**ISSN** 1746-7659, England, UK
**Journal of Information and Computing Science**
Vol. 8, No. 4, 2013, *pp*. 283-290

# Relative Order of Functions of Several Complex Variables Analytic in the Unit Polydisc

Ratan Kumar Dutta

Department of Mathematics, Patulia High School, Patulia, Kolkata-700119, West Bengal, India
E-mail: ratan_3128@yahoo.com

(*Received April 15, 2013, accepted September 23, 2013*)

**Abstract.** Throughout the paper we consider relative order of functions of several complex variables analytic in the unit poly disc with respect to an entire function and after proving several theorems, we show that relative order of analytic function and its partial derivatives are same.

**Keywords:** Analytic function, entire function, relative order, unit polydisc, property (R). **AMS (2010) Mathematics Subject Classification:** 32A15.

## 1. Introduction

A function $f$ analytic in the unit disc $U:\{z:|z|<1\}$, is said to be of finite Nevanlinna order [7] (Juneja and Kapoor 1985) if there exists a number $\mu$ such that Nevanlinna characteristic function $T(r,f)$ of $f$ defined by

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta$$

satisfies

$$T(r,f)=(1-r)^{-\mu}$$

for all $r$ in $0<r_0(\mu)<r<1$.

The greatest lower bound of all such numbers $\mu$ is called Nevanlinna order of $f$. Thus the Nevanlinna order $\rho(f)$ of $f$ is given by

$$\rho(f)=\limsup_{r\to 1}\frac{\log T(r,f)}{-\log(1-r)}.$$

In [1] Banerjee and Dutta introduced the idea of relative order of an entire function which as follows:

**Definition 1.1.** If $f$ be analytic in U and $g$ be entire, then the relative order of $f$ with respect to g, denoted by $\rho_g(f)$ is defined by

$$\rho_g(f)=\inf\{\mu>0:T_f(r)<T_g\left[\left(\frac{1}{1-r}\right)^{\mu}\right]\text{for all } 0<r_0(\mu)<r<1\}.$$

**Note 1.2.** When $g(z)=\exp z$ then the Definition 1.1 coincides with the definition of Nevanlinna order of $f$.

Also in [2] Banerjee and Dutta introduced the idea of relative order of an entire function of two complex variables which as follows:

**Definition 1.3.** Let $f(z_1,z_2)$ be a non-constant analytic function of two complex variables $z_1$ and $z_2$ holomorphic in the closed unit poly disc $P:\{(z_1,z_2):|z_j|\le 1;\ j=1,2\}$ and $g(z_1,z_2)$ be an entire function then relative order of $f$ with respect to $g$ denoted by $\rho_g(f)$ and is defined by

$$\rho_g(f) = \inf\{\mu > 0 : F(r_1, r_2) < G\left(\frac{1}{(1-r_1)^{\mu}}, \frac{1}{(1-r_2)^{\mu}}\right) \text{for all } 0 < r_0(\mu) < r_1, r_2 < 1\}.$$

In a resent paper [3] Dutta introduced the following definition.

**Definition 1.4.** Let $f(z_1, z_2 \ldots\ldots z_n)$ and $g(z_1, z_2 \ldots\ldots z_n)$ be two entire functions of $n$ complex variables $z_1, z_n \ldots\ldots z_n$ with maximum modulus functions $F(r_1, r_2 \ldots\ldots r_n)$ and $G(r_1, r_2 \ldots\ldots r_n)$ respectively then relative order of $f$ with respective to $g$, denoted by $\rho_g(f)$ and is defined by

$$\rho_g(f) = \inf\{\mu > 0 : F(r_1, r_2 \ldots\ldots r_n) < G(r_1^{\mu}, r_2^{\mu} \ldots\ldots r_n^{\mu}) \; for\; r_i \geq R(\mu); i = 1, 2 \ldots\ldots n\}.$$

Also in a paper [4] Dutta introduced the following definition.

**Definition 1.5.** Let $f(z_1, z_2, \ldots\ldots z_n) = \sum_{m_1, m_2, \ldots\ldots m_n = 0}^{\infty} c_{m_1 m_2 \ldots\ldots m_n} z_1^{m_1} z_2^{m_2} \ldots\ldots z_n^{m_n}$ be a function of $n$ complex variables $z_1, z_2, \ldots\ldots z_n$ holomorphic in the unit polydisc

$$P = \{(z_1, z_2, \ldots\ldots z_n) : |z_j| \leq 1; j = 1, 2, \ldots\ldots n\}$$

and

$$F(r_1, r_2, \ldots\ldots r_n) = \max\{|f(z_1, z_2, \ldots\ldots z_n)| : |z_j| \leq r_j; j = 1, 2, \ldots\ldots n\},$$

be its maximum modulus. Then the order $\rho$ and lower order $\lambda$ are defined as

$$\begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{r_1, r_2, \ldots\ldots r_n \to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log F(r_1, r_2, \ldots\ldots r_n)}{-\log(1-r_1)(1-r_2)\ldots\ldots(1-r_n)}.$$

Now we introduce the following definition.

**Definition 1.6.** Let $f(z_1, z_2 \ldots\ldots z_n)$ be a non-constant analytic function of several complex variables $z_1, z_n \ldots\ldots z_n$ holomorphic in the closed unit polydisc

$$P : \{(z_1, z_2 \ldots\ldots z_n) : |z_j| \leq 1; j = 1, 2 \ldots\ldots n\}$$

and $g(z_1, z_2 \ldots\ldots z_n)$ be an entire function then relative order of $f$ with respect to $g$ denoted by $\rho_g(f)$ and defined by

$$\rho_g(f) = \inf\{\mu > 0 : F(r_1, r_2 \ldots\ldots r_n) < G\left(\frac{1}{(1-r_1)^{\mu}}, \frac{1}{(1-r_2)^{\mu}} \ldots\ldots \frac{1}{(1-r_n)^{\mu}}\right)$$
$$\text{for all } 0 < r_0(\mu) < r_1, r_2 \ldots\ldots r_n < 1\}$$

where $G(r_1, r_2 \ldots\ldots r_n) = \max\{|g(z_1, z_2 \ldots\ldots z_n)| : |z_j| = r_j; j = 1, 2 \ldots\ldots n\}$.

**Note 1.7.** When $g(z_1, z_2 \ldots\ldots z_n) = e^{z_1 z_2 \ldots\ldots z_n}$ then Definition 1.6 coincides with the Definition 1.5 and if n=2 then coincide with Definition 1.3.

We require the following definition.

**Definition 1.8.** An entire function $g(z_1, z_2 \ldots\ldots z_n)$ is said to have the property (R) if for any $\sigma > 1, \lambda > 0$ and for all $r_i$ sufficiently close to 1; $i = 1, 2 \ldots\ldots n$,

$$\left[G\left(\frac{1}{(1-r_1)^{\lambda}}, \frac{1}{(1-r_2)^{\lambda}} \ldots\ldots \frac{1}{(1-r_n)^{\lambda}}\right)\right]^2 < G\left(\frac{1}{\left((1-r_1)^{\lambda}\right)^{\sigma}}, \frac{1}{\left((1-r_2)^{\lambda}\right)^{\sigma}} \ldots\ldots \frac{1}{\left((1-r_n)^{\lambda}\right)^{\sigma}}\right).$$

**Note 1.9.** The function $g(z_1, z_2 \ldots\ldots z_n) = e^{z_1 z_2 \ldots\ldots z_n}$ has the property (R) but $g(z_1, z_2 \ldots\ldots z_n) = z_1 z_2 \ldots\ldots z_n$ has not.

Throughout we shall assume that $f, f_1, f_2$ etc, to be functions analytic in P and $g, g_1, g_2$ etc, are non-constant entire functions of several complex variables. We do no explain stander notations and definitions of analytic functions those are available in [5] and [6].

## 2. Lemmas

We require the following lemmas.

**Lemma 2.1.** Let $g(z_1, z_2 .....z_n)$ be an entire function which has the property (R). Then for any positive integer n and for all $\sigma > 1, \lambda > 0$,

$$\left[G\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda} ....... \frac{1}{(1-r_n)^\lambda}\right)\right]^n < G\left(\frac{1}{\left((1-r_1)^\lambda\right)^\sigma}, \frac{1}{\left((1-r_2)^\lambda\right)^\sigma} ...... \frac{1}{\left((1-r_n)^\lambda\right)^\sigma}\right)$$ holds for all

$r_i, 0 < r_i < 1$ sufficiently close to 1; $i = 1, 2.....n.$

The Lemma 2.1 follows from Lemma 2.1 in [3] on replacing $r_i$ by $\frac{1}{(1-r_i)^\lambda}$, where $i = 1, 2.....n.$

**Lemma 2.2.** Let $g(z_1, z_2 .....z_n)$ be an entire and $\alpha > 1, 0 < \beta < \alpha$ then

$$G\left(\frac{\alpha}{(1-r_1)^\lambda}, \frac{\alpha}{(1-r_2)^\lambda} ....... \frac{\alpha}{(1-r_n)^\lambda}\right) > \beta G\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda} ....... \frac{1}{(1-r_n)^\lambda}\right)$$

for all $r_i, 0 < r_i < 1$ sufficiently close to 1; $i = 1, 2.....n.$

The Lemma 2.2 follows from Lemma 2.2 in [3] on replacing $r_i$ by $\frac{1}{(1-r_i)^\lambda}$, where $i = 1, 2.....n.$

## 3. Sum and Product Theorems

**Theorem 3.1.** Let $f_1(z_1, z_2 .....z_n)$ and $f_2(z_1, z_2 .....z_n)$ be analytic in the unit polydisc P having relative order $\rho_g(f_1)$ and $\rho_g(f_2)$ respectively, where $g(z_1, z_2 .....z_n)$ is an entire function having the property (R). Then

(a) $\rho_g(f_1 + f_2) \le \max(\rho_g(f_1), \rho_g(f_2))$ and

(b) $\rho_g(f_1 f_2) \le \max(\rho_g(f_1), \rho_g(f_2))$.

The same inequality holds for the quotient. The equality holds in (a) if $\rho_g(f_1) \ne \rho_g(f_2)$.

***Proof.*** We may assume that $\rho_g(f_1)$ and $\rho_g(f_2)$ both are finite, because if one of them or both are infinite then inequalities are evident.

Let $f = f_1 + f_2$, $\rho_1 = \rho_g(f_1), \rho_2 = \rho_g(f_2)$ and $\rho_1 \le \rho_2$.

For arbitrary $\varepsilon > 0$ and for all $r_i$, $0 < r_i < 1$; $i = 1, 2.....n$, sufficiently close to 1, we have

$$F_1(r_1, r_2 .....r_n) < G\left(\frac{1}{(1-r_1)^{\rho_1+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_1+\varepsilon}} ...... \frac{1}{(1-r_n)^{\rho_1+\varepsilon}}\right)$$

$$\le G\left(\frac{1}{(1-r_1)^{\rho_2+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_2+\varepsilon}} ...... \frac{1}{(1-r_n)^{\rho_2+\varepsilon}}\right)$$

and

$$F_2(r_1, r_2 .....r_n) < G\left(\frac{1}{(1-r_1)^{\rho_2+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_2+\varepsilon}} ...... \frac{1}{(1-r_n)^{\rho_2+\varepsilon}}\right).$$

Now for all $r_i$, $0 < r_i < 1$; $i = 1, 2.....n$ sufficiently close to 1,

$$F(r_1, r_2 \ldots\ldots r_n) = F_1(r_1, r_2 \ldots\ldots r_n) + F_2(r_1, r_2 \ldots\ldots r_n)$$
$$\le 2G\left(\frac{1}{(1-r_1)^{\rho_2+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_2+\varepsilon}} \ldots\ldots \frac{1}{(1-r_n)^{\rho_2+\varepsilon}}\right)$$
$$\le G\left(\frac{3}{(1-r_1)^{\rho_2+\varepsilon}}, \frac{3}{(1-r_2)^{\rho_2+\varepsilon}} \ldots\ldots \frac{3}{(1-r_n)^{\rho_2+\varepsilon}}\right) by\, Lemma\, 2.2$$
$$\le G\left(\frac{1}{(1-r_1)^{\rho_2+3\varepsilon}}, \frac{1}{(1-r_2)^{\rho_2+3\varepsilon}} \ldots\ldots \frac{1}{(1-r_n)^{\rho_2+3\varepsilon}}\right).$$

Therefore $\rho \le \rho_2 - 3\varepsilon$.

Since $\varepsilon > 0$ is arbitrary, so $\rho \le \rho_2$.

Therefore

$$\rho_g(f_1 + f_2) \le \rho_2 = \max(\rho_g(f_1), \rho_g(f_2))$$

which proves (a).

Next let $\rho_1 < \rho_2$ and suppose $\rho_1 < \mu < \lambda < \rho_2$.

Then for all $r_i, 0 < r_i < 1;\ i = 1, 2 \ldots\ldots n$, sufficiently close to 1, we have

$$F_1(r_1, r_2 \ldots\ldots r_n) < G\left(\frac{1}{(1-r_1)^{\mu}}, \frac{1}{(1-r_2)^{\mu}} \ldots\ldots \frac{1}{(1-r_n)^{\mu}}\right) \qquad (1)$$

and there exist non-decreasing sequence $\{r_{ik}\};\ r_{ik} \to 1\_ \quad as\, k \to \infty$ such that

$$F_2(r_1, r_2 \ldots\ldots r_n) > G\left(\frac{1}{(1-r_{1k})^{\lambda}}, \frac{1}{(1-r_{2k})^{\lambda}} \ldots\ldots \frac{1}{(1-r_{nk})^{\lambda}}\right) \qquad (2)$$

for $k = 1, 2, \ldots\ldots$

We see that

$$G\left(\frac{1}{(1-r_1)^{\lambda}}, \frac{1}{(1-r_2)^{\lambda}} \ldots\ldots \frac{1}{(1-r_n)^{\lambda}}\right) > 2G\left(\frac{1}{(1-r_1)^{\mu}}, \frac{1}{(1-r_2)^{\mu}} \ldots\ldots \frac{1}{(1-r_n)^{\mu}}\right) \qquad (3)$$

for all $r_i, 0 < r_i < 1;\ i = 1, 2 \ldots\ldots n.$, sufficiently close to 1.

From (1), (2) and (3) we get

$$F_2(r_{1k}, r_{2k} \ldots\ldots r_{nk}) > 2F_1(r_{1k}, r_{2k} \ldots\ldots r_{nk})$$

for $k = 1, 2, \ldots\ldots$

Therefore

$$F(r_{1k}, r_{2k} \ldots\ldots r_{nk}) \ge F_2(r_{1k}, r_{2k} \ldots\ldots r_{nk}) - F_1(r_{1k}, r_{2k} \ldots\ldots r_{nk})$$
$$> \frac{1}{2} F_2(r_{1k}, r_{2k} \ldots\ldots r_{nk})$$
$$> \frac{1}{2} G\left(\frac{1}{(1-r_{1k})^{\lambda}}, \frac{1}{(1-r_{2k})^{\lambda}} \ldots\ldots \frac{1}{(1-r_{nk})^{\lambda}}\right) from\,(2)$$
$$> G\left(\frac{1}{3(1-r_{1k})^{\lambda}}, \frac{1}{3(1-r_{2k})^{\lambda}} \ldots\ldots \frac{1}{3(1-r_{nk})^{\lambda}}\right) \text{ for all large } k \text{ and by Lemma 2.2}$$
$$> G\left(\frac{1}{(1-r_{1k})^{\lambda-\varepsilon}}, \frac{1}{(1-r_{2k})^{\lambda-\varepsilon}} \ldots\ldots \frac{1}{(1-r_{nk})^{\lambda-\varepsilon}}\right)$$

where $\varepsilon > 0$ is arbitrary.

This gives $\rho \ge \lambda - \varepsilon$ and since $\rho_1 < \mu < \lambda < \rho_2$ and $\varepsilon > 0$ is arbitrary, we get $\rho \ge \rho_2$.

Therefore

$$\rho_g(f_1 + f_2) = \rho_2 = \max(\rho_g(f_1), \rho_g(f_2)).$$

For (b), we consider $f = f_1.f_2, \rho = \rho_g(f)$ and $\rho_1 \le \rho_2$.
Then for any arbitrary $\varepsilon > 0$,

$$F(r_1, r_2 .....r_n) = F_1(r_1, r_2 .....r_n).F_2(r_1, r_2 .....r_n)$$
$$\le \left[ G\left( \frac{1}{(1-r_1)^{\rho_2+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_2+\varepsilon}} ...... \frac{1}{(1-r_n)^{\rho_2+\varepsilon}} \right) \right]^2$$
$$\le G\left( \frac{1}{(1-r_1)^{\sigma(\rho_2+\varepsilon)}}, \frac{1}{(1-r_2)^{\sigma(\rho_2+\varepsilon)}} ...... \frac{1}{(1-r_n)^{\sigma(\rho_2+\varepsilon)}} \right) by\ Lemma\ 2.1,$$

for every $\sigma > 1$.
So

$$\rho \le \sigma(\rho_2 + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we obtain by letting $\sigma \to 1_+$,

$$\rho \le \rho_2.$$

Therefore

$$\rho_g(f_1.f_2) \le \max(\rho_g(f_1), \rho_g(f_2))$$

This proves the theorem.

## 4. Asymptotic Behavior

**Definition 4.1.** Two entire functions $g_1$ and $g_2$ are said to be asymptotic equivalent in the unit polydisc P if there exists $l$, $0 < l < \infty$ such that

$$\frac{G_1\left( \frac{1}{(1-r_1)^{\lambda}}, \frac{1}{(1-r_2)^{\lambda}} ...... \frac{1}{(1-r_n)^{\lambda}} \right)}{G_2\left( \frac{1}{(1-r_1)^{\lambda}}, \frac{1}{(1-r_2)^{\lambda}} ...... \frac{1}{(1-r_n)^{\lambda}} \right)} \to l \text{ as } r_i \to 1_-; i = 1, 2.....n,$$

where $\lambda > 0$ is any number and in this case we write $g_1 \sim g_2$.

**Note 4.2.** If $g_1 \sim g_2$ then clearly $g_2 \sim g_1$.

**Theorem 4.3.** Let $g_1$ and $g_2$ be entire functions having property (R) and $g_1 \sim g_2$ then $\rho_{g_1}(f) = \rho_{g_2}(f)$, where $f$ is analytic in P.

***Proof.*** Let $\varepsilon > 0$ any arbitrary number and for $r_i$, $0 < r_i < 1;\ i = 1, 2.....n.$, sufficiently close to 1, we have

$$G_1\left( \frac{1}{(1-r_1)^{\lambda}}, \frac{1}{(1-r_2)^{\lambda}} ...... \frac{1}{(1-r_n)^{\lambda}} \right) \le (l+\varepsilon) G_2\left( \frac{1}{(1-r_1)^{\lambda}}, \frac{1}{(1-r_2)^{\lambda}} ...... \frac{1}{(1-r_n)^{\lambda}} \right)$$
$$\le G_2\left( \frac{\alpha}{(1-r_1)^{\lambda}}, \frac{\alpha}{(1-r_2)^{\lambda}} ...... \frac{\alpha}{(1-r_n)^{\lambda}} \right)$$

where $\lambda > 0$ and $\alpha > 1$ is such that $l + \varepsilon < \alpha$.
Next let $\rho_{g_1}(f) = \rho_1$ and $\rho_{g_2}(f) = \rho_2$.

Then

$$F\left(r_1, r_2 \ldots\ldots r_n\right) < G_1\left(\frac{1}{(1-r_1)^{\rho_1+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_1+\varepsilon}} \ldots\ldots \frac{1}{(1-r_n)^{\rho_1+\varepsilon}}\right)$$
$$\leq G_2\left(\frac{\alpha}{(1-r_1)^{\rho_1+\varepsilon}}, \frac{\alpha}{(1-r_2)^{\rho_1+\varepsilon}} \ldots\ldots \frac{\alpha}{(1-r_n)^{\rho_1+\varepsilon}}\right)$$
$$\leq G_2\left(\frac{1}{(1-r_1)^{\rho_1+2\varepsilon}}, \frac{1}{(1-r_2)^{\rho_1+2\varepsilon}} \ldots\ldots \frac{1}{(1-r_n)^{\rho_1+2\varepsilon}}\right).$$

Therefore

$$\rho_2 \leq \rho_1 + 2\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, so $\rho_2 \leq \rho_1$.
Therefore

$$\rho_{g_2}(f) \leq \rho_{g_1}(f) .$$

Also from $g_2 \sim g_1$, we obtain $\rho_{g_1}(f) \leq \rho_{g_2}(f)$.
This proves the theorem.

**Note 4.4.** The converse of the above theorem is not always true.

**Example 4.5.** Consider the functions $g_1(z_1, z_2 \ldots\ldots z_n) = e^{z_1 z_2 \ldots\ldots z_n}$, $g_2(z_1, z_2 \ldots\ldots z_n) = e^{2 z_1 z_2 \ldots\ldots z_n}$ and $f(z_1, z_2 \ldots\ldots z_n) = e^{z_1 z_2 \ldots\ldots z_n}$ then $g_1$ is not asymptotic equivalent to $g_2$ but $\rho_{g_1}(f) = \rho_{g_2}(f)$.

## 5. Relative Order of the Partial Derivatives

**Theorem 5.1:** If $f$ is analytic in the unit polydisc P and g be transcendental entire having the property (R), then $\rho_g\left(\frac{\partial f}{\partial z_1}\right) = \rho_g(f)$.

To prove the theorem we require the following lemma.
**Lemma 5.2.** Let $f(z_1, z_2 \ldots\ldots z_n)$ be a transcendental entire function then

$$\frac{F\left(r_1, r_2 \ldots\ldots r_n\right)}{r_1} \leq \overline{F}\left(r_1, r_2 \ldots\ldots r_n\right) \leq \frac{F\left(2r_1, r_2 \ldots\ldots r_n\right)}{r_1}$$

where

$$\overline{F}\left(r_1, r_2 \ldots\ldots r_n\right) = \max_{|z_i|=r_i;\, i=1,2\ldots\ldots n} \left|\frac{\partial f\left(z_1, z_2 \ldots\ldots z_n\right)}{\partial z_1}\right|.$$

***Proof.*** Let $(z_1', z_2' \ldots\ldots z_n')$ be such that

$$|f(z_1', z_2' \ldots\ldots z_n')| = \max\{|f(z_1, z_2 \ldots\ldots z_n)| : |z_j| = r_j;\, j = 1, 2 \ldots\ldots n\}.$$

With out loss of generality we may assume that $f(0, z'_2 \ldots\ldots z'_n) = 0$. Otherwise we set

$$h(z_1, z_2 \ldots\ldots z_n) = z_1 f(z_1, z_2 \ldots\ldots z_n) .$$

Then $h(0, z_2' \ldots\ldots z_n') = 0$ and $\rho_g(f) = \rho_g(h)$.

We may write for fixed $z_i$ $on$ $|z| = r_i;\ i = 2 \ldots\ldots n$

$$f(z_1, z_2 \ldots\ldots z_n) = \int_0^{z_1} \frac{\partial f(t, z_2 \ldots\ldots z_n)}{\partial t}\, dt,$$

where the line of integration is the segment from $z=0$ to $z=re^{i\theta_0}, r>0.$
Now

$$
\begin{aligned}
F(r_1, r_2 \ldots\ldots r_n) &= | f(z_1', z_2' \ldots\ldots z_n') | \\
&= \left| \int_0^{z_1'} \frac{\partial f(t, z_2' \ldots\ldots z_n')}{\partial t} dt \right| \\
&\le r_1 \max_{|z_1|=r_1} \left| \frac{\partial f(z_1, z_2' \ldots\ldots z_n')}{\partial z_1} \right| \\
&= r_1 \overline{F}(r_1, r_2 \ldots\ldots r_n). \qquad (4)
\end{aligned}
$$

Let $(z_1'', z_2'' \ldots\ldots z_n'')$ be such that

$$\left| \frac{\partial f(z_1'', z_2'' \ldots\ldots z_n'')}{\partial z_1} \right| = \max_{|z_i|=r_i\ ; i=1,2,\ldots\ldots n} \left| \frac{\partial f(z_1, z_2 \ldots\ldots z_n)}{\partial z_1} \right|.$$

Let C denote the circle $| t - z_1'' | = r_1$.
So,

$$
\begin{aligned}
\overline{F}(r_1, r_2 \ldots\ldots r_n) &= \max_{|z_i|=r_i\ ; i=1,2,\ldots\ldots n} \left| \frac{\partial f(z_1, z_2 \ldots\ldots z_n)}{\partial z_1} \right| \\
&= \left| \frac{\partial f(z_1'', z_2'' \ldots\ldots z_n'')}{\partial z_1} \right| \\
&= \left| \frac{1}{2\pi i} \oint_C \frac{f(t, z_2'' \ldots\ldots z_n'')}{(t - z_1'')^2} dt \right| \\
&\le \frac{1}{2\pi} \frac{F(2r_1, r_2 \ldots\ldots r_n)}{r_1^2} 2\pi r_1 \\
&= \frac{F(2r_1, r_2 \ldots\ldots r_n)}{r_1}. \qquad (5)
\end{aligned}
$$

From (4) and (5) we obtain

$$\frac{F(r_1, r_2 \ldots\ldots r_n)}{r_1} \le \overline{F}(r_1, r_2 \ldots\ldots r_n) \le \frac{F(2r_1, r_2 \ldots\ldots r_n)}{r_1}.$$

This proves the lemma.

***Proof of the theorem 5.1:*** Let us consider any arbitrary $\varepsilon > 0$ then from definition of $\rho_g\left(\frac{\partial f}{\partial z_1}\right)$, we have for all $r_i, 0 < r_i < 1;\ i = 1, 2 \ldots\ldots n$ sufficiently close to1,

$$\overline{F}(r_1, r_2 \ldots\ldots r_n) \le G\left( \frac{1}{(1-r_1)^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\varepsilon}} \ldots\ldots \frac{1}{(1-r_n)^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\varepsilon}} \right).$$

Now by Lemma 5.2

$$F(r_1, r_2 .....r_n) \le r_1 \overline{F}(r_1, r_2 .....r_n)$$

$$\le \left[ G\left( \frac{1}{(1-r_1)^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\varepsilon}} ...... \frac{1}{(1-r_n)^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\varepsilon}} \right) \right]^2$$

$$\le G\left( \frac{1}{(1-r_1)^{\sigma\left(\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\varepsilon\right)}}, \frac{1}{(1-r_2)^{\sigma\left(\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\varepsilon\right)}} ...... \frac{1}{(1-r_n)^{\sigma\left(\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\varepsilon\right)}} \right)$$

by Lemma 2.1 and for any $\sigma > 1$, since $g$ has the property (R).

So,

$$\rho_g(f) \le \sigma\left(\rho_g\left(\frac{\partial f}{\partial z_1}\right) + \varepsilon\right).$$

Letting $\sigma \to 1_+$, and since $\varepsilon$ >0 is arbitrary

$$\rho_g(f) \le \rho_g\left(\frac{\partial f}{\partial z_1}\right).$$

Using (5) we obtain similarly

$$\rho_g\left(\frac{\partial f}{\partial z_1}\right) \le \rho_g(f).$$

So,

$$\rho_g\left(\frac{\partial f}{\partial z_1}\right) = \rho_g(f).$$

This proves the theorem.

**Note 5.3.** Similar result hold for other partial derivatives.

## 6. References

[1] D. Banerjee and R. K. Dutta. Relative order of functions analytic in the unit disc, *Bull. Cal. Math. Soc.*, 2009, **101(1)**: 95-104.

[2] D. Banerjee and R. K. Dutta. Relative order of functions of two complex variables analytic in the unit disc. *Journal of Mathematics*, 2008, **1**: 37-44.

[3] R. K. Dutta. Relative order of entire functions of several complex variables *Matematiqke Vesnik*, 2013, **65(2):** 222–233.

[4] R. K. Dutta. On order of a function of several complex variables analytic in the unit polydisc. *Journal of Information and Computing Science*, 2011, **6(2):** 97-108.

[5] B. A. Fuks. Theory of analytic functions of several complex variables. Moscow, 1963.

[6] W. K. Hayman. *Meromorphic functions*. The Clarendon Press, Oxford, 1964.

[7] O. P. Juneja and G. P. Kapoor. *Analytic functions -growth aspects.* Pitman Advanced Publishing Program, 1985.

.