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# Biorthogonal Wavelet Packets Associated with Nonuniform Multiresolution

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**Abstract.** In this paper, we introduce the notion of biorthogonal wavelet packets associated with nonuniform multiresoltion analysis and study their characteristics by means of Fourier transform. Three biorthogonal formulas regarding these wavelet packets are established. Moreover, it is shown how to obtain several new Riesz bases of the space $L^2(\mathbb{R})$ by constructing a series of subspaces of these nonuniform wavelet packets.

**Keywords:** Wavelet; nonuniform multiresolution analysis; biorthogonal wavelet packet; Riesz basis; Fourier transform.

## 1. Introduction

In his pioneering paper, Mallat [11] first formulated a new and remarkable idea of multiresolution analysis (MRA) which deals with a general formalism for the construction of an orthonormal basis of wavelet bases. Mathematically, an MRA consist of a sequence of embedded closed subspaces, $\{V_j : j \in \mathbb{Z}$ of $L2\mathbb{R}$ such that $fx\in Vj$ if and only if $f2x\in Vj+1$. Furthermore, there exists an element $\varphi\in V0$ such that the collection of integer translates of function $\varphi$, $\{\varphi(\mathrm{x}-\mathrm{k}): k \in \mathbb{Z}\}$ represents a complete orthonormal system for $V_0$. The function $\varphi$ is called the *scaling function* or the *father wavelet*. Recently, the idea of MRA and wavelets have been generalized in many different settings, for example, one can replace the dilation factor 2 by an integer $M \geq 2$ and in higher dimensions, it can be replaced by a dilation matrix $A$, in which case the number of wavelets required is $|detA| - 1$. But in all these cases, the translation set is always a group. In the two papers [6, 7], Gabardo and Nashed considered a generalization of Mallat's [11] celebrated theory of MRA based on spectral pairs, in which the translation set acting on the scaling function associated with the MRA to generate the subspace $V_0$ is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. More precisely, this set is of the form $\Lambda = \{0, \mathrm{r}/\mathrm{N}\} + 2\,\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N-1$, $r$ is an odd integer relatively prime to $N$. They call this a *nonuniform multiresolution analysis* (NUMRA). Moreover, they have provided the necessary and sufficient conditions for the existence of associated wavelets in $L^2(\mathbb{R})$. Later on, Gabardo and Yu [8, 9] continued their research in this non-standard setting and gave the characterization of all nonuniform wavelets associated with nonuniform multiresolution analysis.

It is well-known that the classical orthonormal wavelet bases have poor frequency localization. To overcome this disadvantage, Coifman *et al.* [5] constructed univariate orthogonal wavelet packets as a generalization of Walsh systems. Wavelet packets give rise to a large class of orthonormal bases of $L^2(\mathbb{R})$, each one corresponding to a different splitting of $L^2(\mathbb{R})$ into direct sum of its closed subspaces. Wavelet packets, due to their nice characteristics have been widely applied to signal processing, coding theory, image compression, fractal theory and solving integral equations and so on. Chui and Li [3] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. The introduction of biorthogonal wavelet packets attributes to Cohen and Daubechies [4]. Shen [14] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Other notable generalizations are the orthogonal version of vector-valued wavelet packets [2], higher dimensional wavelet packets with arbitrary dilation matrix [10], the orthogonal $p$-wavelet packets and the $p$-wavelet frame packets on the positive half-line $\mathbb{R}^+$ [12, 13].

In his recent paper, Behera [1] has constructed nonuniform wavelet packets associated with nonuniform multiresolution analysis. He proved lemmas on the so-called splitting trick and several theorems concerning the construction of nonuniform wavelet packets on $\mathbb{R}$. It is well known that the orthogonal wavelet packets have many desired properties such as compact support, good frequency localization and vanishing moments. However, there is no continuous symmetry which is a much desired property in image and signal processing. To achieve symmetry, several generalizations of scalar orthogonal wavelet packets have been investigated in literature. The biorthogonal wavelet packets achieve symmetry where the orthogonality is replaced by the biorthogonality. Therefore, the main goal of this paper is to introduce the notion of biorthogonal wavelet packets associated with nonuniform multiresoltion analysis and investigate their properties by means of the Fourier transform. Further, we also construct several new Riesz bases of space $L^2(\mathbb{R})$ by constructing a series of subspaces of nonuniform wavelet packets.

## 2. Nonuniform multiresoltion analysis and the wavelet packets

In this section, we state some basic preliminaries and definitions including nonuniform multiresoltion analysis, the associated nonuniform wavelets and wavelet packets. More details are referring to [6–9].

**Definition 2.1.** We say that a pair of functions $f(x), \tilde{f}(x) \in L^2(\mathbb{R})$ are *biorthogonal*, if their translates satisfy

$$\langle f(.), \tilde{f}(.-\lambda) \rangle = \delta_{0,\lambda}, \ \lambda \in \Lambda, \tag{2.1}$$

where $\delta_{0,\lambda}$ is *Kronecker symbol*, i.e., $\delta_{0,\lambda} = 1$ when $\lambda = 0$ and $\delta_{0,\lambda} = 0$, otherwise.

**Definition 2.2.** Let $\mathbb{H}$ be a Hilbert space. A sequence $\{f_k\}_{k=1}^{\infty}$ of $\mathbb{H}$ is said to be a Riesz basis for $\mathbb{H}$ if there exist constants $A$ and $B, 0 < A \leq B < \infty$ such that any $f \in \mathbb{H}$ can be represented as a series $f = \sum_{k=1}^{\infty} c_k f_k = 1$ converging in $\mathbb{H}$ with

$$A \, \|f\|_2^2 \leq \sum_{k=1}^{\infty} |c_k|^2 \leq B \, \|f\|_2^2. \tag{2.2}$$

We first recall the definition of nonuniform multiresolution analysis and some of its properties.

**Definition 2.3.** A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces $L^2(\mathbb{R})$ is called a *nonuniform multiresolution analysis* of $L^2(\mathbb{R})$ if the following hold:

(i) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$,

(ii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,

(iii) $f(x) \in V_j$ if and only if $f(2N.) \in V_{j+1}$,

(iv) there is a function $\varphi$ in $V_0$, called the *scaling function*, such that the system of functions $\{\varphi(.-\lambda) : \lambda \in \Lambda$ forms a Riesz basis for subspace *V0*.

It is worth noticing that, when $N = 1$, one recovers from the definition above the standard definition of a one-dimensional multiresolution analysis with dilation factor equal to 2. When $N > 1$, the dilation factor of $2N$ ensures that $2N\Lambda \subset 2\mathbb{Z} \subset \Lambda$.

Since $\varphi(x) \in V_0 \subset V_1$, there exists sequence $\{a_\lambda\}_{\lambda \in \Lambda}$ with $\sum_{\lambda \in \Lambda} |a_\lambda|^2 < \infty$ such that

$$\varphi\left(\frac{x}{2N}\right) = \sum_{\lambda \in \Lambda} a_\lambda \, \varphi(x - \lambda). \tag{2.3}$$

Taking Fourier transform of (2.3), we get

$$\hat{\varphi}\,(2N\xi) = h_0(\xi)\,\hat{\varphi}\,(\xi), \tag{2.4}$$

where $h_0(\xi) = \sum_{\lambda \in \Lambda} a_\lambda\, e^{-2\pi i\lambda\xi}$, is called the *symbol* of $\varphi(x)$.

Let $W_j : j \in \mathbb{Z}$ be the direct complementary subspaces of $V_j$ in $V_{j+1}$. Assume that there exist a set of $2N-1$ functions $\{\psi_1, \psi_2, \ldots, \psi_{2N-1}\}$ in $L^2(\mathbb{R})$ such that their translates and dilations form a Riesz basis of $W_j$, i.e,

$$W_j = \overline{\text{span}}\left\{\psi_r\big((2N)^j . -\lambda\big) : \lambda \in \Lambda,\ 1 = 1,2,\ldots,2N-1\right\},\ j \in \mathbb{Z}\,. \tag{2.5}$$

Since $\psi_r(x) \in W_0 \subset V_1$, $1 \le r \le 2N-1$, there exists a sequence $\{a_\lambda^r\}_{\lambda \in \Lambda}$ with $\sum_{\lambda \in \Lambda} |a_\lambda^r|^2 < \infty$ such that

$$\frac{1}{2N}\psi_r\left(\frac{x}{2N}\right) = \sum_{\lambda \in \Lambda} a_\lambda^r\ \varphi(x-\lambda). \tag{2.6}$$

Implementing the Fourier transform for both sides of (2.6) gives

$$\hat{\psi}_r(2N\xi) = h_r(\xi)\hat{\varphi}\,(\xi), \tag{2.7}$$

where

$$h_r(\xi) = \sum_{\lambda \in \Lambda} a_\lambda^r\, e^{-2\pi i\lambda\xi},\ \ 1 \le r \le 2N-1. \tag{2.8}$$

If $\varphi(x)$, $\tilde{\varphi}(x) \in L^2(\mathbb{R})$ are a pair of *biorthogonal scaling functions*, then it follows by Definition 2.1 that

$$\langle \varphi(.),\ \tilde{\varphi}(.-\lambda)\rangle = \delta_{0,\lambda},\quad \lambda \in \Lambda. \tag{2.9}$$

Moreover, we say that $\psi_r(x), \tilde{\psi}_r(x) \in L^2(\mathbb{R}), 1 \le r \le 2N-1$ are pair of biorthogonl nonuniform wavelets associated with a pair of biorthogonal scaling functions $\varphi(x)$ and $\tilde{\varphi}(x)$ if the family $\{\psi_r(.-\lambda) : \lambda \in$ *Λ, 1=1,…,2N−1* is a Riesz basis of *W0*, and

$$\langle \varphi(.),\ \tilde{\psi}_r(.-\lambda)\rangle = 0,\ \ \lambda \in \Lambda, 1 \le r \le 2N-1, \tag{2.10}$$

$$\langle \tilde{\varphi}(.),\ \psi_r(.-\lambda)\rangle = 0,\ \ \lambda \in \Lambda, 1 \le r \le 2N-1, \tag{2.11}$$

$$\langle \psi_r(.),\ \tilde{\psi}_s(.-\lambda)\rangle = \delta_{r,s}\delta_{0,\lambda}, \lambda \in \Lambda, 1 \le r,s \le 2N-1. \tag{2.12}$$

Set

$$W_j^r = \overline{\text{span}}\left\{\psi_r\big((2N)^j . -\lambda\big) : \lambda \in \Lambda\ \right\},\ j \in \mathbb{Z}\,, 1 \le r \le 2N-1. \tag{2.13}$$

By definition of $W_j$ and formulae (2.9)– (2.12), we obtain the following proposition.

**Proposition 2.4.** If $\psi_r(x), \tilde{\psi}_r(x) \in L^2(\mathbb{R}), 1 \le r \le 2N-1$, are a pair of biorthogonal nonuniform wavelets associated with a pair of biorthogonal scaling functions $\varphi(x), \tilde{\varphi}(x)$, then

$$L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j = \oplus_{j\in\mathbb{Z}} \oplus_{r=1}^{2N-1} W_j^r, \tag{2.14}$$

where $\oplus$ denotes the direct sum.

Similar to (2.3) and (2.6), there exist finite sequences $\{\tilde{a}_\lambda\}_{\lambda\in\Lambda}$ and $\{\tilde{a}_\lambda^r\}_{\lambda\in\Lambda}, 1 \le r \le 2N-1$ such that $\tilde{\varphi}(x)$ and $\tilde{\psi}_r$ satisfy the following equations:

$$\tilde{\varphi}\left(\frac{x}{2N}\right) = \sum_{\lambda\in\Lambda} \tilde{a}_\lambda \, \tilde{\varphi}(x-\lambda). \tag{2.15}$$

$$\frac{1}{2N}\tilde{\psi}_r\left(\frac{x}{2N}\right) = \sum_{\lambda\in\Lambda} \tilde{a}_\lambda^r \, \tilde{\varphi}(x-\lambda). \tag{2.16}$$

For $n = 0,1,2,\ldots$, the basic *biorthogonal nonuniform wavelet packets* $\omega_n$ and $\widetilde{\omega}_n$ (as defined in [1]) associated with the scaling functions $\varphi(x)$ and $\tilde{\varphi}(x)$, respectively, are defined recursively by

$$\omega_{q+2Np}(x) = \sum_{\lambda\in\Lambda}(2N)a_\lambda^q\,\omega_p(2Nx-\lambda), \tag{2.17}$$

$$\widetilde{\omega}_{q+2Np}(x) = \sum_{\lambda\in\Lambda}(2N)\tilde{a}_\lambda^q\,\widetilde{\omega}_p(2Nx-\lambda), \tag{2.18}$$

where $p \ge 0$ is the unique element such that $n = q + 2Np, 0 \le q \le 2N-1$ holds.

Applying the Fourier transform for the both sides of (2.17) and (2.18) yields, respectively,

$$\widehat{\omega}_{q+2Np}(x) = h_q((2N)^{-1}\xi)\widehat{\omega}_p((2N)^{-1}\xi),\;\; 0 \le q \le 2N-1 \tag{2.19}$$

$$\widehat{\widetilde{\omega}}_{q+2Np}\,(x) = \tilde{h}_q((2N)^{-1}\xi)\;\widehat{\widetilde{\omega}}_p((2N)^{-1}\xi),\;\; 0 \le q \le 2N-1. \tag{2.20}$$

**Lemma 2.5[6].** Let $\varphi(x), \tilde{\varphi}(x)$ be a pair of scaling functions. Then $\varphi(x), \tilde{\varphi}(x)$ are biorthogonal if and only if

$$\sum_{\lambda\in\Lambda} \hat{\varphi}(\xi-\lambda)\overline{\hat{\tilde{\varphi}}\,(\xi-\lambda)} = 1, \quad a.e.\ \xi \in \mathbb{R}.$$

## 3. Properties of biorthogonal wavelet packets

In this section, we characterize the biorthogonality property of the nonuniform wavelet packets by means of Fourier transform.

**Lemma 3.1.** Assume that $\omega_q, \widetilde{\omega}_q \in L^2(\mathbb{R}),\ 1 \le q \le 2N-1$ are a pair of biorthogonal nonuniform wavelets associated with a pair of biorthogonal scaling functions $\omega_0$ and $\widetilde{\omega}_0$. Then, we have

$$\sum_{\boldsymbol{\sigma=0}}^{\boldsymbol{2N-1}} h_p((2N)^{-1}(\xi+2\pi\sigma))\;\overline{\tilde{h}_q((2N)^{-1}(\xi+2\pi\sigma))} = \delta_{p,q}\,,\;\; 0 \le p,q \le 2N-1. \tag{3.1}$$

**Proof.** By using equations (2.9)–(2.13), (2.19), (2.20) and Lemma 2.5, we obtain

$$
\begin{aligned}
\delta_{p,q} &= \sum_{\lambda \in \Lambda} \omega_p(\xi + 2\pi\lambda)\, \overline{\widetilde{\omega}_q(\xi + 2\pi\lambda)} \\
&= \sum_{\lambda \in \Lambda} h_p\big((2N)^{-1}(\xi + 2\pi\lambda)\big)\, \widehat{\omega}_0\big((2N)^{-1}(\xi + 2\pi\lambda)\big) \\
&\qquad\qquad \times\, \overline{\widehat{\widetilde{\omega}}_0((2N)^{-1}(\xi + 2\pi\lambda))}\ \overline{\tilde{h}_q\big((2N)^{-1}(\xi + 2\pi\lambda)\big)} \\
&= \sum_{\boldsymbol{\sigma=0}}^{\boldsymbol{2N-1}} h_p\big((2N)^{-1}(\xi + 2\pi\sigma)\big)\, \overline{\tilde{h}_q((2N)^{-1}(\xi + 2\pi\sigma))} \\
&\qquad \times \sum_{\lambda \in \Lambda} \widehat{\omega}_0((2N)^{-1}(\xi + 2\pi\sigma) + 2\pi\lambda)\, \overline{\widehat{\widetilde{\omega}}_0((2N)^{-1}(\xi + 2\pi\sigma) + 2\pi\lambda)} \\
&= \sum_{\boldsymbol{\sigma=0}}^{\boldsymbol{2N-1}} h_p((2N)^{-1}(\xi + 2\pi\sigma))\, \overline{\tilde{h}_q((2N)^{-1}(\xi + 2\pi\sigma))}\,. \qquad \boxdot
\end{aligned}
$$

**Theorem 3.2.** Suppose $\{\omega_n(x) : n \geq 0\}$ and $\{\widetilde{\omega}_n(x) : n \geq 0\}$ are nonuniform wavelet packets with respect to a pair of biorthogonal scaling functions $\omega_0(x)$ and $\widetilde{\omega}_0(x)$, respectively. Then, for $n \geq 0$, we have

$$\langle \omega_n(.),\ \widetilde{\omega}_n(.-\lambda) \rangle = \delta_{0,\lambda}\,,\ \ \lambda \in \Lambda\,. \tag{3.2}$$

**Proof.** We prove this result by using induction on $n$. It follows from (2.9) and (2.12) that the claim is true for $n = 0$ and $n = 1, 2, \ldots, 2N - 1$. Assume that (3.2) holds when $n < l$, where $\ell > 0$. Then, we prove the result (3.2) for $n = \ell$. Order $n = q + 2Np$, where $p \geq 0,\ 0 \leq q \leq 2N - 1$ and $p < n$. Therefore, by induction assumption, we have

$$\langle \omega_p(.),\ \widetilde{\omega}_p(.-\lambda) \rangle = \delta_{0,\lambda} \quad \Leftrightarrow \quad \sum_{\lambda \in \Lambda} \widehat{\omega}_p(\xi - \lambda)\overline{\widehat{\widetilde{\omega}}_p\,(\xi - \lambda)} = 1\,,\quad \xi \in \mathbb{R}.$$

Using Proposition 2.4, Lemma 2.5, (2.19) and (2.20), we obtain

$$
\begin{aligned}
&\langle \omega_n(.), \widetilde{\omega}_n(.\,-\lambda) \rangle = \langle \widehat{\omega}_n(.),\ \widehat{\widetilde{\omega}}_n(.\,-\lambda) \rangle \\
&= \int_{\mathbb{R}} \widehat{\omega}_{q+2Np}(\xi)\ \overline{\widehat{\widetilde{\omega}}_{q+2Np}\,(\xi)}\, e^{2\pi i\lambda\xi} d\xi \\
&= \int_{\mathbb{R}} h_q((2N)^{-1}\xi)\widehat{\omega}_p((2N)^{-1}\xi)\ \overline{\tilde{h}_q((2N)^{-1}\xi)}\ \overline{\widehat{\widetilde{\omega}}_p((2N)^{-1}\xi)}\, e^{2\pi i\lambda\xi} d\xi \\
&= \sum_{\lambda \in \Lambda} \int_{2N([0,2\pi]+\lambda)} h_q((2N)^{-1}\xi)\widehat{\omega}_p((2N)^{-1}\xi)\ \overline{\tilde{h}_q((2N)^{-1}\xi)}\ \overline{\widehat{\widetilde{\omega}}_p((2N)^{-1}\xi)}\, e^{2\pi i\lambda\xi} d\xi \\
&= \int_{2N[0,2\pi]} \left\{ \sum_{\lambda \in \Lambda} \widehat{\omega}_p\big((2N)^{-1}(\xi + 2\pi\lambda)\big)\, \overline{\widehat{\widetilde{\omega}}_p((2N)^{-1}(\xi + 2\pi\lambda))} \right\} \\
&\qquad\qquad \times\ h_q((2N)^{-1}\xi)\, \overline{\tilde{h}_q((2N)^{-1}\xi)}\ e^{2\pi i\lambda\xi} d\xi \\
&= \int_{2N[0,2\pi]} h_q((2N)^{-1}\xi)\, \overline{\tilde{h}_q((2N)^{-1}\xi)}\ e^{2\pi i\lambda\xi} d\xi
\end{aligned}
$$

$$= \int_{[0,2\pi]} \sum_{\sigma=0}^{2N-1} h_p\big((2N)^{-1}(\xi + 2\pi\sigma)\big) \overline{\tilde{h}_q((2N)^{-1}(\xi + 2\pi\sigma))} \; e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{[0,2\pi]} e^{2\pi i\lambda\xi} d\xi \; = \delta_{0,\lambda} \qquad \boxdot$$

**Theorem 3. 3.** Suppose $\{\omega_n(x) : n \geq 0\}$ and $\{\widetilde{\omega}_n(x) : n \geq 0\}$ are nonuniform wavelet packets with respect to a pair of biorthogonal scaling functions $\omega_0(x)$ and $\widetilde{\omega}_0(x)$, respectively. Then, for $p \geq 0$, we have

$$\langle \omega_{q_1+2Np}(\,.\,), \; \widetilde{\omega}_{q_2+2Np}(\,.-\lambda) \rangle = \delta_{0,\lambda}\, \delta_{q_1,q_2}, \;\; \lambda \in \Lambda, 0 \leq q_1, q_2 \leq 2N-1. \qquad (3.3)$$

**Proof.** By Lemma 2.5, we have

$$\langle \omega_{q_1+2Np}(\,.\,), \widetilde{\omega}_{q_2+2Np}(\,.-\lambda) \rangle = \langle \widehat{\omega}_{q_1+2Np}(\,.\,), \widehat{\widetilde{\omega}}_{q_2+2Np}(\,.-\lambda) \rangle$$

$$= \int_{\mathbb{R}} \widehat{\omega}_{q_1+2Np}(\xi) \; \overline{\widehat{\widetilde{\omega}}_{q_2+2Np}\,(x)}\, e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{\mathbb{R}} h_{q_1}((2N)^{-1}\xi)\widehat{\omega}_p((2N)^{-1}\xi) \; \overline{\tilde{h}_{q_2}((2N)^{-1}\xi)} \; \overline{\widehat{\widetilde{\omega}}_p((2N)^{-1}\xi)}\, e^{2\pi i\lambda\xi} d\xi$$

$$= 2N \sum_{\lambda \in \Lambda} \int_{([0,2\pi]+2\pi\lambda)} h_{q_1}(\xi)\widehat{\omega}_p(\xi) \; \overline{\tilde{h}_{q_2}(\xi)} \; \overline{\widehat{\widetilde{\omega}}_p(\xi)}\, e^{2\pi i(2N)\lambda\xi} d\xi$$

$$= 2N \int_{[0,2\pi]} \sum_{\lambda \in \Lambda} \widehat{\omega}_p(\xi-\lambda) \; \overline{\widehat{\widetilde{\omega}}_p(\xi-\lambda)}\, h_{q_1}(\xi) \; \overline{\tilde{h}_{q_2}(\xi)} \; e^{2\pi i(2N)\lambda\xi} d$$

$$= \int_{2N[0,2\pi]} h_{q_1}((2N)^{-1}\xi) \; \overline{\tilde{h}_{q_2}((2N)^{-1}\xi)} \; e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{[0,2\pi]} \sum_{\sigma=0}^{2N-1} h_{q_1}\big((2N)^{-1}(\xi + 2\pi\sigma)\big) \; \overline{\tilde{h}_{q_2}((2N)^{-1}(\xi + 2\pi\sigma))} \; e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{[0,2\pi]} \delta_{q_1,q_2}\, e^{2\pi i\lambda\xi} d\xi \; = \delta_{0,\lambda}\, \delta_{q_1,q_2}. \qquad \boxdot$$

**Theorem 3.4.** If $\{\omega_n(x) : n \geq 0\}$ and $\{\widetilde{\omega}_n(x) : n \geq 0\}$ are basic nonuniform wavelet packets with respect to a pair of biorthogonal scaling functions $\omega_0(x)$ and $\widetilde{\omega}_0(x)$, respectively. Then, for $\ell, n \geq 0$, we have

$$\langle \omega_\ell(\,.\,), \; \widetilde{\omega}_n(\,.-\lambda) \rangle = \delta_{\ell,n}\, \delta_{0,\lambda}\,, \;\; \lambda \in \Lambda. \qquad (3.4)$$

**Proof.** For $\ell = n$, the result (3.4) follows by Theorem 3.2. When $\ell \neq n$, and $0 \leq \ell, n \leq 2N-1$, the result (3.4) can be established from Theorem 3.3. Assuming that $\ell$ is not equal to $n$, and atleast one of $\{\ell, n\}$ does not lies in $0,1,..,2N-1$, then we can rewrite $\ell, n$ as $\ell = q_1 + 2Np_1$, $n = s_1 + 2Nr_1$, where $p_1,\, r_1 \geq 0,\; 0 \leq q_1,\, s_1 \leq 2N-1$.

**Case 1.** If $p_1 = r_1$, then $q_1 \neq s_1$. Therefore, Eq. (3.4) follows by virtue of (2.19), (2.20), Lemma 2.5 and (3.1), i.e,

$$\langle \omega_\ell(\,.\,),\ \tilde{\omega}_n(\,.-\lambda) \rangle = \langle \omega_{q_1+2Np_1}(\,.\,),\ \tilde{\omega}_{s_1+2N\,r_1}(\,.-\lambda) \rangle$$

$$= \langle \widehat{\omega}_{q_1+2Np_1}(\,.\,),\ \widehat{\tilde{\omega}}_{s_1+2N\,r_1}(\,.-\lambda) \rangle$$

$$= \int_{\mathbb{R}} \widehat{\omega}_{q_1+2Np_1}(\xi)\ \overline{\widehat{\tilde{\omega}}_{s_1+2N\,r_1}(\xi)}\, e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{\mathbb{R}} h_{q_1}((2N)^{-1}\xi)\widehat{\omega}_{p_1}((2N)^{-1}\xi)\ \overline{\widehat{\tilde{\omega}}_{r_1}((2N)^{-1}\xi)}\,\overline{\tilde{h}_{s_1}((2N)^{-1}\xi)}\ e^{2\pi i\lambda\xi} d\xi$$

$$= \sum_{\lambda\in\Lambda} \int_{2N([0,2\pi]+\lambda)} h_{q_1}((2N)^{-1}\xi)\widehat{\omega}_{p_1}((2N)^{-1}\xi)\ \overline{\widehat{\tilde{\omega}}_{r_1}((2N)^{-1}\xi)}$$

$$\times\ \overline{\tilde{h}_{s_1}((2N)^{-1}\xi)}\ e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{2N[0,2\pi]} \left\{ \sum_{\lambda\in\Lambda} \widehat{\omega}_{p_1}\big((2N)^{-1}(\xi+\lambda)\big)\, \overline{\widehat{\tilde{\omega}}_{r_1}((2N)^{-1}(\xi+\lambda))} \right\}$$

$$\times\ h_{q_1}((2N)^{-1}\xi)\, \overline{\tilde{h}_{s_1}((2N)^{-1}\xi)}\ e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{[0,2\pi]} \sum_{\boldsymbol{\sigma}=\mathbf{0}}^{\mathbf{2N-1}} h_{q_1}\big((2N)^{-1}(\xi+2\pi\sigma)\big)\ \overline{\tilde{h}_{s_1}((2N)^{-1}(\xi+2\pi\sigma))}\ e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{[0,2\pi]} \delta_{q_1,s_1}\, e^{2\pi i\lambda\xi} d\xi\ = \delta_{0,\lambda} = 0.$$

**Case 2.** If $p_1 \neq r_1$, order $p_1 = q_2 + 2N\,p_2$, $r_1 = s_2 + 2N\,r_2$, where $p_2, r_2 \geq 0$ and $0 \leq q_2, s_2 \leq 2N - 1$. If $p_2 = r_2$, then $q_2 \neq s_2$. Similar to Case 1, (3.4) can be established. When $p_2 \neq r_2$, we order $p_2 = q_3 + 2N\,p_3$, $r_2 = s_3 + 2N\,r_3$, where $p_3, r_3 \geq 0$ and $0 \leq q_3, s_3 \leq 2N - 1$. Thus, after taking finite steps (denoted by $\kappa$), we obtain $0 \leq p_\kappa, r_\kappa \leq 2N - 1$ and $0 \leq q_\kappa, s_\kappa \leq 2N - 1$. If $p_\kappa = r_\kappa$, then $q_\kappa \neq s_\kappa$. Similar to the Case 1, (3.4) follows. If $p_\kappa \neq r_\kappa$, then it gets from (2.9)–(2.12) that

$$\langle \omega_{p_\kappa}(.),\ \tilde{\omega}_{r_\kappa}(.-\lambda) \rangle = 0,\ \lambda \in \Lambda \iff \sum_{\lambda\in\Lambda} \widehat{\omega}_{p_\kappa}(\xi-\lambda)\overline{\widehat{\tilde{\omega}}_{r_\kappa}(\xi-\lambda)} = 0\,,\quad \xi \in \mathbb{R}.$$

Furthermore, we obtain

$$\langle \omega_p(\,.\,),\ \tilde{\omega}_r(\,.-\lambda) \rangle = \langle \widehat{\omega}_p(\,.\,),\ \widehat{\tilde{\omega}}_r(\,.-\lambda) \rangle$$

$$= \langle \widehat{\omega}_{q_1+2Np_1}(\,.\,),\ \widehat{\tilde{\omega}}_{s_1+2N\,r_1}(\,.-\lambda) \rangle$$

$$= \int_{\mathbb{R}} \widehat{\omega}_{q_1+2Np_1}(\xi)\ \overline{\widehat{\tilde{\omega}}_{s_1+2N\,r_1}(x)}\, e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{\mathbb{R}} h_{q_1}((2N)^{-1}\xi)\ h_{q_2}((2N)^{-2}\xi)\ \hat{\omega}_{p_2}((2N)^{-2}\xi)\overline{\hat{\tilde{\omega}}_{r_2}((2N)^{-2}\xi)} \times \overline{\tilde{h}_{s_1}((2N)^{-1}\xi)}\ \overline{\tilde{h}_{s_2}((2N)^{-2}\xi)}\, e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{\mathbb{R}} \left\{\prod_{\ell=1}^{\kappa} h_{q_\ell}\big((2N)^{-\ell}\xi\big)\right\} \hat{\omega}_{p_\kappa}((2N)^{-\kappa}\xi)\overline{\hat{\tilde{\omega}}_{r_\kappa}((2N)^{-\kappa}\xi)} \left\{\prod_{\ell=1}^{\kappa} \overline{\tilde{h}_{s_\ell}((2N)^{-\ell}\xi)}\right\} e^{2\pi i\lambda\xi} d\xi$$

$$= \sum_{\lambda\in\Lambda} \int_{(2N)^{\kappa}([0,2\pi]+\lambda)} \left\{\prod_{\ell=1}^{\kappa} h_{q_\ell}\big((2N)^{-\ell}\xi\big)\right\} \left\{\hat{\omega}_{p_\kappa}((2N)^{-\kappa}\xi)\overline{\hat{\tilde{\omega}}_{r_\kappa}((2N)^{-\kappa}\xi)}\right\} \times \left\{\prod_{\ell=1}^{\kappa} \overline{\tilde{h}_{s_\ell}((2N)^{-\ell}\xi)}\right\} e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{(2N)^{\kappa}[0,2\pi]} \left\{\prod_{\ell=1}^{\kappa} h_{q_\ell}\big((2N)^{-\ell}\xi\big)\right\} \left\{\prod_{\ell=1}^{\kappa} \overline{\tilde{h}_{s_\ell}((2N)^{-\ell}\xi)}\right\} \times\left\{\sum_{\lambda\in\Lambda} \hat{\omega}_{p_\kappa}((2N)^{-\kappa}(\xi+\lambda))\overline{\hat{\tilde{\omega}}_{r_\kappa}((2N)^{-\kappa}(\xi+\lambda))}\right\} e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{(2N)^{\kappa}[0,2\pi]} \left\{\prod_{\ell=1}^{\kappa} h_{q_\ell}\big((2N)^{-\ell}\xi\big)\right\} .0. \left\{\prod_{\ell=1}^{\kappa} \overline{\tilde{h}_{s_\ell}((2N)^{-\ell}\xi)}\right\} e^{2\pi i\lambda\xi} d\xi \qquad \boxdot$$

## 4. The nonuniform biorthogonal wavelet packet bases of $L^2(\mathbb{R})$

In this section, we will decompose subspaces $V_j$ , $\tilde{V}_j$ and $W_j$ , $\widetilde{W}_j$ by constructing a series of subspaces of nonuniform wavelet packets. Furthermore, we present the direct decomposition for space $L^2(\mathbb{R})$.

For any $n \geq 0$, define

$$U_n = \overline{\text{span}}\,\{\omega_n(x-\lambda) :\ \lambda\in\Lambda\ \}, \qquad (4.1)$$

Then, we have $U_0 = V_0$, $U_s = W_0^q, 1 \le q \le 2N-1$. Assume that $\left\{h_q\ \ ((2N)^{-1}(\xi-\lambda))\right\}_{q,\lambda=0}^{2N-1}$ is a unitary matrix.

**Lemma 4.1.** For $n \geq 0$, the space $\mathcal{D}U_n$ can be decomposed into direct sum of $U_{q+2Nn}$, $0 \le q \le 2N-1$, i.e.

$$\mathcal{D}U_n = \oplus_{q=0}^{2N-1}\ U_{q+2Nn}, \qquad (4.3)$$

where $\mathcal{D}$ is the dilation operator with respect to the dilation $2N$.

**Proof.** First, we claim that

$$\mathcal{D}U_n = \left\{f(x):\ f(x) = \sum_{q=0}^{2N-1}\sum_{\lambda\in\Lambda} b_\lambda^q\ \omega_{q+2Nn}(x-\lambda), \qquad \sum_{\lambda\in\Lambda} \left|b_\lambda^q\right|^2 < \infty\right\}. \qquad (4.4)$$

As for any $0 \le q \le 2N-1$, by (2.17) and (4.1), $\omega_{q+2Nn}(x-\lambda) \in \mathcal{D}U_n$. Assume that $f(x) \in \mathcal{D}U_n$, then there exists a sequence $\{c_\lambda\}_{\lambda\in\Lambda}$ such that

$$f(x) = \sum_{\lambda\in\Lambda} c_\lambda \ \omega_n(2Nx-\lambda). \tag{4.5}$$

Further, if there exists a sequence $\{b_\lambda^q\}_{\lambda\in\Lambda}$, $0 \le q \le 2N-1$, as for $f(x) \in \mathcal{D}U_n$, such that

$$f(x) = \sum_{q=0}^{2N-1} \sum_{\lambda\in\Lambda} b_\lambda^q \ \omega_{q+2Nn}(x-\lambda). \tag{4.6}$$

Taking Fourier transform on the both sides of (4.5) and (4.6), respectively and by using (2.19), we obtain

$$\hat{f}(\xi) = m((2N)^{-1}\xi)\, \widehat{\omega}_n((2N)^{-1}\xi) = \sum_{q=0}^{2N-1} g_q(\xi) h_q((2N)^{-1}\xi)\, \widehat{\omega}_n((2N)^{-1}\xi), \tag{4.7}$$

where

$$m(\xi) = \sum_{\lambda\in\Lambda} c_\lambda\, e^{-2\pi i\lambda\xi},\ g_q(\xi) = \sum_{\lambda\in\Lambda} b_\lambda^q\, e^{-2\pi i\lambda\xi},\ 0 \le q \le 2N-1, \xi \in \mathbb{R}.$$

The above result (4.7) follows if the following equality holds:

$$m((2N)^{-1}\xi) = \sum_{q=0}^{2N-1} g_q(\xi) h_q((2N)^{-1}\xi). \tag{4.8}$$

For any sequence $\{c_\lambda\}_{\lambda\in\Lambda}$, we will prove that there exists a sequence $\{b_\lambda^q\}_{\lambda\in\Lambda}$, $0 \le q \le 2N-1$ such that (4.8) is satisfied. Moreover, Eq. (4.8) is equivalent to the following equation:

$$m((2N)^{-1}(\xi-\lambda)) = \sum_{q=0}^{2N-1} g_q(\xi) h_q((2N)^{-1}(\xi-\lambda)), \quad 0 \le \lambda \le 2N-1. \tag{4.9}$$

The solvability of Eq. (4.9) for every sequence $\{c_\lambda\}_{\lambda\in\Lambda}$ follows from the fact that the matrix $\{h_q((2N)^{-1}(\xi-\lambda))\}_{0\le\lambda,q\le 2N-1}$ is a unitary matrix (see [6, 7]). Hence, equality (4.4) follows. Furthermore, applying Theorem 3.3, it follows that

$$\{\omega_{q+2Nn}(x-\lambda):\ n \ge 0, 0 \le q \le 2N-1, \lambda \in \Lambda\},$$

forms a Riesz basis of $\mathcal{D}U_n$. ⊡

Similar to (4.3), we can establish the following result:

$$\widetilde{U}_0 = \widetilde{V}_0,\ \widetilde{U}_s = \widetilde{W}_0^q, 0 \le q \le 2N-1,$$

and

$$\mathcal{D}\widetilde{U}_n = \oplus_{q=0}^{2N-1}\ \widetilde{U}_{q+2Nn}. \tag{4.10}$$

For $\ell \in \mathbb{N}$, define $\tilde{\Gamma}_\ell = \sum_{j=0}^{\ell} (2N)^j \mu_j, \Gamma_\ell = \tilde{\Gamma}_\ell - \tilde{\Gamma}_{\ell-1}$. In what follows, we will give the direct decomposition of space $L^2(\mathbb{R})$.

**Theorem 4.2.** The family of functions $\{\omega_n(x-\lambda) : n \in \Gamma_\ell, \lambda \in \Lambda\}$ constitutes Riesz basis of $\mathcal{D}^\ell W_0$. In particular, $\{\omega_n(x-\lambda) : n \ge 0,\ \lambda \in \Lambda\}$ constitutes Riesz basis of $L^2(\mathbb{R})$.

**Proof.** By equation (4.3), we have $\mathcal{D}U_0 = \oplus_{q=0}^{2N-1} U_q$ i.e. $\mathcal{D}U_0 = U_0 \oplus_{q=1}^{2N-1} U_q$. Since $U_0 = V_0$ and $W_0 = \oplus_{q=1}^{2N-1} W_0^q = \oplus_{q=1}^{2N-1} U_q$, then $\mathcal{D}U_0 = V_0 \oplus W_0$. It can be inductively inferred from (4.3) that

$$\mathcal{D}^\ell U_0 = \mathcal{D}^{\ell-1} U_0 \oplus_{n\in\Gamma_\ell} U_n, \quad \ell \in \mathbb{N}. \tag{4.11}$$

Since $V_{j+1} = V_j \oplus W_j$, $j \in \mathbb{Z}$, therefore, $\mathcal{D}^\ell U_0 = \mathcal{D}^{\ell-1} U_0 \oplus \mathcal{D}^{\ell-1} W_0$, $\ell \in \mathbb{N}$. It follows from (4.3) and Proposition 2.4 that $\mathcal{D}^\ell W_0 = \oplus_{n\in\Gamma_\ell} U_n$ and

$$L^2(\mathbb{R}) = V_0 \oplus \left(\oplus_{\ell\geq 0}\ \mathcal{D}^\ell W_0\right) = U_0 \oplus \left(\oplus_{\ell\geq 0}\left(\oplus_{n\in\Gamma_\ell} U_n\right)\right) = \oplus_{n=0}^{\infty}\ U_n. \qquad (4.12)$$

In the light of Theorem 3.2, the family $\{\omega_n(x-\lambda) : \lambda \in \Lambda\}$ is a Riesz basis of $\mathcal{D}W_0$. Moreover, according to (4.12), the family $\{\omega_n(x-\lambda) : n \geq 0, \ \lambda \in \Lambda\}$ forms a Riesz basis of $L^2(\mathbb{R})$. ⊡

**Corollary 4.3.** For every $\ell \in \mathbb{N}$, the family of functions

$$\{\tilde{\omega}_n(x-\lambda) : n \in \Gamma_\ell, \lambda \in \Lambda\}$$

forms a Riesz basis of $\mathcal{D}^\ell \widetilde{W}_0$.

**Corollary 4.4.** For every $\ell \in \mathbb{N}$, the family of functions

$$\left\{\omega_n\left((2N)^j x - \lambda\right) : j \in \mathbb{Z}, n \in \Gamma_\ell, \lambda \in \Lambda\right\}$$

forms a Riesz basis of $L^2(\mathbb{R})$.

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