},\mathrm{t}) = \cos(\mathrm{x})$$

Solving now equation (31) with boundary conditions (equation (33) and equation (34)) and by using equation (16) we determine:

$$\mathrm{u}_1(\mathrm{x},\mathrm{t}) = \exp(\mathrm{t})\cos(\mathrm{x}) - \cos(\mathrm{x}).$$

The functions $u_n(\mathrm{x},\mathrm{t});\ \mathrm{n} \geq 2$ are determined recursively by solving the equation (32) with boundary conditions (35). Finally we obtain:

$$u_n(\mathrm{x},\mathrm{t}) = 0; \qquad \mathrm{n} \geq 2.$$

Thus from (36), we have:

$$u(\mathrm{x},\mathrm{t}) = \sum_{n=0}^{\infty} u_n(\mathrm{x},\mathrm{t}) = \exp(\mathrm{t})\cos(\mathrm{x}),$$

which is the exact solution of the problem. Knowing the exact $u(\mathrm{x},\mathrm{t})$, from equation (22), we can easily determine the function describing the diffusion coefficient:

$$a(\mathrm{t}) = 2 + \cos(\mathrm{t})$$

which is equal to the exact $a(\mathrm{t})$ of this example.

Example 3. For this example, we solve the problem (1)-(5) with:
$u_0(\mathrm{x}) = \exp(-\mathrm{x}) + \mathrm{x}$, $g_0(\mathrm{t}) = 1$, $g_1(\mathrm{t}) = \exp(-\mathrm{t}) + \exp(-1)$, $p(\mathrm{t}) = -1$, $g(\mathrm{x},\mathrm{t}) = 0$
$\mathrm{E}(\mathrm{t}) = \exp(-0.25) + 0.25\exp(-\mathrm{t})$, $T = 1$, and $x^* = \frac{1}{4}$. The true solution is
$\mathrm{u}(\mathrm{x},\mathrm{t}) = \exp(-\mathrm{x}) + \mathrm{x}\exp(-\mathrm{t})$ while $a(\mathrm{t}) = 1$.

We can see that:

$$\mathrm{u}_{xx}(\mathrm{x}^*,\mathrm{t}) = \exp(-\mathrm{x}^*) \neq 0 \qquad \forall t \in [\mathrm{o},1].$$

As the initial approximation u_0 we assume the function satisfying the initial condition:

$$u_0(\mathrm{x},\mathrm{t}) = u_0(\mathrm{x}) = \exp(-\mathrm{x}) + \mathrm{x}$$

Solving now equation (31) with boundary conditions (equation (33) and equation (34)) and by using equation (16) we determine:

$u_1(\mathrm{x},\mathrm{t}) = -\mathrm{x}\mathrm{t}$.

The functions $u_n(\mathrm{x},\mathrm{t}); \mathrm{n} \geq 2$ are determined recursively by solving the equation (32) with boundary conditions (35). Finally we obtain:

$$u_2(\mathrm{x},\mathrm{t}) = \mathrm{x}\frac{t^2}{2},$$

$$u_3(\mathrm{x},\mathrm{t}) = -\mathrm{x}\frac{t^3}{3!},$$

$$u_4(\mathrm{x},\mathrm{t}) = \mathrm{x}\frac{t^4}{4!},$$

$$\vdots$$

$$u_n(\mathrm{x},\mathrm{t}) = \mathrm{x}(-1)^n\frac{t^n}{n!}.$$

We known that

$(1 - t + \frac{t^2}{2} + \ldots + (-1)^n\frac{t^n}{n!})$ is the n th order Taylor series of $\exp(-\mathrm{t})$. Thus from (36), we have:

$$\mathrm{u}(\mathrm{x},\mathrm{t}) = \sum_{n=0}^{\infty} u_n(\mathrm{x},\mathrm{t}) = \exp(-\mathrm{x}) + \mathrm{x}\exp(-\mathrm{t}),$$

which is the exact solution of the problem. Knowing the exact $\mathrm{u}(\mathrm{x},\mathrm{t})$, from equation (22), we can easily determine the function describing the diffusion coefficient:

$$a(\mathrm{t}) = 1,$$

which is equal to the exact $a(\mathrm{t})$ of this example.

Example 4. In the next example, we assume:
$u_0(\mathrm{x}) = \exp(\mathrm{x})$, $g_0(\mathrm{t}) = \exp(\mathrm{t})$, $g_1(\mathrm{t}) = \exp(\mathrm{t}+1)$, $p(\mathrm{t}) = -t$, $g(\mathrm{x},\mathrm{t}) = \exp(\mathrm{x}+\mathrm{t})$
$\mathrm{E}(\mathrm{t}) = \exp(0.5 + \mathrm{t})$, $T = 1$, and $x^* = 0.5$. The true solution is $\mathrm{u}(\mathrm{x},\mathrm{t}) = \exp(\mathrm{x}+\mathrm{t})$ while $a(\mathrm{t}) = t$.

We can see that $\mathrm{u}_{xx}(\mathrm{x}^*,\mathrm{t}) \neq 0$. As the initial approximation u_0 we assume the function satisfying the initial condition:

$u_0(\mathrm{x},\mathrm{t}) = \exp(\mathrm{x})$.

Preceding the steps as in the previous example, we can obtain:

$$u_1(x,t) = t\exp(x)(\exp(t) - \frac{1}{2}t),$$

$$u_2(x,t) = \exp(x)(-t^2\exp(t) - \frac{1}{8}t^4 + 2\,t\exp(t) - 2\exp(t) + 2),$$

$$u_3(x,t) = \exp(x)(12 + t^3\exp(t) - \frac{1}{8}t^2\exp(t) + 12\,t\exp(t) - 12\exp(t) - t^2 + \frac{1}{48}t^6).$$

Using the first four computed terms, the approximate solution u is given by:

$$\Phi_3(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t).$$

The absolute error between the approximate solution and the exact solution is reported in Table 1. We must state here that in practice all term of the series in (36) cannot be determined for this example and the approximations $\Phi_n(x,t)$ of $u(x,t)$ will be approximated by series of the form:

$$\Phi_n(x,t) = \sum_{i=0}^{n} u_i(x,t).$$

From (22), one can obtain the n-order approximation of $a(t)$ by:

$$a_n(t) = \frac{F(t)}{\Phi_{nxx}(x^*,t)}.$$

The rest of the terms are obtained using the Maple Package. The obtained numerical results are summarized in Tables 2 and 3. From these results, we conclude that the HPM for this example gives remarkable accuracy in comparison with the exact solution. The numerical behaviour of the error between the exact solution and the solution obtained by HPM is shown in Figure 1.

Table1. The absolute errors of Φ_3 for Example 4.

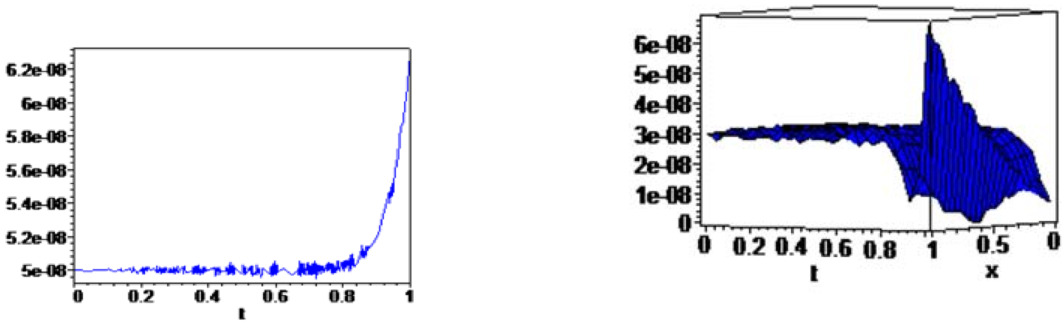
(x/ t)	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.0	1.280E-6	1.745 E-5	2.162 E-4	2.515 E-3	1.272 E-2
0.2	0.0	1.593E-6	2.132 E-5	3.862 E-4	3.072 E-3	1.558 E-2
0.4	0.0	1.890E-6	2.205 E-5	4.717 E-4	3.752 E-3	1.903 E-2
0.6	0.0	2.421 E-6	3.180 E-5	5.762 E-4	4.583 E-3	2.324 E-2
0.8	0.0	2.787 E-6	3.884 E-5	7.038 E-4	5.598 E-3	2.839 E-2
1.0	0.0	3.510 E-6	4.745 E-5	8.596 E-4	6.838 E-3	2.467 E-2

Table2. Absolute error $|u(x,1) - \Phi_n(x,1)|$ for Example 4.

x	Exact solution	$n=4$	$n=6$	$n=8$
0.0	2.71828	1.374E-3	2.022 E-5	3.489 E-8
0.2	3.32011	1.697E-3	1.187 E-5	5.595 E-8
0.4	4.05519	2.051E-3	1.379 E-5	7.052 E-8
0.6	4.95303	2.505E-3	2.015 E-5	3.276 E-8
0.8	6.04946	3.060E-3	2.381 E-5	1.789 E-8
1.0	7.38905	2.737E-3	3.257 E-5	1.868 E-8

Table3. Absolute error $|a(t) - a_n(t)|$ for Example 4.

t	Exact solution	$n=4$	$n=6$	$n=8$
0.0	0.0	0.0	0.00	0.00
0.2	0.2	1.451E-3	1.156 E-5	2.600 E-8
0.4	0.4	3.100E-3	1.176 E-4	7.180 E-8
0.6	0.6	3.871E-3	2.549 E-4	4.105 E-8
0.8	0.8	1.201E-3	4.237 E-4	6.278 E-8
1.0	1.0	2.093E-3	1.520 E-4	5.060 E-8



a) Graph of $|u(x,t)-\Phi_8(x,t)|$. b) Graph of $|a(t)-a_8(t)|$.

Figure1. Graph of absolute error for Example 4.

6. Conclusion

In this paper an application of the homotopy perturbation method for the solution of an inverse parabolic problem is presented. Comparisons with the exact solution reveal that HPM is simple, efficient and reliable. The main advantage of the method is that HPM is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution. Implementation of this method is easy and straightforward. Also, the HPM solve the problem without any discretization of the variables, therefore is free from rounding off errors in computational process. The examples presented in the paper confirm utility of the homotopy perturbation method for solving the discussed problem.

7. References

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