ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 8, No. 4, 2013, *pp*. 299-315

Growth of Iterated Entire Functions in terms of (p, q)-th Order

Ratan Kumar Dutta

Department of Mathematics, Patulia High School, Patulia, Kolkata-700119, West Bengal, India

(*Received September 30, 2012, accepted August 23, 2013*)

Abstract. In this paper we discuss some growth rates of iterated entire functions improving some earlier results.

Keywords: Entire functions, growth, iteration, order, lower order, (p,q)-th order, lower (p,q)-th order.

1. Introduction, Definitions and Notation

Let $f(z)$ and $g(z)$ be two transcendental entire functions defined in the open complex plane **C**. It is well known [1], {[15], p-67, Th-1.46} that

$$\lim_{r\to\infty}\frac{T(r,fog)}{T(r,f)}=\infty \text{ and } \lim_{r\to\infty}\frac{T(r,fog)}{T(r,g)}=\infty .$$

After this Singh [11], Lahiri [7], Song and Yang [13], Singh and Baloria [12], Lahiri and Sharma [8] and Datta and Biswas [3], [4] proved different results on comparative growth property of composite entire functions. In a resent paper [2] Dutta study some comparative growth of iterated entire functions. In this paper, we investigate the comparative growth of iterated entire functions in terms of its (p,q)-th order. We do not explain the standard notations and definitions of the theory of entire functions as those are available in [5], [14] and [15].

The following definitions are well known.

Definition 1.1. The order ρ_f and lower order λ_f of a meromorphic function $f(z)$ is defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}$$

and

$$\lambda_f=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

If $f(z)$ is entire then

$$\rho_f=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r}$$

and

$$\lambda_f=\liminf_{r\to\infty}\frac{\log\log M(r,f)}{\log r}.$$

Notation 1.2. [10] $\log^{[0]}x=x, \exp^{[0]}=x$ and for positive integer m, $\log^{[m]}x=\log(\log^{[m-1]}x)$, $\exp^{[m]}x=\exp(\exp^{[m-1]}x)$.

Definition 1.3. The p-th order ρ_f^p and lower p-th order λ_f^p of a meromorphic function $f(z)$ is defined as

$$\rho_f^p=\limsup_{r\to\infty}\frac{\log^{[p]}T(r,f)}{\log r}$$

and

$$\lambda_f^p = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log r}.$$

If $f(z)$ is entire then

$$\rho_f^p = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}$$

and

$$\lambda_f^p = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}.$$

Clearly $\rho_f^p \le \rho_f^{p-1}$ and $\lambda_f^p \le \lambda_f^{p-1}$ for all *p* and when *p=1* then p-th order and lower p-th order coincide with classical order and lower order respectively.

Definition 1.4. The (p,q)-th order $\rho_f(p,q)$ and lower (p,q)-th order $\lambda_f(p,q)$ of a meromorphic function $f(z)$ is define as

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}$$

and

$$\lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}.$$

If $f(z)$ is an entire function then

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}$$

and

$$\lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}$$

where $p \ge q \ge 1$.

Clearly $\rho_f(p,1) = \rho_f^p$ *and* $\lambda(p,1) = \lambda_f^p$.

Definition 1.5. Let $f(z)$ be an entire function of finite p- th order ρ_f^p then we define σ_f^p as

$$\sigma_f^p = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{r^{\rho_f^p}}.$$

According to Lahiri and Banerjee [6] if $f(z)$ *and* $g(z)$ are entire functions then the iteration of f with respect to g is defined as follows:

$$f_1(z) = f(z)$$
$$f_2(z) = f(g(z)) = f(g_1(z))$$
$$f_3(z) = f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))$$
$$\cdots\cdots\quad \cdots\cdots\quad \cdots\cdots\quad \cdots\cdots$$
$$f_n(z) = f(g(f \ldots\ldots\ldots(f(z) \text{ or } g(z))\ldots\ldots)),$$

according as *n* is odd or even,

and so

$$g_1(z) = g(z)$$
$$g_2(z) = g(f(z)) = g(f_1(z))$$
$$g_3(z) = g(f_2(z)) = g(f(g(z)))$$
$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$
$$g_n(z) = g(f_{n-1}(z)) = g(f(g_{n-2}(z))).$$

Clearly all $f_n(z)$ *and* $g_n(z)$ *are entire functions.*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [5] Let $f(z)$ be an entire function. For $0 \le r < R < \infty$, we have

$$T(r,f) \le \log^+ M(r,f) \le \frac{R+r}{R-r} T(R,f).$$

Lemma 2.2. [1] If $f(z)$ and $g(z)$ are any two entire functions, for all sufficiently large values of *r*,

$$M\left(\frac{1}{8}M\left(\frac{r}{2},g\right)-|g(0)|,f\right) \le M(r,fog) \le M(M(r,g),f).$$

Lemma 2.3. [9] Let $f(z)$ and $g(z)$ be two entire functions. Then we have

$$T(r,fog) \ge \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4},g\right)+O(1),f\right).$$

Lemma 2.4. Let $f(z)$ and $g(z)$ be two entire functions of non zero finite (p,q)-th order $\rho_f(p,q)$ and $\rho_g(p,q)$ respectively, then for any $\varepsilon > 0$ *and* $p \ge q \ge 1$,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r,f_n) \le \begin{cases} (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r,g)+O(1) & \text{when } n \text{ is even,} \\ (\rho_g(p,q)+\varepsilon)\log^{[q]} M(r,f)+O(1) & \text{when } n \text{ is odd} \end{cases}$$

for all sufficiently large values of *r*.

Proof. First suppose that *n* is even. Then from second part of Lemma 2.2 and Definition of (p,q)-th order, it follows that for all sufficiently large values of *r*,

$$M(r,f_n) \le M(M(r,g_{n-1}),f)$$

$$i.e.,\ \log^{[p+1]} M(r,f_n) \le \log^{[p+1]} M(M(r,g_{n-1}),f)$$
$$\le (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r,g_{n-1}).$$

$$So,\ \log^{[p+2]} M(r,f_n) \le \log^{[q+1]} M(r,g(f_{n-2}))+O(1)$$

$$\log^{[p+2-q]} M(r,f_n) \le \log M(r,g(f_{n-2}))+O(1).$$

Taking repeated logarithms p times, we get

$$\log^{[2p+2-q]} M(r, f_n) \le \log^{[p+1]} M(M(r, f_{n-2}), g) + O(1)$$

$$\le (\rho_g(p,q)+\varepsilon)\log^{[q]} M(r, f_{n-2}) + O(1),$$

$$i.e., \log^{[2p+3-q]} M(r, f_n) \le \log^{[q+1]} M\left(r, f_{n-2}\right) + O(1)$$

$$\log^{[2p+3-2q]} M(r, f_n) \le \log M\left(r, f_{n-2}\right) + O(1).$$

Again taking repeated logarithms p times, we get

$$\log^{[3(p+1)-2q]} M(r, f_n) \le (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r, g_{n-3}) + O(1).$$

Finally, after taking repeated logarithms *(n-4)(p+1)* times more, we have for all sufficiently large values of r,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \le (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r, g) + O(1).$$

Similarly if n is odd then for all sufficiently large values of r,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \le (\rho_g(p,q)+\varepsilon)\log^{[q]} M(r, f) + O(1).$$

This proves the lemma.

Lemma 2.5. Let $f(z)$ and $g(z)$ be two entire functions of non zero finite lower (p,q)-th order $\lambda_f(p,q)$ *and* $\lambda_g(p,q)$ respectively, then for any

$$0 < \varepsilon < \min\{\lambda_f(p,q), \lambda_g(p,q)\} \text{ and } p \ge q \ge 1,$$

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \ge \begin{cases} (\lambda_f(p,q)-\varepsilon)\log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right) + O(1) & \text{when } n \text{ is even} \\ (\lambda_g(p,q)-\varepsilon)\log^{[q]} M\left(\frac{r}{2^{n-1}}, f\right) + O(1) & \text{when } n \text{ is odd,} \end{cases}$$

for all sufficiently large values of r.

Proof. First suppose that n is even. Then from first part of Lemma 2.2 we have for all sufficiently large values of r and for any $0 < \varepsilon < \min\{\lambda_f(p,q), \lambda_g(p,q)\}$,

$$M(r, f_n) = M\left(r, f(g_{n-1})\right)$$

$$\ge M\left(\frac{1}{8} M\left(\frac{r}{2}, g_{n-1}\right) - \left|g_{n-1}(0)\right|, f\right)$$

$$\ge M\left(\frac{1}{16} M\left(\frac{r}{2}, g_{n-1}\right), f\right).$$

$$\therefore \log^{[p+1]} M(r, f_n) \ge \left(\lambda_f(p,q) - \varepsilon\right)\log^{[q]}\left[\frac{1}{16} M\left(\frac{r}{2}, g_{n-1}\right)\right],$$

using the Definition 1.4,

$$i.e.,\ \log^{[p+1]} M(r,f_n) \geq \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]} M\left(\frac{r}{2},g_{n-1}\right)+O(1)$$

$$i.e.,\ \log^{[p+2]} M(r,f_n) \geq \log^{[q+1]} M\left(\frac{r}{2},g(f_{n-2})\right)+O(1)$$

$$i.e.,\ \log^{[p+2-q]} M(r,f_n) \geq \log M\left(\frac{1}{16}M\left(\frac{r}{2^2},f_{n-2}\right),\mathrm{g}\right)+O(1).$$

Taking repeated logarithms *p* times, we get

$$\log^{[2p+2-q]} M(r,f_n) \geq \log^{[p+1]} M\left(\frac{1}{16}M\left(\frac{r}{2^2},f_{n-2}\right),\mathrm{g}\right)+O(1)$$

$$\geq \left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]}\left(\frac{1}{16}M\left(\frac{r}{2^2},f_{n-2}\right)\right)+O(1)$$

$$\log^{[2p+3-2q]} M(r,f_n) \geq \log M\left(\frac{r}{2^2},f_{n-2}\right)+O(1).$$

Again taking repeated logarithms *p* times, we get

$$\log^{[2p+2-2q]} M(r,f_n) \geq \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{16}M\left(\frac{r}{2^3},g_{n-3}\right)\right]+O(1)$$

$$\geq \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]} M\left(\frac{r}{2^3},g_{n-3}\right)+O(1).$$

Finally, after taking repeated logarithms *(n-4)(p+1)* times more, we have for all sufficiently large values of *r*,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r,f_n) \geq \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{16}M\left(\frac{r}{2^{n-1}},g\right)\right]+O(1)$$

$$i.e.,\ \log^{[(n-1)(p+1)-(n-2)q]} M(r,f_n) \geq \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]} M\left(\frac{r}{2^{n-1}},g\right)+O(1).$$

Similarly if *n* is odd then for all sufficiently large values of *r*,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r,f_n) \geq \left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]} M\left(\frac{r}{2^{n-1}},\mathrm{f}\right)+O(1).$$

This proves the lemma.

Lemma 2.6. Let *f(z)* and *g(z)* be two non- constant entire functions, such that $0<\rho_f(p,q)<\infty$ *and* $0<\rho_g(p,q)<\infty$. Then for all sufficiently large *r* and $\varepsilon > 0$,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r,f_n) \leq \begin{cases}\left(\rho_f(p,q)+\varepsilon\right)\log^{[q]} M(r,g)+O(1) & \text{when } n \text{ is even},\\ \left(\rho_g(p,q)+\varepsilon\right)\log^{[q]} M(r,f)+O(1) & \text{when } n \text{ is odd}\end{cases}$$

where $p \geq q \geq 1.$

The lemma follows from Lemma 2.1 and Lemma 2.4.

Lemma 2.7. Let $f(z)$ and $g(z)$ be two entire functions such that $0<\lambda_f(p,q)<\infty$ and $0<\lambda_g(p,q)<\infty$. Then for any $\varepsilon\,(0<\varepsilon<\min\{\lambda_f(p,q),\ \lambda_g(p,q)\})$ and $p\geq q\geq 1$,

$$\log^{[(n-1)p+(n-2)(1-q)]}T(r,f_n)\geq\begin{cases}\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}M\left(\frac{r}{4^{n-1}},g\right)+O(1) & \text{when } n \text{ is even,}\\ \left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]}M\left(\frac{r}{4^{n-1}},f\right)+O(1) & \text{when } n \text{ is odd}\end{cases}$$

for all sufficiently large values of *r*.

Proof. To prove this lemma we first suppose that *n* is even. Then from Lemma 2.1and Lemma 2.3 we get for any $\varepsilon\,(0<\varepsilon<\min\{\lambda_f(p,q),\ \lambda_g(p,q)\})$ and for all sufficiently large values of *r*,

$$T(r,f_n)=T\left(r,f(g_{n-1})\right)$$
$$\geq\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+O(1),f\right).$$
$$\therefore\ \log^{[p]}T(r,f_n)\geq\log^{[p+1]}M\left(\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+O(1),f\right)+O(1)$$
$$\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+O(1)\right]+O(1)$$
$$\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{9}M\left(\frac{r}{4},g_{n-1}\right)\right]+O(1)$$
$$\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}M\left(\frac{r}{4},g_{n-1}\right)+O(1)$$
$$\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q-1]}T\left(\frac{r}{4},g_{n-1}\right)+O(1)$$
$$\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q-1]}\left[\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+O(1),g\right)\right]+O(1),$$

i.e., $\log^{[p+1]}T(r,f_n)\geq\log^{[q+1]}M\left(\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+O(1),g\right)+O(1)$

i.e., $\log^{[p+1-q]}T(r,f_n)\geq\log M\left(\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+O(1),g\right)+O(1)$

i.e., $\log^{[2p+1-q]}T(r,f_n)\geq\log^{[p+1]}M\left(\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+O(1),g\right)+O(1)$

$$\geq\left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+O(1)\right]+O(1)$$
$$\geq\left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{9}M\left(\frac{r}{4^2},f_{n-2}\right)\right]+O(1).$$

i.e., $\log^{[2p+1-q]} T(r, f_n) \geq (\lambda_g(p,q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1)$

....

....

$\therefore \log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4^{n-1}}, g\right) + O(1)$ *when n is even.* Similarly,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \geq (\lambda_g(p,q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4^{n-1}}, f\right) + O(1) \text{ when } n \text{ is odd.}$$

This proves the lemma.

3. Theorems

Theorem 3.1. Let *f* and *g* be two non-constant entire functions of non- zero finite (p,q)-th order and lower (p,q)-th order, also $0 < \sigma_f^q, \sigma_g^q < \infty$. Then

$$(i) \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \leq \frac{4^{\rho_g^q} \rho_f(p,q)}{\lambda_f(p,q)},$$

$$(ii) \quad \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \geq \frac{\lambda_f(p,q)}{\left(2^{n-1}\right)^{\rho_g^q} \rho_f(p,q)}$$

when *n* is even and

$$(iii) \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} \leq \frac{4^{\rho_f^q} \rho_g(p,q)}{\lambda_g(p,q)},$$

$$(iv) \quad \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} \geq \frac{\lambda_g(p,q)}{\left(2^{n-1}\right)^{\rho_f^q} \rho_g(p,q)}$$

when *n* is odd.

Proof. First we suppose that *n* is even, then from Lemma 2.4 and the Definition 1.5 we have for all large *r* and $\varepsilon > 0$,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \leq (\rho_f(p,q) + \varepsilon) \log^{[q]} M(r, g) + O(1)$$

$$\leq (\rho_f(p,q) + \varepsilon)\left(\sigma_g^q + \varepsilon\right) r^{\rho_g^q} + O(1). \qquad (3.1)$$

From Lemma 2.3 we get

$$T(r, f(g)) \geq \frac{1}{3} \log M\left(\frac{1}{16} M\left(\frac{r}{4}, g\right), f\right).$$

Since $\lambda_f(p,q)$ is the lower (p,q)-th order of f so for given $\varepsilon (0 < \varepsilon < \lambda_f(p,q))$ and for all large values of *r*,

$$\log^{[p]} T(r, f(g)) \geq \log^{[p+1]} M\left(\frac{1}{16} M\left(\frac{r}{4}, g\right), f\right) + O(1)$$

$$\therefore \log^{[p]} T(r, f(g)) \geq \left(\lambda_f(p,q) - \varepsilon\right) \log^{[q]} M\left(\frac{r}{4}, g\right) + O(1). \quad (3.2)$$

Again for a sequence of values of *r* tending to infinity,

$$\log^{[q]} M\left(\frac{r}{4}, g\right) > \left(\sigma_g^q - \varepsilon\right)\left(\frac{r}{4}\right)^{\rho_g^q}. \quad (3.3)$$

Therefore from (3.2) and (3.3) we get for a sequence of values of *r* tending to infinity,

$$\log^{[p]} T(r, f(g)) \geq \left(\lambda_f(p,q) - \varepsilon\right)\left(\sigma_g^q - \varepsilon\right)\left(\frac{r}{4}\right)^{\rho_g^q} + O(1) \quad (3.4)$$

where $0 < \varepsilon < \min\{\lambda_f(p,q), \sigma_g^q\}$.

Now from (3.1) and (3.4) we have for a sequence of values of *r* tending to infinity,

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \leq \frac{\left(\rho_f(p,q) + \varepsilon\right)\left(\sigma_g^q + \varepsilon\right) r^{\rho_g^q} + O(1)}{\left(\lambda_f(p,q) - \varepsilon\right)\left(\sigma_g^q - \varepsilon\right)\left(\frac{r}{4}\right)^{\rho_g^q} + O(1)}$$

$$= \frac{\left(\rho_f(p,q) + \varepsilon\right)\left(\sigma_g^q + \varepsilon\right) 4^{\rho_g^q} + o(1)}{\left(\lambda_f(p,q) - \varepsilon\right)\left(\sigma_g^q - \varepsilon\right) + o(1)}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\liminf_{r \to \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \leq \frac{4^{\rho_g^q} \rho_f(p,q)}{\lambda_f(p,q)}.$$

Also when *n* is even then from Lemma 2.5 we get for all sufficiently large values of *r*

$$\log^{[(n-1)(p-1)-(n-2)q]} M(r, f_n) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right) + O(1).$$

Now for a sequence of values of *r* tending to infinity, we have

$$\log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right) > \left(\sigma_g^q - \varepsilon\right)\left(\frac{r}{2^{n-1}}\right)^{\rho_g^q}.$$

Therefore for a sequence of values of *r* tending to infinity, we get

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \geq (\lambda_f(p,q) - \varepsilon)\left(\sigma_g^q - \varepsilon\right)\left(\frac{r}{2^{n-1}}\right)^{\rho_g^q} + O(1) \quad (3.5)$$

where $0 < \varepsilon < \min\{\lambda_f(p,q), \sigma_g^q\}$.

Again by Lemma 2.1 we have for all large values of *r*, and $\varepsilon > 0$,

$$T(r, f(g)) \leq \log M(r, f(g))$$

$$\leq \log M(M(r,g), f).$$

$$\therefore \log^{[p]} T(r, f(g)) \le (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r,g)$$
$$\le (\rho_f(p,q)+\varepsilon)\left(\sigma_g^q+\varepsilon\right) r^{\rho_g^q}. \qquad (3.6)$$

Therefore from (3.5) and (3.6) we have for a sequence of values of r tending to infinity,

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \ge \frac{\left(\lambda_f(p,q)-\varepsilon\right)\left(\sigma_g^q-\varepsilon\right)\left(\frac{r}{2^{n-1}}\right)^{\rho_g^q}+O(1)}{\left(\rho_f(p,q)+\varepsilon\right)\left(\sigma_g^q+\varepsilon\right) r^{\rho_g^q}}$$

$$= \frac{\left(\lambda_f(p,q)-\varepsilon\right)\left(\sigma_g^q-\varepsilon\right)+o(1)}{\left(2^{n-1}\right)^{\rho_g^q}\left(\rho_f(p,q)+\varepsilon\right)\left(\sigma_g^q+\varepsilon\right)}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \ge \frac{\lambda_f(p,q)}{\left(2^{n-1}\right)^{\rho_g^q} \rho_f(p,q)}.$$

Similarly for odd n we get the second part of this theorem.
This proves the theorem.

Remark 3.2 If f is of regular growth i.e. $\rho_f(p, q) = \lambda_f(p, q)$ and n is even then

$$(i) \qquad \liminf_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \le 4^{\rho_g^q},$$

$$(ii) \qquad \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \ge \frac{1}{\left(2^{n-1}\right)^{\rho_g^q}}.$$

Also if g is of regular growth i.e. $\rho_g(p, q) = \lambda_g(p, q)$ and n is odd then

$$(iii) \qquad \liminf_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} \le 4^{\rho_f^q},$$

$$(iv) \qquad \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} \ge \frac{1}{\left(2^{n-1}\right)^{\rho_f^q}}.$$

Remark 3.3. The conditions non zero lower (p,q)-th order and finite (p,q)-th order are necessary for Theorem 3.1, which are shown by the following examples.

Example 3.4. Let $f(z) = \exp^{[p-q+1]} z, g(z) = \exp^{[p-q]} z \text{ and } 2q \ge p+1$. Then $\rho_f(p, q) = \lambda_f(p, q) = 1$ and $\rho_g(p, q) = \lambda_g(p, q) = 0$.

Here $f(g) = \exp^{[2p-2q+1]} z$ and

$$3T(2r, f(g)) \geq \log M(r, f(g)) = \exp^{[2p-2q]} r$$

$$i.e., \; T(r, f(g)) \geq \frac{1}{3} \exp^{[2p-2q]} \frac{r}{2}.$$

$$\therefore \; \log^{[p]} T(r, f(g)) \geq \exp^{[p-2q]} \frac{r}{2} + O(1).$$

Now

$$f_n = \begin{cases} \exp^{\left[np - nq + \frac{n}{2}\right]} z & when\ n\ is\ even \\ \exp^{\left[np - nq + \frac{n+1}{2}\right]} z & when\ n\ is\ odd. \end{cases}$$

So when n is even,

$$M(r, f_n) = \exp^{\left[np - nq + \frac{n}{2}\right]} r$$

$$\therefore \log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) = \log^{[(n-1)(p+1)-(n-2)q]} \exp^{\left[np - nq + \frac{n}{2}\right]} r$$

$$= \exp^{\left[p - 2q - \frac{n}{2} + 1\right]} r.$$

Therefore

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \leq \frac{\exp^{[p-2q-\frac{n}{2}+1]} r}{\exp^{[p-2q]} r} + o(1)$$

$$= \frac{1}{\exp^{\left[\frac{n}{2}-1\right]} r} + o(1) \to 0 \; not\ gater\ then\ 1 \text{ as } r \to \infty.$$

Similarly for odd n,

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T\left(r, g(f)\right)} \leq \frac{\exp^{[p-2q-\frac{n-1}{2}+1]} r}{\exp^{[p-2q]} r} + o(1)$$

$$\to 0 \; not\ gater\ then\ 1 \text{ as } r \to \infty.$$

Example 3.5. Let $f(z) = \exp^{[p-q+1]} z$, $g(z) = \exp^{[p-q+2]} z$ and $2q \geq p+1$. Then $\rho_f(p, q) = \lambda_f(p, q) = 1$ and $\rho_g(p, q) = \lambda_g(p, q) = \infty$.

Here $g(f) = \exp^{[2p-2q+3]} z$ and

$$T(r, g(f)) \leq \log M(r, g(f)) = \exp^{[2p-2q+2]} r$$

$$\therefore \; \log^{[p]} T(r, g(f)) \leq \exp^{[p-2q+2]} r.$$

Now

$$f_n = \begin{cases} \exp^{\left[n\ p-nq+\frac{3n}{2}\right]} z & when\ n\ is\ even \\ \exp^{\left[n p-nq+\frac{3n-1}{2}\right]} z & when\ n\ is\ odd. \end{cases}$$

So when n is even,

$$M(r, f_n) = \exp^{\left[n\ p-nq+\frac{3n}{2}\right]} r$$

$$\text{i.e., } \log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) = \log^{[(n-1)(p+1)-(n-2)q]} \exp^{\left[n\ p-nq+\frac{3n}{2}\right]} r$$

$$= \exp^{\left[p-2q+\frac{n}{2}+1\right]} r.$$

Therefore

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} \geq \frac{\exp^{\left[p-2q+\frac{n}{2}+1\right]} r}{\exp^{[p-2q+2]} r}$$

$$= \exp^{\left[\frac{n}{2}-1\right]} r \to \infty\ not\ leas\ then\ 1 \text{ as } r \to \infty.$$

When n is odd,

$$M(r, f_n) = \exp^{\left[n\ p-nq+\frac{3n-1}{2}\right]} r$$

$$\text{i.e., } \log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) = \log^{[(n-1)(p+1)-(n-2)q]} \exp^{\left[n\ p-nq+\frac{3n-1}{2}\right]} r$$

$$= \exp^{\left[p-2q+\frac{n+1}{2}\right]} r.$$

Therefore

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} \geq \frac{\exp^{\left[p-2q+\frac{n+1}{2}\right]} r}{\exp^{[p-2q+2]} r}$$

$$= \exp^{\left[\frac{n+1}{2}-2\right]} r \to \infty\ not\ leas\ then\ 1 \text{ as } r \to \infty.$$

Theorem 3.6. Let $f\ and\ g$ be two non-constant entire functions of non- zero finite (p,q)-th order and lower (p,q)-th order, also $0 < \sigma_f^q, \sigma_g^q < \infty$. Then

$$(i)\quad \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \le \frac{\left(4^{n-1}\right)^{\rho_g^q} \rho_f(p,q)}{\lambda_f(p,q)},$$

$$(ii)\quad \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \ge \frac{\lambda_f(p,q)}{2^{\rho_g^q} \rho_f(p,q)}$$

when n is even and

$$(iii)\quad \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r, g(f))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \le \frac{\left(4^{n-1}\right)^{\rho_f^q} \rho_g(p,q)}{\lambda_g(p,q)},$$

$$(iv)\quad \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \ge \frac{\lambda_g(p,q)}{2^{\rho_f^q} \rho_g(p,q)}$$

when n is odd.

Proof. when n is even, then from Lemma 2.7, we get for all large values of r and any $\varepsilon(0<\varepsilon<\lambda_f(p,q))$,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \ge \left(\lambda_f(p,q)-\varepsilon\right) \log^{[q]} M\left(\frac{r}{4^{n-1}}, g\right) + O(1). \quad (3.7)$$

Again for a sequence of values of r tending to infinity,

$$\log^{[q]} M\left(\frac{r}{4^{n-1}}, g\right) > \left(\sigma_g^q - \varepsilon\right)\left(\frac{r}{4^{n-1}}\right)^{\rho_g^q}. \qquad (3.8)$$

Therefore from (3.7) and (3.8) we have for a sequence of values of r tending to infinity,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \ge (\lambda_f(p,q)-\varepsilon)\left(\sigma_g^q-\varepsilon\right)\left(\frac{r}{4^{n-1}}\right)^{\rho_g^q} + O(1) \qquad (3.9)$$

where $0<\varepsilon<\min\{\lambda_f(p,q), \sigma_g^q\}$.

Now from second part of Lemma 2.2 we get for large values of r,

$$\begin{aligned}\log^{[p+1]} M(r, f(g)) &\le \log^{[p+1]} M(M(r,g), f) \\ &\le (\rho_f(p,q)+\varepsilon) \log^{[q]} M(r,g) \\ &\le (\rho_f(p,q)+\varepsilon)\left(\sigma_g^q+\varepsilon\right) r^{\rho_g^q}. \qquad (3.10)\end{aligned}$$

Now from (3.9) and (3.10) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \le \frac{(\rho_f(p,q)+\varepsilon)\left(\sigma_g^q+\varepsilon\right) r^{\rho_g^q}}{(\lambda_f(p,q)-\varepsilon)\left(\sigma_g^q-\varepsilon\right)\left(\frac{r}{4^{n-1}}\right)^{\rho_g^q} + O(1)}$$

$$= \frac{(\rho_f(p,q)+\varepsilon)\left(\sigma_g^q+\varepsilon\right)\left(4^{n-1}\right)^{\rho_g^q}}{(\lambda_f(p,q)-\varepsilon)\left(\sigma_g^q-\varepsilon\right)+o(1)}.$$

Since $\varepsilon > 0$ *is arbitrary,*

$$\therefore \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \le \frac{\left(4^{n-1}\right)^{\rho_g^q} \rho_f(p,q)}{\lambda_f(p,q)}.$$

Again for all sufficiently large values of *r*, we get from first part of Lemma 2.2,

$$\log^{[p+1]} M(r, f(g)) \ge \log^{[p+1]} M\left(\frac{1}{16} M\left(\frac{r}{2}, g\right), f\right)$$

$$\ge \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]} M\left(\frac{r}{2}, g\right) + o(1). \qquad (3.11)$$

Also for a sequence of values of *r* tending to infinity,

$$\log^{[q]} M\left(\frac{r}{2}, g\right) > \left(\sigma_g^q-\varepsilon\right)\left(\frac{r}{2}\right)^{\rho_g^q}. \qquad (3.12)$$

Therefore from (3.11) and (3.12) for a sequence of values of *r* tending to infinity,

$$\log^{[p+1]} M(r, f(g)) \ge \left(\lambda_f(p,q)-\varepsilon\right)\left(\sigma_g^q-\varepsilon\right)\left(\frac{r}{2}\right)^{\rho_g^q} + O(1) \qquad (3.13)$$

where $0 < \varepsilon < \min\{\lambda_f(p,q), \sigma_g^q\}$.

Also when *n* is even then from Lemma 2.6 we get for *r* trending to infinity,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \le (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r,g)+O(1)$$

$$\le (\rho_f(p,q)+\varepsilon)\left(\sigma_g^q+\varepsilon\right) r^{\rho_g^q} + O(1). \qquad (3.14)$$

Now from (3.13) and (3.14) for a sequence of values of *r* tending to infinity,

$$\frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \ge \frac{\left(\lambda_f(p,q)-\varepsilon\right)\left(\sigma_g^q-\varepsilon\right)\left(\frac{r}{2}\right)^{\rho_g^q} + O(1)}{\left(\rho_f(p,q)+\varepsilon\right)\left(\sigma_g^q+\varepsilon\right) r^{\rho_g^q} + O(1)}$$

$$= \frac{\left(\lambda_f(p,q)-\varepsilon\right)\left(\sigma_g^q-\varepsilon\right)+o(1)}{2^{\rho_g^q}\left(\rho_f(p,q)+\varepsilon\right)\left(\sigma_g^q+\varepsilon\right)+o(1)}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{\lambda_f(p,q)}{2^{\rho_g^q} \rho_f(p,q)}.$$

Similarly for odd *n* we get the second part of this theorem.
This proves the theorem.

Remark 3.7. If *f* is of regular growth i.e. $\rho_f(p,q) = \lambda_f(p,q)$ and *n* is even then

$$(i) \quad \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq \left(4^{n-1}\right)^{\rho_g^q},$$

$$(ii) \quad \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{1}{2^{\rho_g^q}}.$$

Also if *g* is of regular growth i.e. $\rho_g(p,q) = \lambda_g(p,q)$ and *n* is odd then

$$(iii) \quad \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r, g(f))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq \left(4^{n-1}\right)^{\rho_f^q},$$

$$(iv) \quad \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r, g(f))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{1}{2^{\rho_f^q}}.$$

The next theorem is the generalization of the above theorems.

Theorem 3.8. Let *f* and *g* be two non-constant entire functions of non- zero finite (p,q)-th order and lower (p,q)-th order, also $0 < \sigma_f^q, \sigma_g^q < \infty$. Then

$$(i) \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq \frac{\left(4^{n-1}\right)^{\rho_g^q} \rho_f(p,q)}{\lambda_f(p,q)},$$

$$(ii) \quad \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{\lambda_f(p,q)}{\left(2^{n-1}\right)^{\rho_g^q} \rho_f(p,q)}$$

when *n* is even and

$$(iii) \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r,f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r,f_n)} \le \frac{\left(4^{n-1}\right)^{\rho_f^q} \rho_g(p,q)}{\lambda_g(p,q)},$$

$$(iv) \quad \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r,f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r,f_n)} \ge \frac{\lambda_g(p,q)}{\left(2^{n-1}\right)^{\rho_f^q} \rho_g(p,q)}$$

when n is odd.

Proof. When n is even then from (3.1) and (3.9) we get for a sequence of values of r tending to infinity and for $0<\varepsilon<\min\left\{\lambda_f(p,q),\sigma_g^q\right\}$

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r,f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r,f_n)} \le \frac{\left(\rho_f(p,q)+\varepsilon\right)\left(\sigma_g^q+\varepsilon\right) r^{\rho_g^q} + O(1)}{\left(\lambda_f(p,q)-\varepsilon\right)\left(\sigma_g^q-\varepsilon\right)\left(\frac{r}{4^{n-1}}\right)^{\rho_g^q} + O(1)}$$

$$= \frac{\left(\rho_f(p,q)+\varepsilon\right)\left(\sigma_g^q+\varepsilon\right)\left(4^{n-1}\right)^{\rho_g^q} + o(1)}{\left(\lambda_f(p,q)-\varepsilon\right)\left(\sigma_g^q-\varepsilon\right) + o(1)}.$$

Since is $\varepsilon>0$ arbitrary,

$$\therefore \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r,f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r,f_n)} \le \frac{\left(4^{n-1}\right)^{\rho_g^q} \rho_f(p,q)}{\lambda_f(p,q)}.$$

Also from (3.5) and (3.14) we have for a sequence of values of r tending to infinity and for $0<\varepsilon<\left\{\lambda_f(p,q),\sigma_g^q\right\}$,

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r,f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r,f_n)} \ge \frac{\left(\lambda_f(p,q)-\varepsilon\right)\left(\sigma_g^q-\varepsilon\right)\left(\frac{r}{2^{n-1}}\right)^{\rho_g^q} + O(1)}{\left(\rho_f(p,q)+\varepsilon\right)\left(\sigma_g^q+\varepsilon\right) r^{\rho_g^q} + O(1)}$$

$$= \frac{\left(\lambda_f(p,q)-\varepsilon\right)\left(\sigma_g^q-\varepsilon\right) + o(1)}{\left(\rho_f(p,q)+\varepsilon\right)\left(\sigma_g^q+\varepsilon\right)\left(2^{n-1}\right)^{\rho_g^q} + o(1)}.$$

Since is $\varepsilon>0$ arbitrary,

$$\therefore \quad \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{\lambda_f(p,q)}{\left(2^{n-1}\right)^{\rho_g^q} \rho_f(p,q)}.$$

Similarly for odd n we get the second part of the theorem.
This proves the theorem.

Remark 3.9. If f is regular growth i.e. $\rho_f(p,q) = \lambda_f(p,q)$ and n is even then

$$(i) \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq \left(4^{n-1}\right)^{\rho_g^q},$$

$$(ii) \quad \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{1}{\left(2^{n-1}\right)^{\rho_g^q}}.$$

Also if g is regular growth i.e. $\rho_g(p,q) = \lambda_g(p,q)$ and n is odd then

$$(iii) \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq \left(4^{n-1}\right)^{\rho_f^q},$$

$$(iv) \quad \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{1}{\left(2^{n-1}\right)^{\rho_f^q}}.$$

4. References

[1] J. Clunie, *The composition of entire and meromorphic functions*, Mathematical essays dedicated to A. J. Macintyre, Ohio Univ. Press, 1970, 75-92.

[2] R. K. Dutta, *Further growth of iterated entire functions-I*, Journal of Mathematical Inequalities, 2011, **5(4)**: 533-550.

[3] S. K. Datta and T. Biswas, *On the growth estimate of composite entire and meromorphic functions*, Bulletin of Mathematical Analysis and Applications, 2010, **2(2):** 1-17.

[4] S. K. Datta and T. Biswas, *Some comparative growth rate of composite entire and meromorphic functions*, Journal of Information and Computing Science, 2012, **7(2)**: 111-120.

[5] W. K. Hayman, *Meromorphic Functions*, Oxford University Press, 1964.

[6] B. K. Lahiri and D. Banerjee, *Relative fix points of entire functions*, J. Indian Acad. Math., 1997, **19(1)**: 87-97.

[7] I. Lahiri, *Growth of composite integral functions*, Indian J. Pure Appl. Math., 1997, **20(9)**: 899-907.

[8] I. Lahiri and D. K. Sharma, *Growth of composite entire and meromorphic functions*, Indian J. Pure Appl. Math., 1995, **26(5)**: 451-458.

[9] K. Niino and C. C. Yang, *Some growth relationships on factors of two composite entire functions, factorization theory of meromorphic functions and related topics*, Marcel Dekker Inc. (New York and Basel). 1982, 95-99.

[10] D. Sato, *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc., 1963, **69**: 411-414.

[11] A. P. Singh, *Growth of composite entire functions*, Kodai Math. J., 1985, **8**: 99-102.

JIC email for contribution: editor@jic.org.uk

[12] A. P. Singh and M. S. Baloria, *On maximum modulus and maximum term of composition of entire functions*, Indian J. Pure Appl. Math., 1991, **22(12)**: 1019-1026.

[13] G. D. Song and C. C. Yang, *Further growth properties of composition of entire meromorphic functions*, Indian J. Pure Appl. Math., 1984, **15(1)**: 67-82.

[14] G. Valiron, *Lectures on the general theory of Integral functions*, Chelsea Publishing Company, 1949.

[15] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers and Science Press, Beijing, 2003.