

An Improvement on the Hardy-Hilbert Integral Type Inequality

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(Received January 23, 2013, accepted August 22, 2013)

Abstract. In this paper, a new Hardy-Hilbert integral inequality is obtained by proving an inequality on weight coefficients.

Keywords: Hardy-Hilbert inequality, Hölder inequality, weight coefficient.

1. Introduction

In this paper, we study the famous Hardy Hilbert inequality. $f \in L^p$ denotes a measurable function f satisfying $\int_0^{+\infty} |f(x)|^p dx < \infty$, where $p > 0$. If $f \in L^p$, $g \in L^q$, $f \geq 0$, $g \geq 0$, $p^{-1} + q^{-1} = 1$, $p > 1$, $q > 1$, then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y+1} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x) dx \right)^{1/p} \left(\int_0^{+\infty} g^q(y) dy \right)^{1/q}. \quad (1)$$

Inequality (1) is called the Hardy Hilbert integral inequality, that is, as follows:

$$\int_0^{+\infty} \left(\int_0^{+\infty} \frac{f(x)}{x+y+1} dx \right)^p dy < \left(\frac{\pi}{\sin(\pi/p)} \right)^p \int_0^{+\infty} f^p(x) dx. \quad (2)$$

In 2003, Yang[1] obtained the following result.

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y+1} dx dy < \left(\int_0^{+\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{1}{13(x+1)(2x+1)^{1/p}} \right] f^p(x) dx \right)^{1/p} \left(\int_0^{+\infty} \left[\frac{\pi}{\sin(\pi/q)} - \frac{1}{13(y+1)(2y+1)^{1/q}} \right] g^q(y) dy \right)^{1/q}. \quad (3)$$

In the special case $p = q = 2$,

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y+1} dx dy < \left(\int_0^{+\infty} \left[\pi - \frac{1}{13(x+1)(2x+1)^{1/2}} \right] f^2(x) dx \right)^{1/2} \left(\int_0^{+\infty} \left[\pi - \frac{1}{13(y+1)(2y+1)^{1/2}} \right] g^2(y) dy \right)^{1/2}. \quad (4)$$

In the paper, we obtain an improvement form of the above by means of the Hölder inequality and related lemmas.

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2. New Results and their Proofs

2.1. Some lemmas

Lemma 1 Let $r > 1$, $y > 0$, then we have the following inequality on weight coefficients:

$$\omega(y, r) \stackrel{\text{def}}{=} \int_0^{+\infty} \frac{1}{x+y+1} \left(\frac{2y+1}{2x+1} \right)^{1/r} dx < \frac{\pi}{\sin(\pi/r)} - \frac{2}{3(r-1)(2y+1)^{1-1/r}}. \quad (5)$$

Proof: Let $f_y(x) = \frac{1}{x+y+1} \left(\frac{2y+1}{2x+1} \right)^{1/r}$, then

$$\int_0^{+\infty} f_y(x) dx = \int_{\frac{1}{2y+1}}^{+\infty} (1+u)^{-1} u^{-1/r} du = \frac{\pi}{\sin(\pi/r)} - \int_0^{\frac{1}{2y+1}} (1+u)^{-1} u^{-1/r} du,$$

$$\int_0^{\frac{1}{2y+1}} (1+u)^{-1} u^{-1/r} du > \frac{1}{(2y+1)^{1-1/r}} \left[\frac{r}{r-1} \frac{2y+1}{2(y+1)} + \frac{r^2}{(r-1)(2r-1)} \frac{2y+1}{4(y+1)^2} \right],$$

$$\omega(y, r) < \frac{\pi}{\sin(\pi/r)} - \frac{1}{(2y+1)^{1-1/r}} \left[\left(\frac{r}{r-1} - 1 - \frac{1}{3r} \right) \frac{2y+1}{2(y+1)} + \frac{r^2}{(r-1)(2r-1)} \frac{2y+1}{4(y+1)^2} - \frac{2y+1}{12(y+1)^2} \right],$$

where,

$$\begin{aligned} \left(\frac{r}{r-1} - 1 - \frac{1}{3r} \right) \frac{2y+1}{2(y+1)} &= \frac{2}{3(r-1)} + \frac{2y+1}{6r(r-1)(y+1)} - \frac{1}{3(r-1)(y+1)}, \\ \frac{r^2}{(r-1)(2r-1)} \frac{2y+1}{4(y+1)^2} &> \frac{1}{4(y+1)} + \frac{3}{8(r-1)(y+1)} - \frac{1}{8(y+1)^2} - \frac{3}{16(r-1)(y+1)^2}, \\ \frac{2y+1}{12(y+1)^2} &= \frac{1}{6(y+1)} - \frac{1}{12(y+1)^2}, \end{aligned}$$

Hence,

$$\omega(y, r) < \frac{\pi}{\sin(\pi/r)} - \frac{1}{(2y+1)^{1-1/r}} \left[\frac{2}{3(r-1)} + \frac{2y+1}{6r(r-1)(y+1)} + \frac{2r-1}{24(r-1)(y+1)} - \frac{2r+7}{48(r-1)(y+1)^2} \right],$$

by computation, we obtain

$$\frac{2y+1}{6r(r-1)(y+1)} + \frac{2r-1}{24(r-1)(y+1)} - \frac{2r+7}{48(r-1)(y+1)^2} > \frac{6r^2 - 11r + 48}{48r(r-1)(y+1)^2} > 0.$$

So we complete the proof of lemma 1.

Lemma 2 (Hölder inequality)[2] If $f \in L^p(E)$, $g \in L^q(E)$, E is measurable set, $p > 1$, $p^{-1} + q^{-1} = 1$, then $fg \in L^1(E)$ and

$$\int_E |f(x)g(x)| dx \leq \left(\int_E |f(x)|^p dx \right)^{1/p} \left(\int_E |g(x)|^q dx \right)^{1/q}.$$

2.2. Main results

Theorem 1 If $f \in L^p$, $g \in L^p$, $f \geq 0$, $g \geq 0$, $p^{-1} + q^{-1} = 1$, $p > 1$, $q > 1$, then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y+1} dx dy$$

$$< \left(\int_0^{+\infty} \left[\frac{\pi}{\sin(\pi/q)} - \frac{2}{3(q-1)(2x+1)^{1/q}} \right] f^p(x) dx \right)^{1/p} \left(\int_0^{+\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{2}{3(p-1)(2y+1)^{1/p}} \right] g^q(y) dy \right)^{1/q}. \quad (6)$$

Proof: By lemma 2, we get

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y+1} dx dy \\ & < \left(\int_0^{+\infty} \int_0^{+\infty} \left[\left(\frac{2x+1}{2y+1} \right)^{\frac{1}{pq}} \frac{f(x)}{(x+y+1)^{1/p}} \right]^p dx dy \right)^{1/p} \left(\int_0^{+\infty} \int_0^{+\infty} \left[\left(\frac{2y+1}{2x+1} \right)^{\frac{1}{pq}} \frac{g(y)}{(x+y+1)^{1/q}} \right]^q dx dy \right)^{1/q} \\ & = \left(\int_0^{+\infty} \int_0^{+\infty} \left[\left(\frac{2x+1}{2y+1} \right)^{\frac{1}{q}} \frac{1}{(x+y+1)} \right] dy f^p(x) dx \right)^{1/p} \left(\int_0^{+\infty} \int_0^{+\infty} \left[\left(\frac{2y+1}{2x+1} \right)^{\frac{1}{p}} \frac{1}{(x+y+1)} \right] dx g^q(y) dy \right)^{1/q} \\ & = \left(\int_0^{+\infty} \omega(x, q) f^p(x) dx \right)^{1/p} \left(\int_0^{+\infty} \omega(y, p) g^q(y) dy \right)^{1/q}. \end{aligned}$$

Reminding that $\frac{\pi}{\sin(\pi/p)} = \frac{\pi}{\sin(\pi/q)}$ holds for $p^{-1} + q^{-1} = 1$, and by lemma 1, we complete the proof of theorem 1.

Theorem 2 If $f \in L^p$, $f \geq 0$, $p^{-1} + q^{-1} = 1$, $p > 1$, $q > 1$, then we have

$$\int_0^{+\infty} \left(\int_0^{+\infty} \frac{f(x)}{x+y+1} dx \right)^p dy < \left(\frac{\pi}{\sin(\pi/p)} \right)^{p-1} \int_0^{+\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{2}{3(q-1)(2x+1)^{1/p}} \right] f^p(x) dx. \quad (7)$$

Proof: By lemma 2, we get

$$\begin{aligned} & \int_0^{+\infty} \frac{f(x)}{x+y+1} dx \leq \left(\int_0^{+\infty} \frac{f^p(x)}{x+y+1} \left(\frac{2x+1}{2y+1} \right)^{1/q} dx \right)^{1/p} \left(\int_0^{+\infty} \frac{1}{x+y+1} \left(\frac{2y+1}{2x+1} \right)^{1/p} dx \right)^{1/q} \\ & = \left(\int_0^{+\infty} \frac{f^p(x)}{x+y+1} \left(\frac{2x+1}{2y+1} \right)^{1/q} dx \right)^{1/p} \omega^q(y, p), \end{aligned}$$

where $\omega(y, p) < \frac{\pi}{\sin(\pi/p)}$, by lemma 1.

Thus we get

$$\begin{aligned} & \int_0^{+\infty} \left(\int_0^{+\infty} \frac{f(x)}{x+y+1} dx \right)^p dy \leq \int_0^{+\infty} \int_0^{+\infty} \frac{f^p(x)}{x+y+1} \left(\frac{2x+1}{2y+1} \right)^{1/q} \omega^q(y, p) dx dy \\ & < \int_0^{+\infty} \int_0^{+\infty} \frac{f^p(x)}{x+y+1} \left(\frac{2x+1}{2y+1} \right)^{1/q} \left[\frac{\pi}{\sin(\pi/p)} \right]^{p/q} dx dy \\ & = \left[\frac{\pi}{\sin(\pi/p)} \right]^{p/q} \int_0^{+\infty} \int_0^{+\infty} \omega(x, q) dy f^p(x) dx, \end{aligned}$$

by lemma 1 again, we complete the proof of theorem 2.

Remark: Taking $p = q = 2$ in (6), we obtain

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y+1} dx dy$$

$$< \left(\int_0^{+\infty} \left[\pi - \frac{2}{3(2x+1)^{1/2}} \right] f^2(x) dx \right)^{1/2} \left(\int_0^{+\infty} \left[\pi - \frac{2}{3(2y+1)^{1/2}} \right] g^2(y) dy \right)^{1/2},$$

which is an improvement of (4).

3. Acknowledgements

This paper was supported in part by a Natural Science Research Program of the Education Department of Jiangsu Province of P.R.China, Approved No. 07KJB110069 and a grant of NUIST, Approved No. 20080290. TaiZhong Zhang is the corresponding author.

4. References

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