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On Stability of a Second-Order System

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Abstract. The stability of a second-order system is studied under diffusion process modeling for damping ratio and natural frequency parameters. The probability of stability is derived and it is described how this probability is computed for various selects of parameters of diffusion processes. Finally, a conclusion section is also given.

Keywords: Brownian motion; Damping ratio; Diffusion process; Natural frequency; Second-order system

1. Introduction

The characteristic equation plays important role for checking the stability of a system. The stability requires all roots of characteristic equation to be in the open left half *s*-plane.
For a system represented as n-order differential equation, the characteristic equation is an algebraic equation of degree *n* which is obtained by replacing simply the solution with *exp{sx}* where *x* is the input process and *s* is an arbitrary complex number. For example, for a classical second order system, the characteristic equation is given by

$$s^2+2\varphi w_n s+w_n^2=0$$

φ and w_n are damping ratio and natural frequency, respectively.

The damping ratio φ is a positive coefficient with no dimension which describes how oscillations in a system decay after a disturbance. In practice, there are four cases related to $\varphi<1, \varphi>1, \varphi=1$ and $\varphi\to 0$ referred as under-damped, over-damped, critically damped and un-damped (see Strømmen Melbøand Higham (2004)). When system differential equation is linear, homogenous with constant coefficients, the damping ratio and natural frequency are independent of time *t*. However, these parameters may depend on *t*, in practical situations. Also, systems with random parameters appear frequently in industry. For deterministic system, parameters of systems determine that they are stable or unstable. For stochastic system, the probability of stability specifies a score for chance of stability.

Stengel and Ray (1991) studied the effects of parameter uncertainty on system stability by considering φ and w_n as normally distributed random variables. In this paper, we consider a stochastic process model for random time-dependent damping ratio φ_t and w_n^t.

. This paper is organized as follows. In the next section, we study the suitable model for φ_t. The stability analysis is proposed in section 3. Some examples are given in section 4. Conclusions are proposed in section 5.

2. Models of φ_t and w_n^t

Here, we propose two stochastic process models for φ_t and w_n^t. To extend work of Stengel and Ray (1991), the first attempt is to assume that

$$B_1^t=\theta\int_0^t \varphi_s ds$$

i.e., $\varphi_t dt=\theta' dB_t^1$, where B_1^t is the standard Brownian motion on [0,1) and $\theta'=\theta^{-1}>0$. For the next step, we may assume that φ_t satisfies in a stochastic differential equation given by

$$d\varphi_s=a(\varphi_t)dt+b(\varphi_t)dB_t^1$$

(see Kloeden and Platen (1999)). In this note, we assume that $ax = \alpha x$ and $b(x) = \sigma$, where $\sigma > 0$ and $-1 < \alpha < 0$. To see the reason of this selection, note that for $-1 < \alpha < 0$ and small σ 's, we make sure that φ_t remains on (0, 1) and the related system is under-damped.

To describe more, we use the Euler discrete sampling scheme to find that φ_k is presented as

$$\varphi_k = (\alpha + 1)\varphi_{k-1} + \sigma Z_k$$

where Z_k is a sequence of iid normally distributed random variables. Note that $\alpha + 1 \in (0,1)$ and for small σ 's, all of φ_k remains in (0, 1). A formal model to this end in the literatures is the Ornstein-Uhlenbeck process defined by

$$d\varphi_t = -\beta\varphi_t dt + \sigma dB_t^1$$

where $\beta \in$ (0, 1). The stability analysis under this choice is studied under this modeling.

Finally, we simply let

$$B_2^t = \vartheta \int_0^t w_n^s ds$$

i.e., $w_n^t dt = \vartheta^{-1} dB_2^t$, and B_2^t is another standard Brownian motion on [0,1) independent of B_t^1

3. Stability analysis

In this section, we study the probability of stability under the modeling assumption of section 2. We first drop the index *t* and after calculating the probability of stability, we add *t* to the results. Let us, first, discuss about the zeros which are

$$-\varphi w_n \pm |w_n| \sqrt{\varphi^2 - 1}$$

First, suppose that $\varphi^2 > 1$, then stability condition is satisfied if $w_n > 0$ Because in this case,

$$-\varphi w_n < |w_n| \sqrt{\varphi^2 - 1} < \varphi w_n$$

Therefore, for this case, the stability is equivalent to $\varphi^2 > 1$ and $w_n > 0$ Next, suppose that $\varphi^2 < 1$, then roots are

$$\varphi w_n \pm |w_n| \sqrt{1 - \varphi^2}\, j, \qquad j = \sqrt{-1}$$

Thus, we should have $-\varphi w_n < 0$ or equivalently $\varphi w_n > 0$

Hence, in this case, the stability is $\varphi^2 < 1$ and $\varphi w_n > 0$. Finally, the probability of stability is

$$p = P(\varphi^2 > 1, w_n > 0) + P(\varphi^2 < 1, \varphi w_n > 0)$$

Example 1. Suppose that φ and w_n are independent come form N(0.7, 1) and N(1, 1) distributions, respectively. Then, using a Monte Carlo simulation study with 10000 repetitions, we have p = 0.7058.

Example 2. Let $\varphi > 0$, then p = P($w_n > 0$). Or, if $w_n > 0$, then

$$p = P(\varphi^2 > 1) + P(\varphi^2 < 1, w_n > 0)$$

In section 2, we assumed that φ and w_n are independent. Therefore,

$$p_t = P(\varphi_t^2 > 1)P(w_n^t > 0) +$$
$$P(0 < \varphi_t < 1)P(w_n^t > 0) +$$
$$P(-1 < \varphi_t < 0)P(w_n^t > 0)$$

The p_t is computed using the models of φ_t and w_n^t. It is advised to plot 1 − p_t against time to display the probability of instability for a given instant *t*.

4. Computations

To compute p_t for, say, $t \in [0,1]$, it is enough to compute the $P(\varphi_t < x_1)$ and $P(w_n^t < x_2)$ for t = 0(0.02)1 using a Monte Carlo method for a given set of hyper-parameters β, σ and *v*. Therefore, in this way, the plot of instability probability is drawn. When $\beta = 0$ (a unit root problem for discrete time case) or $= \varepsilon$, close to zero (a near unit root (perturbation) case), the probability pt is given by

$$p_t = 0.5 - (1 - 2P(w_n^t > 0))\Phi(\frac{-1}{\sigma\sqrt{t}})$$

where $\Phi(\cdot)$ is the distribution function of standard normal law. The last quantity which is calculated here is the probability of under-damping. This is given by

$$q_t = P(\varphi_t < 1)$$

which may be calculated using a Monte Carlo method for given hyper-parameters.

5. Conclusions

A second-order system is analyzed while damping ratio and natural frequency are considered as two independent diffusion process. The probability of stability is derived and it is described how this probability is computed for various selects of parameters of diffusion processes.

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