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Numerical solution of the boundary value problems in calculus of variations using parametric cubic spline method

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Abstract. In this paper, a numerical solution based on parametric cubic spline is used for finding the solution of boundary value problems arising in the calculus of variations. The present approach has less computational coast and gives better approximation. This approximation reduce the problems to an explicit system of algebraic equations. Some numerical examples are also given to illustrate the accuracy and applicability of the presented method.

Keywords: Calculus of variation, Parametric spline, Numerical method.

1. Introduction

The calculus of variations and its extensions are devoted to finding the optimum function that gives the best value of the economic model and satisfies the constraints of a system. The need for an optimum function, rather than an optimal point, arises in numerous problems from a wide range of fields in engineering and physics, which include optimal control, transport phenomena, optics, elasticity, vibrations, statics and dynamics of solid bodies and navigation[1]. In computer vision the calculus of variations has been applied to such problems as estimating optical flow [2] and shape from shading [3]. Several numerical methods for approximating the solution of problems in the calculus of variations are known. Galerkin method is used for solving variational problems in [4]. The Ritz method [5], usually based on the subspaces of kinematically admissible complete functions, is the most commonly used approach in direct methods of solving variational problems. Chen and Hsiao [6] introduced the Walsh series method to variational problems. Due to the nature of the Walsh functions, the solution obtained was piecewise constant. Some orthogonal polynomials are applied on variational problems to find the continuous solutions for these problems [7-9]. A simple algorithm for solving variational problems via Bernstein orthonormal polynomials of degree six is proposed by Dixit et al. [10]. Razzaghi et al. [11] applied a direct method for solving variational problems using Legendre wavelets. Adomian decomposition method has been employed for solving some problems in calculus of variations in [12].

Spline functions are special functions in the space of which approximate solutions of ordinary differential equations. In other words spline function is a piecewise polynomial satisfying certain conditions of continuity of the function and its derivatives. The applications of spline as approximating, interpolating and curve fitting functions have been very successful[13-16]. In [17], a cubic non-polynomial spline technique has been developed for the numerical solutions of a system of fourth order boundary value problems associated with obstacle, unilateral and contact problems. Quadratic and cubic polynomial and non-polynomial spline functions based methods have been presented to find approximate solutions to second order boundary value problems[18]. Parametric spline method for a class of singular two-point boundary value problems has been developed by Rashidinia et al. [19]. The main purpose of the present paper is to use parametric cubic spline method for numerical solution of boundary value problems

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which arise from problems of calculus of variations. The method consists of reducing the problem to a set of algebraic equations.

The outline of the paper is as follows. First, in Section 2 we introduce the problems in calculus of variations and explain their relations with boundary value problems. Section 3 outlines parametric cubic spline and basic equations that are necessary for the formulation of the discrete system. Also in this section, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering two numerical examples.

2. Statement of the problem

The genaral form of a variational problem is finding extremum of the functional

$$J[u_1(t),u_2(t),\dots,u_n(t)]=\int_a^b G\Big(t,u_1(t),u_2(t),\dots,u_n(t),u_1'(t),u_2'(t),\dots,u_n'(t)\Big)dt. \tag{1}$$

To find the extreme value of J, the boundary conditions of the admissible curves are known in the following form:

$$u_i(a)=\gamma_i,\qquad i=1,2,\dots,n, \tag{2}$$

$$u_i(b)=\delta_i,\qquad i=1,2,\dots,n. \tag{3}$$

The necessary condition for $u_i(t)$, $i=1,2,\dots,n$ to extremize $J[u_1(t),u_2(t),\dots,u_n(t)]$ is to satisfy the Euler-Lagrange equations that is obtained by applying the well known procedure in the calculus of variation [5],

$$\frac{\partial G}{\partial u_i}-\frac{d}{dt}\Big(\frac{\partial G}{\partial u_i'}\Big)=0,\qquad i=1,2,\dots,n, \tag{4}$$

subject to the boundary conditions given by Eqs. (2)-(3).

In this paper, we consider the spacial forms of the variational problem (1) as

$$J[u(t)]=\int_a^b G\Big(t,u(t),u'(t)\Big)dt, \tag{5}$$

with boundary conditions

$$u(a)=\gamma,\ \ u(b)=\delta, \tag{6}$$

and

$$J[u_1(t),u_2(t)]=\int_a^b G\Big(t,u_1(t),u_2(t),u_1'(t),u_2'(t)\Big)dt, \tag{7}$$

subject to boundary conditions

$$u_1(a)=\gamma_1,\ \ u_1(b)=\delta_1, \tag{8}$$

$$u_2(a)=\gamma_2,\ \ u_2(b)=\delta_2. \tag{9}$$

Thus, for solving the variational problems (5), we consider the second-order differential equation

$$\frac{\partial G}{\partial u}-\frac{d}{dt}\Big(\frac{\partial G}{\partial u'}\Big)=0, \tag{10}$$

with the boundary condition (6). And also, for solving the variational problems (7), we find the solution of the system of second-order differential equations

$$\frac{\partial G}{\partial u_i}-\frac{d}{dt}\left(\frac{\partial G}{\partial u_i'}\right)=0, \quad i=1,2, \tag{11}$$

with the boundary conditions (8)-(9). Therefore, by applying parametric cubic spline method for the Euler-Lagrange equations (10) and (11), we can obtain an approximate solution to the variational problems (5) and (7).

3. Parametric Cubic spline method

Consider the partition $\Delta=\{t_0,t_1,t_2,\ldots,t_n\}$ of $[a,b]\subset R$. Let $S_k(\Delta)$ denote the set of piecewise polynomials of degree k on subinterval $I_i=[t_i,t_{i+1}]$ of partition Δ. In this work, we consider parametric cubic spline method for finding approximate solution of variational problems.

Consider the grid points t_i on the interval $[a,b]$ as follows:

$$a=t_0<t_1<t_2<\ldots,t_{n-1}<t_n=b, \tag{12}$$

$$t_i=t_0+ih, \quad i=0,1,2,\ldots,n, \tag{13}$$

$$h=\frac{b-a}{n}, \tag{14}$$

where n is a positive integer. Let $S_\Delta(t,\tau)$ be cubic spline function of class $C^2[a,b]$ that interpolates $u(t)$ at the grid points $\{t_i\}_{i=0}^n$. Also, $S_\Delta(t,\tau)$ depends on a parameter $\tau>0$ that is called a parametric spline function also, $S_\Delta(t,\tau)$ reduces to a cubic spline as $\tau\to 0$. By considering parametric cubic spline $S_\Delta(t,\tau)=S_\Delta(t)$, the spline function $S_\Delta(t)$ satisfies in the following equation:

$$S_\Delta''(t)+\tau S_\Delta(t)=\left[S_\Delta''(t_i)+\tau S_\Delta(t_i)\right]\left(\frac{t_{i+1}-t}{h}\right)+\left[S_\Delta''(t_{i+1})+\tau S_\Delta(t_{i+1})\right]\left(\frac{t-t_i}{h}\right), \tag{15}$$

where $t\in[t_i,t_{i+1}]$, $S_\Delta(t_i)=u(t_i)$ and $h=t_{i+1}-t_i$. The Eq.(15) is a inhomogeneous ordinary differential equation. We solve the Eq.(15) and obtain the constants of integration by using interpolation conditions at the endpoints of the interval $[t_i,t_{i+1}]$, then we get the result as follows:

$$S_\Delta(t)=\frac{-h^2}{w^2\sin w}\left[M_{i+1}\sin\left(\frac{w(t-t_i)}{h}\right)+M_i\sin\left(\frac{w(t_{i+1}-t)}{h}\right)\right]$$
$$+\left(\frac{h}{w}\right)^2\left[\left(\frac{t-t_i}{h}\right)\left(M_{i+1}+\left(\frac{w}{h}\right)^2u_{i+1}\right)+\left(\frac{t_{i+1}-t}{h}\right)\left(M_i+\left(\frac{w}{h}\right)^2u_i\right)\right] \tag{16}$$

where $S_\Delta(t_i)=u(t_i)=u_i, S_\Delta''(t_i)=M_i$ and $w=h\sqrt{\tau}$. We use the continuity of first derivative of spline function at t_i, and obtain the following result:

$$h^2\left(\alpha M_{i+1}+2\beta M_i+\alpha M_{i-1}\right)=u_{i+1}-2u_i+u_{i-1}, \quad i=1,2,\ldots,n-1, \tag{17}$$

where

$$\alpha=\frac{1}{w^2}\left(w\csc w-1\right), \quad \beta=\frac{1}{w^2}\left(1-w\cot w\right). \tag{18}$$

4. Numerical examples

In this section, in order to illustrate the performance of the parametric cubic spline method, we present two examples.

Example 1. We first consider the following variational problem with the exact solution $u(t)=e^{3t}$ in [12]:

$$\min J=\int_0^1\left(u(t)+u'(t)-4e^{3t}\right)^2dt, \tag{19}$$

subject to boundary conditions

$$u(0)=1, \quad u(1)=e^3. \tag{20}$$

Considering the Eq. (19), the Euler-Lagrange equation of this problem can be written in the following form:

$$u''(t)-u(t)-8e^{3t}=0. \tag{21}$$

The solution of the second-order differential equation (21) with boundary conditions (20) is approximated by the presented spline method. For our purpose, We consider the boundary value problem (21) in general form as follows:

$$u''(t)=g(t)u(t)+f(t), \tag{22}$$

where $g(t)=1$ and $f(t)=8e^{3t}$. The exact solution of this problem is $u(t)=e^{3t}$. For a numerical solution of the boundary-value problem (22), the interval $[0,1]$ is divided into a set of grid points with step size h. Setting $t=t_i=t_0+ih$, in Eq. (22), we obtain

$$u''(t_i)=g(t_i)u(t_i)+f(t_i), \tag{23}$$

by using the assumption $S''_\Delta(t_i)=M_i$ in (23), we have

$$M_i=g(t_i)u(t_i)+f(t_i). \tag{24}$$

Replacing M_i as Eq. (24) in Eq. (17), for $i=1,2,\ldots,n-1,$ we get

$$h^2\Big(\alpha(g(t_{i-1})u_{i-1}+f(t_{i-1}))+2\beta(g(t_i)u_i+f(t_i))+\alpha(g(t_{i+1})u_{i+1}+f(t_{i+1}))\Big)=u_{i+1}-2u_i+u_{i-1}, \tag{25}$$

or

$$\big(\alpha h^2 g(t_{i-1})-1\big)u_{i-1}+2\big(\beta h^2 g(t_i)+1\big)u_i+\big(\alpha h^2 g(t_{i+1})-1\big)u_{i+1}=-h^2\big(\alpha f(t_{i-1})+2\beta f(t_i)+\alpha f(t_{i+1})\big), \tag{26}$$

where $u_0=1, u_n=e^3$. The linear system (26) consists of $(n-1)$ equations with $(n-1)$ unknowns $u_1,u_2,\ldots,u_{n-1}$. Solving this linear system, we obtain the approximations $u_1,u_2,\ldots,u_{n-1}$ of the solution $u(t)$ at the grid points $t_1,t_2,\ldots,t_{n-1}$. Using Taylor's series for Eq. (26), we can obtain truncation error as follows:

$$t_i=\big(1-2(\alpha+\beta)\big)h^2u''_i+(1-12\alpha)\frac{h^4}{12}u_i^{(4)}+(1-30\alpha)\frac{h^6}{360}u_i^{(6)}+\ldots, \quad i=1,2,\ldots,n-1. \tag{27}$$

In Eq. (27), if $\alpha=\frac{1}{12}$ and $\beta=\frac{5}{12}$, the presented method is a fourth-order convergence method[20].

The errors are reported on the set of uniform grid points $S=\{a=t_0,\ldots,t_1,\ldots,t_n=b\}$

$$t_i=t_0+ih, \quad i=0,1,2,\ldots,n, \; h=\frac{b-a}{n}. \tag{28}$$

The maximum error on the uniform grid points S is

$$\| E_u(h)\|_\infty=\max_{0\le j\le n}\left|u(t_j)-u_n(t_j)\right|, \tag{29}$$

where $u(t_j)$ is the exact solution of the given example, and u_j is the computed solution by the parametric cubic spline method. The maximum absolute errors in numerical solution of the Example 1 are tabulated in table 1. These results show the efficiency and applicability of the presented method.

n	h	$\Vert E_u(h)\Vert_\infty$
5	0.200	3.10326×10^{-3}
10	0.100	1.96017×10^{-4}
20	0.050	1.22896×10^{-5}
30	0.033	2.43458×10^{-6}
40	0.025	7.70612×10^{-7}

Table 1. Results for example 1.

Example 2. In this example, consider the following problem of finding the extremals of the functional[11]:

$$J[u_1(t),u_2(t)]=\int_0^{\frac{\pi}{2}}\left(u'^2_1(t)+u'^2_2(t)+2u_1(t)u_2(t)\right)dt, \tag{30}$$

with boundary conditions

$$u_1(0)=0,\ \ u_1(\frac{\pi}{2})=1, \tag{31}$$

$$u_2(0)=0,\ \ u_2(\frac{\pi}{2})=-1, \tag{32}$$

which has the exact solution given by $\left(u_1(t),u_2(t)\right)=\left(\sin(t),-\sin(t)\right)$. For this problem, the corresponding Euler-Lagrange equations are

$$\begin{cases} u''_1(t)-u_2(t)=0, \\ u''_2(t)-u_1(t)=0, \end{cases} \tag{33}$$

with boundary conditions (31) and (32). In a similar manner and applying (15) and (16), we assume that functions $u_1(t)$ and $u_2(t)$ defined over the interval $[0,\frac{\pi}{2}]$ are approximated by

$$\begin{aligned} u_1(t)\cong S_{1\Delta}(t)&=\frac{-h^2}{w^2\sin w}\left[M_{1,i+1}\sin\left(\frac{w(t-t_i)}{h}\right)+M_{1,i}\sin\left(\frac{w(t_{i+1}-t)}{h}\right)\right] \\ &+\left(\frac{h}{w}\right)^2\left[\left(\frac{t-t_i}{h}\right)\left(M_{1,i+1}+\left(\frac{w}{h}\right)^2u_{1,i+1}\right)+\left(\frac{t_{i+1}-t}{h}\right)\left(M_{1,i}+\left(\frac{w}{h}\right)^2u_{1,i}\right)\right], \end{aligned} \tag{34}$$

$$\begin{aligned} u_2(t)\cong S_{2\Delta}(t)&=\frac{-h^2}{w^2\sin w}\left[M_{2,i+1}\sin\left(\frac{w(t-t_i)}{h}\right)+M_{2,i}\sin\left(\frac{w(t_{i+1}-t)}{h}\right)\right] \\ &+\left(\frac{h}{w}\right)^2\left[\left(\frac{t-t_i}{h}\right)\left(M_{2,i+1}+\left(\frac{w}{h}\right)^2u_{2,i+1}\right)+\left(\frac{t_{i+1}-t}{h}\right)\left(M_{2,i}+\left(\frac{w}{h}\right)^2u_{2,i}\right)\right] \end{aligned} \tag{35}$$

where $w=h\sqrt{\tau}$ and

$$S_{j\Delta}(t_i)=u_j(t_i)=u_{j,i},\quad S''_{j\Delta}(t_i)=M_{j,i},\quad j=1,2. \tag{36}$$

Having used the continuity of first derivatives of the spline functions $S_{1\Delta}(t)$ and $S_{2\Delta}(t)$, and substituted $t=t_i$ for $i=1,2,\ldots,n-1$, where t_i are uniform grid points and also, we can obtain the following results:

$$h^2\left(\alpha M_{1,i+1}+2\beta M_{1,i}+\alpha M_{1,i-1}\right)=u_{1,i+1}-2u_{1,i}+u_{1,i-1},\quad i=1,2,\ldots,n-1, \tag{37}$$

$$h^2\left(\alpha M_{2,i+1}+2\beta M_{2,i}+\alpha M_{2,i-1}\right)=u_{2,i+1}-2u_{2,i}+u_{2,i-1}, \quad i=1,2,\ldots,n-1, \tag{38}$$

where α and β are defined as (18). Now, consider the system (33) and substitute $t=t_i$, then we get:

$$u''_{1,i}=u_{2,i}, \quad u''_{2,i}=u_{1,i}. \tag{39}$$

By considering Eq. (39) and assumption (36), we have:

$$M_{1,i}=u_{2,i}, \quad M_{2,i}=u_{1,i}. \tag{40}$$

Now, by using relations (37)-(40) , for $i=1,2,\ldots,n-1$, we can write

$$\begin{cases} h^2\left(\alpha u_{2,i+1}+2\beta u_{2,i}+\alpha u_{2,i-1}\right)=u_{1,i+1}-2u_{1,i}+u_{1,i-1}, \\ h^2\left(\alpha u_{1,i+1}+2\beta u_{1,i}+\alpha u_{1,i-1}\right)=u_{2,i+1}-2u_{2,i}+u_{2,i-1}, \end{cases} \tag{41}$$

The above linear system contains $2(n-1)$ equations with $2(n-1)$ unknown coefficients $u_{j,i}, j=1,2, i=1,\ldots,n-1$. Solving this linear system, we can obtain the approximate solution of the system of second-order boundary value problems (33).

Suppose $\|E_{u_1}(h)\|_\infty$ and $\|E_{u_2}(h)\|_\infty$ be the maximum absolute errors. We solved example 2 for different values of n.The maximum of absolute errors on the uniform grid points (28) are tabulated in Table 2.

n	h	$\|E_{u_1}(h)\|_\infty$	$\|E_{u_2}(h)\|_\infty$
5	0.314159	1.12874×10^{-5}	1.12874×10^{-5}
10	0.157080	7.05084\$$\times10^{-7}$	7.05084\$$\times10^{-7}$
20	0.078540	4.44889×10^{-8}	4.44889\$$\times10^{-8}$
30	0.052360	8.77860×10^{-9}	8.77860×10^{-9}
40	0.039270	2.78005×10^{-9}	2.78005×10^{-9}

Table 2. Results for example 1

5. Conclusion

In this paper parametric cubic spline method employed for finding the extremum of a functional over the specified domain. The main purpose is to find the solution of boundary value problems which arise from the variational problems. The parametric cubic spline method reduce the computation of boundary value problems to some algebraic equations. The proposed scheme is simple and computationally attractive. Applications are demonstrated through illustrative examples

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