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Solving Fractional Nonlinear Fredholm Integro-differential Equations via Hybrid of Rationalized Haar Functions

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Abstract. This paper presents hybrid of rationalized Haar (HRH) functions method for approximate the numerical solution of the fractional nonlinear Fredholm integro-differential equations (FNFIDEs). The fractional derivatives are considered in the sense of Caputo. The fractional operational matrix of hybrid of block-pulse and rationalized Haar functions are presented. This matrix together with the dual operational matrix are used to reduce the computation of FNFIDEs into a system of algebraic equations. Some numerical examples are given and the results of applying this method demonstrate time and computational are small.

Keywords: Fredholm integro-differential equation, Riemann-Liouville integral, Caputo fractional derivative, Fractional operational matrix, Rationalized Haar, Hybrid.

1. Introduction

The main purpose of this paper is to consider the numerical solution of the FNFIDEs of the types

$$y'(x) - \lambda \int_0^1 k(x,t)\,{}_*D^{\alpha} y(t)dt = f(x), \tag{1}$$

$$0 \le x \le 1, \quad 0 < \alpha < 1,$$

with the initial condition

$$y(0) = \gamma,$$

and

Error! Not a valid embedded object.

$${}_*D^{\alpha} y(x) - \lambda \int_0^1 k(x,t) G(y(t))dt = f(x), \tag{2}$$

$$0 \le x \le 1, \quad n-1 < \alpha \le, \quad n \in N,$$

$y^{(i)}(0) = \delta_i,$ with n initial conditions $i = 0,1,2,3,\ldots,n-1.$

Here ${}_*D^{\alpha}$ is Caputo's fractional derivative, α is a parameter describing the order of fractional derivative and λ is a real known constant. Also, $f \in L^2([0,1])$ and $k \in L^2([0,1]^2)$ are given functions, y(x) is the solution to be determined and G(y(x)) is a polynomial of the unknown function y(x), we assume $G(y(x)) = [y(x)]^q$ where $1 < q \in N$. The fractional calculus has been applied in many mathematica models. For example the nonlinear oscillation of earthquake [1], fluid-dynamic traffic [2], continuum and statistical mechanics [3] can be modeled with fractional derivatives. There are several methods that are used to solve the fractional integro-differential equations such as, Adomian decomposition method [4], collocation method [5], CAS wavelet method [6], hybrid functions and the collocation method [7] and second kind Chebyshev

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wavelet method [8]. In this work we report application of HRH functions to solve the FNFIDEs, for this purpose, in problems (1) and (2) we expanding the high order of derivative by HRH functions with unknown coefficients, then we can evaluate the unknown coefficients and obtain an approximate solution to problems (1) and (2). In this technic time and computational are small and this is a good and useful property of the HRH functions method.
The article is organized as follows:

In section 2, we introduce some necessary fundamentals of the fractional calculus theory. in section 3, we present the properties of HRH functions required for our subsequent development. In section, 4 we describe the solution of problems (1) and (2) by using HRH functions, and in section 5 we give some numerical examples to demonstrate the accuracy of the proposed method.

2. Fundamentals of fractional calculus

In this section, we give some definitions and fundamentals of the fractional calculus theory.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ is defined as [9,10]

$$I^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t) dt, \quad \alpha > 0, \quad x > 0,$$

$$I^0 y(x) = y(x),$$

whrere $\Gamma(.)$ is Gamma function.
It has the following properties:

$$I^{\alpha} y(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}, \quad \gamma > -1.$$

Definition 2.2. The Caputo definition of fractional derivative operator is given by [11,12]

$${}_*D^{\alpha} y(x) = I^{n-\alpha} {}_*D^{\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} y^{(n)}(t) dt,$$

where $n-1 \leq \alpha < n, \quad n \in N, \quad \alpha > 0.$
It has following properties:

$${}_*D^{\alpha} I^{\alpha} y(x) = y(x),$$

$$I^{\alpha} {}_*D^{\alpha} y(x) = y(x) - \sum_0^{n-1} y^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

3. Properties of hybrid functions

3.1 Hybrid functions of block-pulse and rationalized Haar functions

The HRH functions $h_{nr}(x)$, $n = 1,2,\ldots,N$, $R = 0,1,2,\ldots,M-1$, $M = 2^{\beta+1}$, $\beta = 0,1,2,\ldots,$ where n, r are the order of block-pulse functions and rationalized Haar functions respectively is defined on the interval [0,1) as [13]

$$h_{nr}(x) = \begin{cases} h_{nr}(Nx-n+1), & \frac{n-1}{N} \leq x < \frac{n}{N}, \\ 0, & otherwise, \end{cases} \quad (3)$$

In (3), $h_r(x)$ are the orthogonal set of rationalized Haar functions and can be defined on the interval [0,1) as [14]

$$h_r(x)=\begin{cases}1, & J_1 \le x < J_{\frac{1}{2}},\\ -1, & J_{\frac{1}{2}} \le x < J_{0,}\\ 0, & otherwise,\end{cases} \tag{4}$$

Where, $J_u=\dfrac{j-u}{2^i}, \quad u=0,\dfrac{1}{2},1.$

The value of r is defined by two parameters i and j as

$$r=2^i+j-1, \quad i=0,1,2,3,..., \quad j=1,2,3,...,2^i,$$

$h_0(x)$ is defined for $i=j=0$ and given by

$$h_0(x)=1, \quad 0\le x<1.$$

since $h_{nr}(x)$ is the combination of rationalized Haar functions and block-pulse functions which are both complete and orthogonal, thus the set of hybrid functions are complete orthogonal set. The orthogonality property of HRH functions is given by [13]

$$\int_0^1 h_{nr}(x)h_{n'r'}(x)dx=\begin{cases}\dfrac{2^{-i}}{N}, & n=n', r=r,'\\ 0, & oterwise,\end{cases}$$

where

$$r=2^i+j-1, \quad r'=2^{i'}+j'-1.$$

3.2 Function approximation

A function $f(x)\in L^2([0,1])$ may be expanded into HRH functions as [13]

$$f(x)=\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}c_{nr}h_{nr}(x), \tag{6}$$

where c_{nr} given by

$$c_{nr}=\frac{<f,h_{nr}>}{\|h_{nr}\|^2}=2^i N\int_0^1 f(x)h_{nr}(x)dx,$$

and $<.,.>$ denote the inner product.

If, the infinite series in (6) is truncated, then (6) can be written as

$$f(x)\cong\sum_{n=1}^{N}\sum_{r=0}^{M-1}c_{nr}h_{nr}(x)=C^T H(x), \tag{7}$$

The HRH function coefficient vector C and RH function vector H(x) are defined as

$$C=[c_{10,}c_{11},...,c_{1M-1}\big|c_{20},c_{21},...,c_{2M-1}\big|...\big|c_{N0},c_{N1},...,c_{NM-1}]^T, \tag{8}$$

$$H(x)=[H_1^T(x)\big|H_2^T(x)\big|...\big|H_N^T]^T, \tag{9}$$

where

$$H_i^T(x)=[h_{i0},h_{i1},...,h_{1M-1}], \quad i=1,2,...,N.$$

Also, we can expand the function $k(x,t)\in L^2([0,1]^2)$ into HRH function as follow

$$k(x,t)\cong H^T(x)KH(t), \tag{10}$$

where $K=(k_{vr})^T$ is an $MN\times MN$ matrix such that

$$k_{vr}=\frac{< H_v(x),k(x,t),H_r(x)>}{\|H_v(x)\|^2\|H_r(x)\|^2}, \quad v=1,2,...,N, \quad r=0,1,...,M-1.$$

Taking the Newton-Cotes nodes as following [15]

$$x_i=\frac{2i-1}{2MN}, \quad i=1,2,...,MN. \tag{11}$$

We have

$$\varphi_{MN}=[H(\frac{1}{2MN}),H(\frac{3}{2MN}),...,H(\frac{2MN-1}{2MN})]. \tag{12}$$

Then, from (12) the square hybrid matrix φ_{MN} can be expressed as

$$\varphi_{MN}=diag(\hat{\phi}_{M\times M},...,\hat{\phi}_{M\times M},...,\hat{\phi}_{M\times M}), \tag{13}$$

where $\hat{\phi}_{M\times M}$ M-square Haar matrix ([14]).

For example if M=2 and N=2 we have

$$\varphi_{22}=\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{bmatrix}.$$

Using (7) we get

$$[f(\frac{1}{2MN}),f(\frac{3}{2MN}),...,f(\frac{2MN-1}{2MN})]=C^T\varphi_{MN}. \tag{14}$$

From (10) and (14) we have

$$K=(\varphi_{MN}^{-1})^T\hat{K}\varphi_{MN}^{-1}, \tag{15}$$

where

$$\hat{K} = \left(\hat{k}_{lp}\right)_{MN\times MN}, \quad \hat{k}_{lp}\left(\frac{2l-1}{2MN}, \frac{2p-1}{2MN}\right), \quad p, l = 1,2,\ldots,MN.$$

3.3 Operational matrix of the fractional integration

The integration of the vector H(x) defined in (9) can be defined as [13]

$$\int_0^x H(t)dt \cong PH(x), \tag{16}$$

where P is the $MN \times MN$ operational matrix of integration.

In this section, we want to derive the HRH functions operational matrix of the fractional integration [16]. For this purpose, we consider an m-set of block-pulse functions as

$$b_i(x) = \begin{cases} 1, & \frac{i+1}{m} \le x < \frac{i+1}{m}, \\ 0, & otherwise, \end{cases}$$

where $m = MN, \quad i = 0,1,2,\ldots,m-1.$

The function $b_i(x)$, are disjoint and orthogonal. That is

$$b_i(x)b_j(x) = \begin{cases} 0, & i \ne j, \\ b_i(x), & i = j, \end{cases}$$

$$\int_0^1 b_i(x)b_j(x) = \begin{cases} 0, & i \ne j, \\ \frac{1}{m}, & i = j, \end{cases}$$

Rationalized Haar functions can be expanded into an m-set of block-pulse functions [16]. Similarly, HRH functions can be expanded into their block-pulse functions as

$$H(x) = \varphi_{MN} B(x), \tag{17}$$

where $B(x) = [b_0(x), b_1(x), \ldots, b_{m-1}(x)]^T$ and φ_{MN} is $MN \times MN$ matrix defined in (12).

In [17], Kilicman and Alzhour have given the block-pulse operational matrix of the fractional integration F^{α} as follows:

$$I^{\alpha} B(x) \cong F^{\alpha} B(x), \tag{18}$$

where

$$F^{\alpha} = \frac{1}{m}\frac{1}{\Gamma(\alpha+1)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \vdots & 1 & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \xi_1 \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \tag{19}$$

with m=MN and $\xi_k = (k+1)^{\alpha} - 2k^{\alpha+1} + (k-1)^{\alpha-1}$.

Now, we obtain the HRH function operational matrix of the fractional integration.

Let

$$I^{\alpha} H(x) \cong P^{\alpha} H(x), \tag{20}$$

where P^{α} is called the HRH functions operational matrix of the fractional integration.

Using (17) and (18), we have

$$I^{\alpha} H(x) = I^{\alpha} \varphi_{MN} B(x) = \varphi_{MN} I^{\alpha} B(x) \cong \varphi_{MN} F^{\alpha} B(x),$$

from (17) and (20), we get

$$P^{\alpha} H(x) = P^{\alpha} \varphi_{MN} B(x) = \varphi_{MN} F^{\alpha} B(x),$$

Then, P^{α} is given by

$$P^{\alpha} = \varphi_{MN} F^{\alpha} \varphi_{MN}^{-1}, \tag{21}$$

Furthermore, we have found the operational matrix of fractional integration for HRH functions.

For example, let $N = M = 2$, then we have

$$P^{\alpha} = \frac{1}{4^{\alpha}} \frac{1}{\Gamma(\alpha+1)} \begin{bmatrix} \frac{\xi_1}{2}+1 & -\frac{\xi_1}{2} & \frac{\xi_1}{2}+\xi_2+\frac{\xi_3}{2} & \frac{\xi_1}{2}-\frac{\xi_3}{2} \\ \frac{\xi_1}{2} & 1-\frac{\xi_1}{2} & \frac{\xi_3}{2}-\frac{\xi_1}{2} & \xi_2-\frac{\xi_1}{2}-\frac{\xi_3}{2} \\ 0 & 0 & \frac{\xi_1}{2}+1 & -\frac{\xi_1}{2} \\ 0 & 0 & \frac{\xi_1}{2} & 1-\frac{\xi_1}{2} \end{bmatrix},$$

and for $\alpha = 0.25$, the operational matrix $P^{0.25}$ can be expressed as following

$$P^{0.25} = \begin{bmatrix} 0.7422 & -0.1181 & 0.2809 & 0.0747 \\ 0.1181 & 0.5060 & -0.0747 & -0.0419 \\ 0.0000 & 0.0000 & 0.7422 & -0.1181 \\ 0.0000 & 0.0000 & 0.1181 & 0.5060 \end{bmatrix},$$

3.4 The dual operational matrix

The integration of the cross, product of two hybrid vector is [13]

$$W = \int_0^1 H(x) H^T(x) dx = \frac{1}{N} diag(D, D, \ldots, D), \tag{22}$$

where W is a $MN \times MN$ matrix and D is a $MN \times MN$ matrix given by [14]

$$D = diag(1,1,\frac{1}{2},\frac{1}{2},\underbrace{\frac{1}{2^2},\ldots,\frac{1}{2^2}}_{2^2},\ldots,\underbrace{\frac{1}{2^\beta},\ldots,\frac{1}{2^\beta}}_{2^\beta}).$$

Then we obtain the matrix W that is dual matrix of HRH functions.

4. Solution of FNFIDEs

In this section we first consider the FNFIDEs given in problem (1).To solve for y(x), we first approximate y'(x) as

$$y^{'}(x) \cong C^T H(x) = H^T(x)C, \tag{23}$$

where C is the HRH functions coefficient vector and H(x) is HRH functions vector.

For simplicity, we can let that the initial condition $\gamma = 0$. By using the properties of Caputo derivative we have

$${}_*D^\alpha y(x) = I^{1-\alpha} y^{'}(x) \cong C^T P^{1-\alpha} H(x), \tag{24}$$

$$y(x) = C^T \int_0^x H(t)dt \cong C^T PH(x), \tag{25}$$

Suppose f(x) and k(x,t) can be expressed approximately as

$$f(x) = H^T(x)F, \quad k(x,t) = H^T(x)KH(t), \tag{26}$$

where F and K are given in (7) and (15) respectively.

Using (22), (24) and (26), we have

$$\int_0^1 k(x,t)_* D^\alpha y(t)dt \cong \int_0^1 H^T(x)KH(t)H^T(t)(P^{1-\alpha})^T Cdt = H^T(x)KW(P^{1-\alpha})^T C, \tag{27}$$

with substituting (23), (26) and (27) in problem (1), we obtain

$$H^T(x)C - \lambda H^T(x)KW(P^{1-\alpha})^T C = H^T(x)F, \tag{28}$$

therefore,

$$(I - \lambda KW(P^{1-\alpha})^T)C - F = 0, \tag{29}$$

where I is $MN \times MN$ identity matrix. (29) is a system of linear equations and can be solved for the unknown vector C, easily.

In second case, we consider the FNFIDEs given in problem (2). we first assume

$${}_*D^\alpha y(x) \cong C^T H(x) = H^T(x)C, \tag{30}$$

where C is the HRH functions coefficient vector and H(x) is HRH functions vector.

For simplicity, we can let that the initial conditions $\delta_i = 0$. So by using the properties of Caputo derivative in section 2, (20) and (30) we have

$$y(x) = C^T P^\alpha H(x), \tag{31}$$

Also, we let

$$z(x)=G(y(x)). \tag{32}$$

Suppose z(x), f(x) and k(x,t) can be expressed approximately as

$$z(x)=Z^T H(x), \quad f(x)=H^T(x)F, \quad k(x,t)=H^T(x)KH(t), \tag{33}$$

where Z, F and K are given in (7) and (15) respectively.

Using (22), (32) and (33), we have

$$\int_0^1 k(x,t)G(y(t))dt=\int_0^1 H^T(x)KH(t)Zdt=H^T(x)KWZ, \tag{34}$$

with substituting (30), (33) and (34) in problem (2), we have

$$H^T(x)C-\lambda H^T(x)KWZ=H^T(x)F, \tag{35}$$

then, (35) can be written as

$$C-\lambda KWZ-F=0, \tag{36}$$

from (31), (32), and (33) we get

$$Z^T H(x)=G(C^T P^\alpha H(x)). \tag{37}$$

In order to construct the approximation for y(x) we collocate (37) in MN points. For a suitable collocation points we choose Newton-Cotes nodes defined in (11). By using (9), (12) and (13) we have

$$H(x_i)=\varphi_{MN}e_i, \quad i=1,2,\ldots,MN,$$

where

$$e_i=(\underbrace{0,0,\ldots,0}_{i-1},1,\underbrace{0,\ldots,0}_{MN-i})^T.$$

Then, from (36) and (37) we obtain the following nonlinear systems of algebraic equations

$$\begin{cases} C-\lambda KWZ-F=0, \\ Z^T\varphi_{MN}e_i=G(C^T P^\alpha \varphi_{MN}e_i), \quad i=1,2,3,\ldots,MN. \end{cases} \tag{38}$$

Therefore, (38) can be solved for the unknowns C and Z, then required approximation to the solution y(x) in problem (2) is obtained.

5. Numerical examples

In this section, we apply the present method and solve some examples where given in the different papers. All calculations were performed using MATLAB software.

Example 1. Consider the following FNFIDE ([18])

$$y'(x)=f(x)+\int_0^1 k(x,t)_* D^{\frac{1}{4}} y(t)dt, \quad y(0)=0,$$

Where $k(x,t)=x^2t^2$, $f(x)=8x^3-\frac{3}{2}x^{\frac{1}{2}}-(\frac{48}{6.25\Gamma(4.75)}-\frac{\Gamma(2.5)}{4.25\Gamma(2.25)})x^2$, and the exact solutionis

$y(x)=2x^4-x^{\frac{3}{2}}$. Table 1 shows the absolute error for example 1.

Table 1 Absolute error for different values of M, N for example 1

x	Peresent method N=3			Method of [18]	
	M=4	M=8	M=16	d=3 (N=15)	d=10 (N=50)
0.0	2.6233×10^{-3}	1.0571×10^{-3}	3.1259×10^{-4}	-	-
0.1	2.5300×10^{-3}	8.4142×10^{-4}	3.0031×10^{-4}	-	-
0.2	1.8580×10^{-3}	7.0771×10^{-4}	2.4749×10^{-4}	-	-
0.3	1.7035×10^{-3}	4.3479×10^{-4}	2.2967×10^{-4}	-	-
0.4	3.6676×10^{-3}	1.1471×10^{-3}	3.6822×10^{-4}	-	-
0.5	4.75079×10^{-5}	2.6090×10^{-4}	1.5864×10^{-4}	1.2476×10^{-1}	2.0087×10^{-3}
0.6	3.7515×10^{-3}	1.2903×10^{-4}	8.0527×10^{-5}	-	-
0.7	5.3464×10^{-3}	1.8156×10^{-4}	2.0443×10^{-4}	-	-
0.8	2.1215×10^{-3}	1.6582×10^{-3}	1.7129×10^{-5}	-	-
0.9	9.2554×10^{-3}	2.5623×10^{-3}	7.2916×10^{-4}	-	-
1.0				7.2403×10^{-2}	7.3243×10^{-3}

We have solved this example for N=3 for different M, the result in Table 1, show that our method is better than method of [18] and computational with our method is small.

Example 2. Consider the following FNFIDE ([6-8])

$$_{*}D^{\alpha}y(x)-\int_0^1 (x+t)^2[y(t)]^3\,dt=g(x),\quad 0\le x<1,$$

where

$$g(x)=\frac{6}{\Gamma(\frac{1}{3})}\sqrt[3]{x}-\frac{x^2}{7}-\frac{x}{7}-\frac{1}{9}.$$

With the initial conditions:

$$y(0)=y'(0)=0.$$

The exact solution is $y(x)=x^2$.

We have solved this example for N=3 and M=32 and have compared it with method of [7]. The comparison is shown in Table 2.

Table 2 Comparison of present method and method of [7] of example 2

x	Present method		Method of [7]		Exact
	Numerical solution	Absolute error	Numerical solution	Absolute error	solution
0.2	0.04009273	9.2725×10^{-5}	0.04023540	2.3540×10^{-4}	0.04000000
0.4	0.16014505	1.4505×10^{-4}	0.16196810	1.9681×10^{-3}	0.16000000
0.6	0.36021653	2.1653×10^{-4}	0.36025632	2.5632×10^{-4}	0.36000000
0.8	0.64031407	3.1407×10^{-4}	0.64035128	3.5128×10^{-4}	0.64000000
1.0	1.00041220	4.1220×10^{-4}	1.00133210	1.3321×10^{-4}	1.00000000
CPU	7.4012070s	-	-	-	-

From Table 2 we can see the numerical results that are obtained with our method are in a good agreement with the exact solution. Although with present method number of value must be very large but time and computational are small.

Example 3. Consider the following FNFIDE ([6-8])

$$ {}_*D^{\alpha} y(x) - \int_0^1 (xt)[y(t)]^2\,dt = 1 - \frac{x}{4}, \quad 0 \le x < 1, \quad 0 < \alpha \le 1, $$

with the initial condition:

$$ y(0) = 0. $$

The only case which we know the exact solution for $\alpha = 1$ is $y(x) = x.$

In Table 3 comparison present method for different N and M with method of [6] in the case of $\alpha = 1,$ show the error by using HRH functions method is smaller than the method of [6]. From Table 3 and Fig 1 we conclude that the numerical results is in good agreement with the exact solution when $\alpha = 1$. Therefore, for the cases $\alpha = \frac{1}{4},$ $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{4}$ that exact solution is unknown and numerical results are shown in Table 4 and Fig 1 for N=3 , HRH functions method is powerful and reliable tool and as $\alpha \to 1$, numerical results tend to exact solution of $\alpha = 1$.

Table 3 Maximum absolute error for example 3

Present method		Method of [6]	
N=3, M=4 (m=12)	N=3, M=8 (m=24)	k=2, M=1 (m'=12)	k=3, M=1 (m'=24)
5.4233×10^{-4}	1.3561×10^{-4}	2.7133×10^{-3}	6.5179×10^{-4}

Table 4 Approximate and exact solution for different α of example 3

x	$\alpha=\frac{1}{4}$	$\alpha=\frac{1}{2}$	$\alpha=\frac{3}{4}$	$\alpha=1$	Exact solution for (α=1)
0.0	0	0	0	0	0.0
0.1	0.650962	0.362260	0.194154	0.099998	0.1
0.2	0.821678	0.525361	0.329896	0.199995	0.2
0.3	0.959520	0.657181	0.450540	0.299988	0.3
0.4	1.084520	0.774336	0.563051	0.399978	0.4
0.5	1.203260	0.883175	0.670474	0.499966	0.5
0.6	1.1318270	0.986267	0.774073	0.599951	0.6
0.7	1.431310	1.085710	0.875049	0.699934	0.7
0.8	1.543170	1.182460	0.973941	0.799913	0.8
0.9	1.654420	1.277270	1.071220	0.899890	0.9
1.0	1.765420	1.370720	1.167250	0.999864	1.0

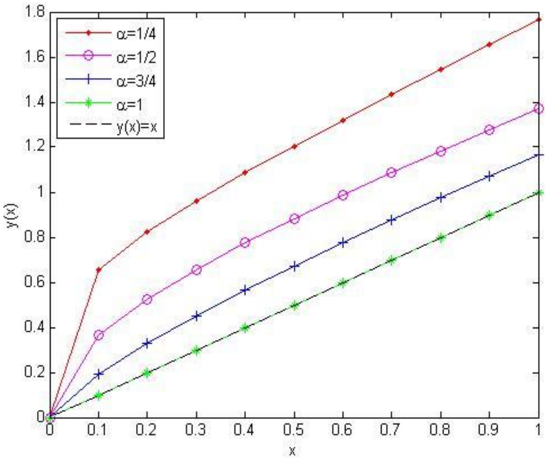

Fig. 1: Plot of example 3 for different α

6. Conclusion

In the present work HRH functions are used to solve the FNFIDEs. We reduce the FNFIDEs to a system of algebraic equations by using the HRH functions together Newton-Cotes nodes. In this method time and computations are small, because the matrix φ_{MN} introduces in (12) contain many zeros, and these zeros make HRH functions faster than other methods. Numerical examples with satisfactory results are given to demonstrate that it is reliable and useful tool to solve the FNFIDEs.

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7. References

[1] J. H. He, Nonlinear oscillation with fractional derivative and its applicatons, International conference on vibrating engineering 98, Dalian, China, (1998) 288-291.

[2] J. H. He, Some application of nonlinear fractional diffrential equations and their approximations, Bulletin of Science Technology 15 (2) (1999) 86-90.

[3] F. Mainardi, Fractals and Fractional Calculus Continuum Mechanics, Springer Verlag, New York, (1997) 291-348.

[4] S. Momani, M. A. Noor, Numerical methods for fourth-order fractional integro-differential equations, Applied Mathmatics and Computation 182 (2006) 754-760.

[5] E. A. Rawashdeh, Numrical solution of fractional integro-differential equations by collocation method, Applied Mathmatics and Computation 176 (2006) 1-6.

[6] H. Saeedi, M. M. Moghadam, N. Mollahasani, G. N. Chuev, A CAS wavelet method for solving nonlinear Fredholm integro-diffrential equations of fractional oder,Communications in Nonlinear Science and Numerical Simulation 16 (2011) 1154-1163.

[7] J. Hou, B. Qin, C. Yang, Numerical solution of nonlinear fredholm integro differential equations of fractional order by using hy brid functions and the collocation method, Journal of Applid Mathematics, 687030 (2012) 1-9.

[8] L. Zhu, Q. Fan, Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet, Communications in Nonlinear Science and Numerical Simulation 17 (6) (2012) 2333-1341.

[9] I. Podlubny, Fractional Differential Equatio, Academic Press, New York, (1999).

[10] J. D. Munkhammar, Riemann-Liouville Fractional Derivatives and the Taylor-Riemmann Series, U. U. D. M. Project Report, 2004.

[11] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematic Studies, Elsvier 204 (2006).

[12] A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional -order differential equations, Computers and Mathematics with Applications 59 (2010), 1326-1336.

[13] B. Arabzadeh, M. Razzaghi, Y. Ordukhani, Numerical solution of linear time-varying differential equations using hybrid of block-pulse and rationalized Haar functions, Journal of Vibration and Control, 12 (10) (2006) 1081-1092.

[14] Y. Ordokhani, Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via rationalized Haar functions, Applied Mathmatics and Computation 187 (2006) 436-443.

[15] G. M. Philips, P.J. Taylor, Theory and Application of Numerical Analysis, Academic Press, New York, 1973.

[16] Y. Li, W. Zhao, Haar Wavelet operational matrix of fractional order integration and its application in solving the fractional order differential equations, Applied Mathmatics and Computation 216 (2010) 2276-2285.

[17] A. Kilicman, Z. A. A. Al Zhour, Kronecker operational matrices for fractional calculus and some applications, Applied Mathematics and Computation 187 (2007) 250-265.

[18] M. F. Al-Jamal, E. A. Rawashdeh,The approximate solution of fractional integro-differential equations, Int. J. Contemp. Math. Sciences 4 (22) (2009) 1067-1078.