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Multiplicity of positive solutions for a class of quasilinear elliptic p-Laplacian problems with nonlinear boundary conditions

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Abstract. In this paper, we deal with the existence and multiplicity of positive weak solutions for a class of quasilinear elliptic p-Laplacian problems with nonlinear boundary conditions. By extracting the Palais-Smale sequences in the Nehari manifold and using the fibering maps, it is proved that there exists λ^* such that for $\lambda \in (0,\lambda^*)$, the given boundary value problem has at least two positive solutions.

Keywords: critical point, quasilinear p-Laplacian problem, nonlinear boundary value problem, fibering map, Nehari manifold.

1. Introduction

We study the existence and multiplicity of positive solutions for the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u + m(x)|u|^{p-2}u = \lambda f(x,u) - g(x)u^{q-1} & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = h(x,u) & x \in \partial\Omega, \end{cases} \quad (1)$$

where $\lambda > 0$, Δ_p denotes the p-Laplacian operator defined by $\Delta_p = div(|\nabla u|^{p-2}\nabla u)$, $2 \le q \le p < p^*$ ($p^* = \frac{pN}{N-p}$ if $N > p$, $P^* = \infty$ if $N \le p$), $\frac{\partial}{\partial N}$ is the outer normal derivative, Ω is a bounded region in R^N with the smooth boundary $\partial\Omega$, $N > p$ and $m(x), g(x) \in C(\overline{\Omega})$ are nonnegative functions. Also the basic assumptions for the functions $f(x,u)$ and $h(x,u)$ are the following:

(f1) $f(x,u) \in C^1(\Omega \times R)$ such that $f(x,0) \ge 0$, $f(x,0) \not\equiv 0$ and there exists $C_1 > 0$ such that $|f_u(x,u)| \le C_2 u^{P-2}$ for all $(x,u) \in \Omega \times R^+$.

(f2) For $u \in L^p(\Omega)$, the integral $\int_\Omega f_u(x,t|u|)u^2dx$ has the same sign for every $t > 0$

(h1) $h(x,u) \in C^1(\partial\Omega \times R)$ and for $u \in L^p(\partial\Omega)$, $\int_{\partial\Omega} h_u(x,t|u|)u^2dx$ has the same sign for every $t > 0$.

(h2) $h(x,0) \ge 0$, $\lim_{t\to\infty}\frac{h(x,t|u|)|u|}{t^{r-1}} = \eta(x,u)$ uniformly respect to (x,u), where $\eta(x,u) \in C(\partial\Omega \times R^+)$ and $|\eta(x,u)| > \theta > 0$, a.e. for all $(x,u) \in \partial\Omega \times R^+$.

(h3) There exists $C_2 > 0$ such that $H(x,u) \le \frac{1}{r}h(x,u)u \le \frac{1}{r(r-1)}h_u(x,u)u^2 \le C_2 u^r$ for all $(x,u) \in \partial\Omega \times R^+$, where $p < r < p^*$ and

$$H(x,u) = \int_0^u h(x,s)ds. \quad (2)$$

The problem of existence of the positive solutions for the quasilinear elliptic equations (systems) with nonlinear boundary conditions of different types has received considerable attention, for example see [4, 8, 10, 12, 17, 18, 19, 20, 21, 23, 24, 25] and the references cited therein.

When $f(x,u) = a(x)u^k$ or $h(x,u) = a(x)u^k$, the problem (1) has also been studied by some authors and the existence of multiple positive solutions has been established. For instance, Drabek and Schindler [14] showed the existence of positive, bounded and smooth solutions of the following p-Laplacian equation

$$\begin{cases} -\Delta_p u + b|u|^{p-2}u = f(.,u) & in \quad \Omega, \\ \Re u = 0 & on \quad \partial\Omega, \end{cases}$$

where $\Re u = |\nabla u|^{p-2}\frac{\partial u}{\partial v} + b_0|u|^{p-2}u$, $\Omega \subset R^N$ is a bounded domain and $1 < p < N$.

In the regular case; with $p = 2$, Szulkin and Weth in [22] considered Dirichlet boundary value problem

$$\begin{cases} -\Delta u - \lambda u = f(x,u) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $\lambda < \lambda_1$, λ_1 denotes the first Dirichlet eigenvalue of $-\Delta$ in Ω and $f \in C(\Omega \times R, R)$ satisfies some growth restrictions and proved the existence of a ground state solution under some appropriate conditions, by using the method of Nehari manifold.

In unbounded domain, the following semilinear elliptic problem

$$\begin{cases} -\Delta u + \lambda u = g(x,u) + f(x) & x \in \mathrm{R}^{\mathrm{N}}, \\ u(x) > 0 & u \in H^1(R^N), \end{cases}$$

where g satisfies some suitable conditions and $f \in H^{-1}(R^N)\backslash\{0\}$ is nonnegative, has been the focus of a great deal of research by several authors [1, 11, 16] and the existence of at least two positive solutions was proved.

The main idea in our proofs lies in dividing the Nehari manifold associated with the Euler functional for problem (1) into two disjoint parts and then considering the infima of this functional on each part and by extracting Palais-Smale sequences we show that there exists at least one solution on each part. The main difficulty will be the nonlinearity of $f(x,u)$ and $h(x,u)$ in problem (1) and the lack of separability, but clearly, the problems in [2, 5, 6, 7, 11], possess this assumption. To overcome this difficulty, we need to restrict the problem (1) to assumptions (f2) and (h1).

Here we present some examples for $f(x,u)$ satisfying the conditions (f1) and (f2).

$f_1(x,u) = \frac{-a_1(x)u^{p+r}}{1+a_2(x)u^2} + a_3(x),\ a_i(x) \in C(\bar{\Omega}),\ a_i(x) \geq 0,\ a_3(x) \not\equiv 0,\ max\{2-p,-1\} \leq r \leq 1.$

$f_2(x,u) = b_1(x)\tan^{-1}\left(b_2(x)u^{p+k}\right)\ln\left(1+u^{2k}\right) + b_3(x), b_i(x) \in C(\bar{\Omega}),\ b_i(x) \geq 0,\ b_3(x) \not\equiv 0,\ \frac{p}{2} \leq k.$

$f_3(x,u) = c_1(x)\sqrt[r]{(1+c_2(x)u^{2k})^{p-1}},\ c_i(x) \in C(\bar{\Omega}),\ c_i(x) \geq 0,\ c_1(x) \not\equiv 0,\ k \in N,\ 0 \leq 2k \leq r.$

$f_4(x,u) = \frac{-e_1(x)u^{P-1}}{4+\cot^{-1}(e_2(x)u^k)} + e_3(x),\ e_i(x) \in C(\bar{\Omega}),\ e_i(x) \geq 0,\ e_3(x) \not\equiv 0, k \geq 0.$

Also the following are the examples of functions that satisfy the conditions (h1)–(h3):

$h_1(x,u) = a(x)u^{r-1},\ a(x) \in C(\partial\Omega),\ a(x) \geq 0.$

$h_2(x,u) = b(x)\frac{u^{q+r-1}}{1+u^q},\ b(x) \in C(\partial\Omega),\ b(x) \geq 0,\ q \geq 0.$

$h_3(x,u) = c_1(x)\left(-c_2(x) + \sqrt[q]{\left(c_2(x)\right)^q + u^{q(r-1)}}\right),\ c_i(x) \in C(\partial\Omega),\ c_i(x) \geq 0,\ q \in N.$

Before stating our main results, we mention the following remarks.

Remark 1.1. Notice that using conditions (f1) and (f2), we conclude that there exists $C_3 > 0, C_4 > 0$ such that for all $(x,u) \in (\Omega \times R^+)$,

$$f(x,u) \leq C_3(1+u^{p-1}) \quad and \quad F(x,u) \leq C_4(1+u^p),$$

where

$$F(x,u) = \int_0^u f(x,s)ds. \tag{3}$$

Remark 1.2. It should be mentioned that using condition (h2) we have

$$|h(x,tw)w| \leq (1+|\eta(x,w)|)t^{r-1},$$

for t sufficiently large and $(x,w) \in \partial\Omega \times R^+$, hence taking $w = 1$ and $t = |u|$ for $|u|$ sufficiently large we arrive at

$$|h(x,|u|)|u|| \leq (1+|\eta(x,1)|)|u|^r \leq A_0|u|^r,$$

where $A_0 = \max\{1 = |\eta(x,1)| : x \in \partial\Omega\}$. Furthermore from (h1), $h(x,u) \in C^1(\partial\Omega \times R)$, consequently there exists $A_1 > 0$ such that

$$|h(x,u)u| \leq A_1(1+|u|^r), \qquad (x,u) \in \partial\Omega \times R^+. \tag{4}$$

Also using (h3) and (4) there exists $A_2 > 0$ such that

$$|h_u(x,u)|u^2 \leq A_2(1+|u|^r), \qquad (x,u) \in \partial\Omega \times R^+. \tag{5}$$

This paper is organized as follows. In section 2 we point out some notations and preliminary results and give some properties of Nehari manifold and fibering maps. In section 3, a fairly complete description of the fibering maps associated with the problem is given. Finally in section 4, we will prove the existence of positive solutions of problem (1) by establishing the existence of local minimas for the Euler functional, associated with problem (1) on the Nehari manifold.

2. Preliminaries and auxiliary results

Problem (1) is posed in the framework of the Sobolev space $W^{1,p}(\Omega)$ with the norm

$$\|u\|_{W^{1,p}} = \left(\int_\Omega (|\nabla u|^p + m(x)|u|^p)dx\right)^{\frac{1}{p}},$$

which is equivalent to the standard one and we use the standard $L^p(\Omega)$ spaces whose norms are denoted by $\|u\|_p$. Throughout this paper, we denote S_r and $\overline{S_r}$ the best Sobolev and the best Sobolev trace constants for the embedding of $W^{1,p}(\Omega)$ into $L^r(\Omega)$ and $W^{1,p}(\Omega)$ into $L^r(\partial\Omega)$, respectively. So we have

$$\frac{\left(\|u\|^p_{W^{1,p}}\right)^r}{\left(\int_\Omega |u|^r dx\right)^p} \geq \frac{1}{S_r^{pr}} \quad \text{and} \quad \frac{\left(\|u\|^p_{W^{1,p}}\right)^r}{\left(\int_{\partial\Omega} |u|^r dx\right)^p} \geq \frac{1}{\overline{S}_r^{pr}}. \tag{6}$$

Now we will show the existence and multiplicity results of nontrivial solutions of problem (1) by looking for critical points of the associated Euler functional

$$I_\lambda(u) = \frac{1}{p}M(u) - \lambda\int_\Omega F(x,|u|)dx + \frac{1}{q}G(u) - \int_{\partial\Omega} H(x,|u|)dx, \tag{7}$$

where

$$M(u) := \int_\Omega (|\nabla u|^p + m(x)|u|^p)dx = \|u\|^p_{W^{1,p}}\ , \quad G(u) := \int_\Omega g(x)|u|^q dx\,,$$

and the functions $F(x,u)$ and $H(x,u)$ are introduced in (3) and (2) respectively. Also from assumptions on problem (1), we know that $g(x) \geq 0$, so

$$G(u) = \int_\Omega g(x)|u|^q dx \geq 0. \tag{8}$$

The critical points of the functional I_λ are in fact weak solutions of problem (1). It is said that $u \in W^{1,p}(\Omega)$ is a weak solution of problem (1), if for any $\varphi \in W^{1,p}(\Omega)$

$$\int_\Omega (|\nabla u|^{p-2}\nabla u.\nabla\varphi + m(x)|u|^{p-2}u\varphi)dx = \lambda\int_\Omega f(x,|u|)\varphi dx - \int_\Omega g(x)|u|^{q-2}u\varphi dx + \int_{\partial\Omega} h(x,|u|)\varphi dx.$$

If I_λ is bounded below and has a minimizer on $W^{1,p}(\Omega)$, then this minimizer is a critical point of I_λ, so it is a solution of the corresponding elliptic problem. However, the energy functional I_λ, is not bounded below on the whole space $W^{1,p}(\Omega)$, but is bounded on an appropriate subset of $W^{1,p}(\Omega)$ and a minimizer on this set gives rise to a solution of problem (1). In order to obtain the existence results, we introduce the Nehari manifold

$$N_\lambda(\Omega) = \{u \in W^{1,p}(\Omega)\backslash\{0\} :\ \langle I'_\lambda(u), u\rangle = 0\},$$

where $\langle , \rangle$ denotes the usual duality between $W^{1,p}(\Omega)$ and W^{-1} (W^{-1} is the dual of the sobolev space $W^{1,p}(\Omega)$), hence $u \in N_\lambda(\Omega)$ if and only if

$$M(u) = \lambda\int_\Omega f(x,|u|)|u|dx - G(u) + \int_{\partial\Omega} h(x,|u|)|u|dx. \tag{9}$$

So we have the following theorem.

Theorem 2.1. There exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, the energy functional $I_\lambda(u)$ is coercive and bounded from below on $N_\lambda(\Omega)$.

Proof. It follows from (6)–(9), (h3) and Remark 1.1

$$I_\lambda(u) \geq \left(\frac{1}{p} - \frac{1}{r}\right)M(u) - \lambda\left(\int_\Omega F(x,|u|)dx - \frac{1}{r}\int_\Omega f(x,|u|)|u|dx\right) + \left(\frac{1}{q} - \frac{1}{r}\right)G(u)$$

$$\geq \frac{r-p}{rp}\|u\|^p_{W^{1,p}} - \left(C_4 + \frac{2}{r}C_3\right)\lambda\left(|\overline{\Omega}| + S_p^p\|u\|^p_{W^{1,p}}\right) + \frac{r-q}{rq}\int_\Omega g(x)|u|^q dx\,,$$

thus $I_\lambda(u)$ is coercive and bounded from below on $N_\lambda(\Omega)$ for $0<\lambda<\lambda_0$ where $\lambda_0=\frac{r-p}{pS_p^p(rC_4+C_3)}$.

The Nehari manifold is closely linked to the behavior of functions of the form $\phi_u : t \to I_\lambda(tu)$ $(t>0)$. Such maps are known as fibering maps. They were introduced by Drabek and Pohozaev in [13] and also were discussed in Brown and Zhang [9]. for $u\in W^{1,p}(\Omega)$, we have

$$\begin{cases}\phi_u(t)=I_\lambda(tu)=\frac{t^p}{p}M(u)-\lambda\int_\Omega F(x,t|u|)dx+\frac{t^q}{q}G(u)-\int_{\partial\Omega}H(x,t|u|)dx,\\ \phi_u'(t)=\langle I_\lambda'(tu),u\rangle=t^{p-1}M(u)-\lambda\int_\Omega f(x,t|u|)|u|dx+t^{q-1}G(u)-\int_{\partial\Omega}h(x,t|u|)|u|dx,\\ \phi_u''(t)=(p-1)t^{p-2}M(u)-\lambda\int_\Omega f_u(x,t|u|)u^2dx+(q-1)t^{q-2}G(u)-\int_{\partial\Omega}h_u(x,t|u|)u^2dx.\end{cases} \quad (10)$$

It is easy to see that $\phi_u'(t)=0$ if and only if $tu\in N_\lambda(\Omega)$ and in particular $u\in N_\lambda(\Omega)$ if and only if $\phi_u'(1)=0$, i.e. elements in $N_\lambda(\Omega)$ correspond to stationary points of fibering maps. Thus, we consider the split of $N_\lambda(\Omega)$ in three parts corresponding to local minima, local maxima and points of inflection, so we define

$$\begin{cases}N_\lambda^+=\{u\in N_\lambda(\Omega):\ \phi_u''(1)>0\},\\ N_\lambda^-=\{u\in N_\lambda(\Omega):\ \phi_u''(1)<0\},\\ N_\lambda^0=\{u\in N_\lambda(\Omega):\ \phi_u''(1)=0\}.\end{cases} \quad (11)$$

The following lemma shows that minimizers for $I_\lambda(u)$ on $N_\lambda(\Omega)$ are usually critical points for $I_\lambda(u)$, as proved by Brown and Zhang in [9] or in Aghajani et al. [3].

Lemma 2.2. Let u_0 be a local minimizer for $I_\lambda(u)$ on $N_\lambda(\Omega)$, if $u_0\notin N_\lambda^0(\Omega)$, then u_0 is a critical point of $I_\lambda(u)$.

Motivated by Lemma 2.2, we give conditions for $N_\lambda^0=\emptyset$.

Lemma 2.3. There exists $\lambda_1>0$ such that for $0<\lambda<\lambda_1$, we have $N_\lambda^0=\emptyset$.

proof. Suppose otherwise, then for $u\in N_\lambda^0$, by (10) and (11) we have

$$\phi_u'(1)=M(u)-\lambda\int_\Omega f(x,|u|)|u|dx+G(u)-\int_{\partial\Omega}h(x,|u|)|u|dx=0, \quad (12)$$

and

$$\phi_u''(1)=(p-1)M(u)-\lambda\int_\Omega f_u(x,|u|)u^2dx+(q-1)G(u)-\int_{\partial\Omega}h_u(x,|u|)u^2dx=0. \quad (13)$$

By (13) and (h3) we get

$$(p-1)M(u)\ge\lambda\int_\Omega f_u(x,|u|)u^2dx-(q-1)G(u)-(r-1)\int_{\partial\Omega}h(x,|u|)|u|dx=0,$$

using (8), (12), (f1) and Remark 1.1 we obtain

$$\begin{aligned}(r-p)M(u)&\le(r-1)\int_\Omega f(x,|u|)|u|dx-\lambda\int_\Omega f_u(x,|u|)u^2dx-(r-q)G(u)\\ &\le(2(r-1)C_3+C_1)\lambda\int_\Omega(1+|u|^p)dx.\end{aligned} \quad (14)$$

Thus, for any $u\epsilon N_\lambda^0$ using (14) and (6) we get

$$(r-p)\|u\|_{w1,p(\Omega)}^p\le(2(r-1)C_3+C_1)\lambda|\overline{\Omega}|+S_p^p\|u\|_{w1,p}^p),$$

which concludes

$$\|u\|_{W^{1,p}}\le\left(\frac{\lambda(2(r-1)C_3+C_1)|\overline{\Omega}|}{r-p-\lambda(2(r-1)C_3+C_1)S_p^p}\right)^{\frac{1}{p}}. \quad (15)$$

Moreover, (6) together with (h3) imply

$$\int_{\partial\Omega}h_u(x,|u|)|u|^2dx=r(r-1)\int_{\partial\Omega}C_3|u|^rdx\le r(r-1)C_3\bar{S}_r^r\|u\|_{W^{1,p}}^r, \quad (16)$$

hence using (16) in (13) and from (f1), (6) and (8) we get

$$(p-1)M(u)\le L\|u\|_{W^{1,p}}^r+\lambda L'\|u\|_{W^{1,p}}^p,$$

where $L=r(r-1)C_2\bar{S}_r^r$ and $L'=C_1S_P^p$, so

$$(p-1-\lambda L')\|u\|_{W^{1,p}}^p\le L\|u\|_{W^{1,p}}^r,$$

which concludes

$$\|u\|_{W^{1,p}}\ge(\frac{p-1-\lambda L'}{L})^{\frac{1}{r-p}}. \quad (17)$$

Now from (15) and (17) we infer that

$$\left(\frac{p-1-\lambda L'}{L}\right)^p \le \left(\frac{\lambda(2(r-1)C_3+C_1)|\bar{\Omega}|}{r-p-\lambda(2(r-1)C_3+C_1)S_p^p}\right)^{r-p},$$

which is a contradiction for λ sufficiently small, so there exists $\lambda_1 > 0$ such that for $0 < \lambda < \lambda_1$, $N_\lambda^0 = \emptyset$.

Definition 2.4. A sequence $\{u_n\} \subset W^{1,p}(\Omega)$ is called a Palais-Smale sequence if $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. If $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$, then u_n is a $(PS)_c$ − sequence. It is said that the functional I_λ satisfies the Palais-Smale condition (or $(PS)_c$ − condition), if each Palais-Smale sequence ($(PS)_c$ − sequence) has a convergent subsequence.

Now we will prove the boundedness of Palais-Smale sequence.

Lemma 2.5. If $\{u_n\}$ is a $(PS)_c$ − sequence for I_λ, then $\{u_n\}$ is a bounded sequence in $W^{1,p}(\Omega)$ provided that $0 < \lambda < \lambda_0$.

Proof. Using Remark 1.1, (h3), (6), (8) and (10) we have

$$I_\lambda(u_n) - \frac{1}{r}\langle I'_\lambda(u_n), u_n\rangle \ge \frac{r-p}{rp} M(u_n) - \lambda \int_\Omega \left(F(x,|u_n|) - \frac{1}{r} f(x,|u_n|)|u_n|\right) dx + \frac{r-q}{rq} G(u_n)$$
$$\ge \frac{r-p}{rp}\|u_n\|^p_{W^{1,p}(\Omega)} - \left(C_4 + \frac{2}{r}C_3\right)\lambda\left(|\bar{\Omega}| + S_p^p\|u_n\|^p_{W^{1,p}}\right),$$

so for $0 < \lambda < \frac{r-p}{p(rC_4+C_3)S_p^p} = \lambda_0$, $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$.

Lemma 2.6. There exists $\lambda_2 > 0$ such that if $0 < \lambda < \lambda_2$, then $\int_{\partial\Omega} h_u(x,|u|)u^2 dx <> 0$, provided that $u \in N_\lambda^-$.

Proof. Suppose otherwise, then $\int_{\partial\Omega} h_u(x,|u|)u^2 dx \le <0$, and by (11)

$$\phi''_u(1) = (p-1)M(u) - \lambda\int_\Omega f_u(u,|u|)u^2 dx + (q-1)G(u) - \int_{\partial\Omega} h_u(x,|u|)u^2 dx < < 0, \quad (18)$$

so by (f1), (6), (8) and (18) we have

$$(p-1)\,M(u) = (p-1)\|u\|^p_{W^{1,p}} < <\lambda\int_\Omega f_u(x,|u|)u^2\ dx \le \lambda\, C_1 S_p^p \|u\|^p_{W^{1,p}}. \quad (19)$$

Therefore, we must have $(p-1) < \lambda C_1 S_p^p$, which is a contradiction for $\lambda < \lambda_2 = \frac{p-1}{C_1 S_p^p}$.

3. Properties of fibering maps

In this section we shall describe the nature of the derivative of the fibering maps for all possible signs of $\int_{\partial\Omega} h_u(x,|u|)u^2 dx <$. We begin by recalling that, $\phi'_u(t) = 0$ if and only if $tu \in N_\lambda(\Omega)$. It will be useful to consider the functions

$$k_u(t) := \frac{1}{p}t^p\|u\|^p_{W^{1,p}} - \int_{\partial\Omega} H(x,t|u|)\ dx \quad (t>0), \quad (20)$$

$$\ell_u(t) := \lambda\int_\Omega F(x,t|u|)\ dx - \frac{1}{q}t^q G(u) \quad (t>0), \quad (21)$$

hence using (10) we have $\phi_u(t) = k_u(t) - \ell_u(t)$. Moreover, $\phi'_u(t) = 0$ if and only if $k'_u(t) = \ell'_u(t)$, where

$$\begin{cases} k'_u(t) = t^{p-1}\|u\|^p_{W^{1,p}} - \int_{\partial\Omega} h(x,t|u|)|u|\ dx, \\ \ell'_u(t) = \lambda\int_\Omega f(x,t|u|)|u|\ dx - t^{q-1}G(u). \end{cases} \quad (22)$$

In the next result we see that ϕ_u has positive values for all nonzero $u \in W^{1,p}(\Omega)$ whenever, λ is sufficiently small.

Lemma 3.1. There exists $\lambda_3 > 0$ such that $\phi_u(t)$ takes on positive values for all non-zero $u \in W^{1,p}(\Omega)$, whenever $0 < \lambda < \lambda_3$.

Proof. Using (20) and condition (h3) we obtain that

$$k_u(t) \geq \frac{1}{p} t^p M(u) - C_2 t^r \int_{\partial\Omega} |u|^r \, dx.$$

Define

$$\bar{k}_u(t) := \frac{1}{p} t^p M(u) - C_2 t^r \int_{\partial\Omega} |u|^r \, dx \qquad (t > 0), \tag{23}$$

we obtain $k_u(t) \geq \bar{k}_u(t)$, and by elementary calculus, we see that $\bar{k}_u(t)$ takes a maximum value at

$$t_{max} = \left(\frac{\|u\|^p_{W^{1,p}}}{rC_2 \int_{\partial\Omega} |u|^r dx} \right)^{\frac{1}{r-p}}, \tag{24}$$

then follow by (20), (24) and (6) that

$$k_u(t_{max}) \geq \bar{k}_u(t_{max}) = \frac{r-p}{rp} \left(\frac{\left(\|u\|^p_{W^{1,p}(\Omega)} \right)^r}{\left(rC_2 \int_{\partial\Omega} |u|^r dx \right)^p} \right)^{\frac{1}{r-p}} \geq \frac{r-p}{rp} \left(\frac{1}{(rC_2)^p \bar{S}_r^{pr}} \right)^{\frac{1}{r-p}} = \delta_1 > 0, \tag{25}$$

Where δ_1 is independent of u. Now, we will prove that there exists $\lambda_3 > 0$ such that for all $u \in W^{1,p} \backslash \{0\}$, $\phi_u(t_{max}) > 0$ provided that $\lambda < \lambda_3$. To do this, first note that from (24), (25) and (6)

$$\begin{aligned} &(t_{max})^\gamma \int_\Omega |u|^\gamma dx \\ &\leq S_\gamma^\gamma \left(\frac{\|u\|^p_{W^{1,p}(\Omega)}}{rC_2 \int_{\partial\Omega} |u|^r dx} \right)^{\frac{\gamma}{r-p}} \left(\|u\|^p_{W^{1,p}}(\Omega) \right)^{\frac{\gamma}{p}} \\ &= S_\gamma^\gamma \left\{ \frac{\left(\|u\|^p_{W^{1,p}}(\Omega) \right)^r}{\left(rC_2 \int_{\partial\Omega} |u|^r dx \right)^p} \right\}^{\frac{\gamma}{p(r-p)}} \\ &\leq S_\gamma^\gamma \left(\frac{rp}{r-p} \right)^{\frac{r}{p}} (k_u(t_{max}))^{\frac{\gamma}{p}} = \alpha_1 (k_u(t_{max}))^{\frac{\gamma}{p}}, \end{aligned} \tag{26}$$

for $1 \leq \gamma < p*$. Then by Remark 1.1, (8), (21) and (26) we find

$$\begin{aligned} \ell_u(t_{max}) &\leq \lambda \int_\Omega C_4 (1 + |(t_{max}) u|^p) \, dx - \frac{1}{q} (t_{max})^q G(u) \\ &\leq \lambda C_4 (|\bar{\Omega}| + \alpha_1 k_u(t_{max})), \end{aligned} \tag{27}$$

hence using (25) and (27) we observe that

$$\begin{aligned} \phi_u(t_{max}) &= k_u(t_{max}) - \ell_u(t_{max}) \\ &\geq k_u(t_{max}) \left(1 - \lambda C_4 \left\{ (|\bar{\Omega}| \left(k_u(t_{max}) \right)^{-1} + \alpha_1 \right\} \right) \\ &\geq \delta_1 \left(1 - \lambda \, C_4 \left(|\bar{\Omega}| {\delta_1}^{-1} + \alpha_1 \right) \right) = \delta_1 (1 - \lambda\alpha). \end{aligned} \tag{28}$$

So we conclude that $\phi_u(t_{max}) > 0$ for all nonzero u, if $\lambda < \lambda_3 = \frac{1}{2\alpha}$ and this completes the proof.

Corollary 3.2. If $0 < \lambda < \min\{\lambda_2, \lambda_3\}$, then there exists $\varepsilon > 0$ such that $I_\lambda(u) \geq \varepsilon$ for all $u \in N_\lambda^-$.

Proof. If $u \in N_\lambda^-$, then by lemma 2.6, $\int_{\partial\Omega} h_u \, (x, |u|) u^2 dx > 0$. Also due to (f2) and (h1), $I_\lambda(tu)$ has a positive global maximum at t = 1 and so by using (28)

$$I_\lambda(u) = \phi_u(1) \geq \phi_u(t_{max}) \geq \delta_1 (1 - \lambda\alpha) \geq \delta_1 (1 - \alpha\lambda_3) = \varepsilon > 0.$$

To state our main results, we now present some important properties of N_λ^- and N_λ^+.

Lemma 3.3. There exists $\lambda_4 > 0$ such that $\phi_u'(t)$ takes on positive values for all non-zero $u \in W^{1,p}(\Omega)$ whenever $\lambda < \lambda_4$.

Proof. By elementary calculus and using (23), we can show that $tk_u'(t)$ achieves its maximum at

$$\tau_{max} = \left(\frac{p\|u\|^p_{W^{1,p}(\Omega)}}{C_2 r^2 \int_{\partial\Omega} |u|^r dx} \right)^{\frac{1}{r-p}}. \tag{29}$$

Therefore, by (6), (22), (23) and (29) we obtain that

$$\tau_{max}k'_u(\tau_{max}) \geq \tau_{max}\bar{k}'_u(\tau_{max}) = (\frac{p}{C_2 r^2})^{\frac{p}{r-p}}\left(\frac{r-p}{r}\right)\left(\frac{\left(\|u\|^p_{W^{1,p}}\right)^r}{\left(\int_\Omega |u|^r dx\right)^p}\right)^{\frac{1}{r-p}}$$
$$\geq (\frac{p}{C_2 r^2})^{\frac{p}{r-p}}\left(\frac{r-p}{r}\right)\left(\frac{1}{\bar{S}_r^{pr}}\right)^{\frac{1}{r-p}} = \delta_2 > 0\,, \tag{30}$$

Where δ_2 is independent of u. Now from (6), (29) and (30), and by some calculation very similar to (26) we get

$$(\tau_{max})^\gamma \int_{\partial\Omega} |u|^\gamma \, dx \leq \alpha_2\left(\tau_{max}k'_u(\tau_{max})\right)^{\frac{\gamma}{p}}, \tag{31}$$

for $1 \leq \gamma < p^*$. Then using (8), (22), (31) and Remark 1.1 we conclude that

$$\tau_{max}\ell'_u(\tau_{max}) \leq \lambda C_3 \int_\Omega (|\tau_{max}u| + |\tau_{max}u|^p)\, dx - (\tau_{max})^q G(u)$$
$$\leq \lambda\left(\beta_1\left(\tau_{max}k'_u(\tau_{max})\right)^{\frac{1}{p}} + \beta_2\tau_{max}k'_u(\tau_{max})\right), \tag{32}$$

where β_1 and β_2 are independent of u, so from (30) and (32) we get

$$\tau_{max}\phi'_u(\tau_{max}) = \tau_{max}k'_u(\tau_{max}) - \tau_{max}\ell'_u(\tau_{max})$$
$$\geq \tau_{max}k'_u(\tau_{max})\left(1 - \lambda\left\{\beta_1\left(\tau_{max}k'_u(\tau_{max})\right)^{\frac{1-p}{p}} + \beta_1\right\}\right)$$
$$\geq \delta_2\left(1 - \lambda\,(\beta_1\delta_2^{\frac{1-p}{p}} + \beta_2)\right) \geq \delta_2(1 - \lambda\beta).$$

Clearly for all nonzero u, $t_{max}\phi'_u\,(t_{max}) > 0$ provided that $\lambda < \lambda_4$, where $\lambda_4 = 1/2\beta$ and this completes the proof.

Corollary 3.4. If $\int_{\partial\Omega} h_u\,(x, t\,|u|)u^2 dx \leq 0$ for $u \in W^{1,p}(\Omega) \setminus \{0\}$, then there exists t_1 such that $t_1 u \in N^+_\lambda$ and $\phi_u(t_1) < 0$.

Proof. By (10), (f1) and (h2), we know that $\phi'_u(0) < 0$ and $\lim_{t\to\infty}\phi'_u(t) = \infty$, so by the intermediate value theorem, there exists $t_1 > 0$ such that $\phi'_u(t_1) = 0$. Now using (f2) and (h1), for $0 < t < t_1$, $\phi'_u(t) < 0$ and for $t_1 < t$, $\phi'_u(t) > 0$, hence $t_1 u \in N^+_\lambda$ and $\phi_u(t_1) < \phi_u(0) = 0$.

Corollary 3.5. If $\int_{\partial\Omega} h_u\,(x, t\,|u|)u^2 dx \geq 0$ for $u \in W^{1,p}(\Omega) \setminus \{0\}$ and $\lambda < \lambda_3$ then there exist $0 < t_1 < t_2$ such that $t_1 u_1 \in N^+_\lambda$, $t_2 u_2 \in N^-_\lambda$ and $\phi_u\,(t_1) < 0$.

Proof. From the definition of $\phi'_u(t)$ together with (f1) and (h2) we have $\phi'_u(0) < 0$, $\lim_{t\to\infty}\phi'_u(t) = -\infty$ and by Lemma 3.3, $\phi'_u(\tau) > 0$ for suitable τ, so using again the intermediate value theorem concludes that there exist t_1, t_2 such that $0 < t_1 < t_2$ and $\phi'_u(t_1) = \phi'_u(t_2) = 0$. Also using the same argument as in the proof of the Corollary 3.4 and using (f2) and (h1) we have $t_1 u_1 \in N^+_\lambda$, $t_2 u_2 \in N^-_\lambda$ and $\phi_u\,(t_1) < 0$.

4. Existence of multiple Solutions

In this section, we will show the existence and multiplicity of solutions of problem (1), for this, we need the following remark:

Remark 4.1. Using the compactness of the embeddings $W^{1,p}(\Omega) \hookrightarrow L^m(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^m(\partial\Omega)$ for $1 \leq m < p^*$ (the Rellich-Kondrachov Theorem [7]) together with (4), (5), (f1) and (f2) we conclude that the functionals $J_1(u) = \int_\Omega F(x,u)dx$ and $J_2(u) = \int_{\partial\Omega} H(x,u)dx$ are weakly continuous, i.e. if $u_n \rightharpoonup u$, then $J_i(u_n) \to J_i(u)$ $(i = 1,2)$. Moreouer The operators $J'_1(u) = \int_\Omega f(x,u)udx$, $J''_1(u) = \int_\Omega f_u(x,u)u^2 dx$, $J'_2(u) = \int_{\partial\Omega} h(x,u)udx$ and $J''_2(u) = \int_{\partial\Omega} h_u(x,u)u^2 dx$ are weak to strong continuous, i.e. if $u_n \rightharpoonup u$, then $J'_i(u_n) \to J'_i(u)$ and $J''_i(u_n) \to J''_i(u)$ $(i = 1,2)$.

Theorem 4.2. For $0 < \lambda < min\{\lambda_0, \lambda_3, \lambda_4\}$, there exists a minimizer of I_λ on $N_\lambda^+(\Omega)$.

Proof. As in Theorem 2.1, I_λ is bounded from below on $N_\lambda(\Omega)$, and so on $N_\lambda^+(\Omega)$. Let $\{u_n\}$ be a minimizing sequence for I_λ on $N_\lambda^+(\Omega)$, i.e.

$$\lim_{n\to\infty} I_\lambda(u_n) = inf_{u\in N_\lambda^+} I_\lambda(u),$$

and by Ekeland's variational principle [15] we may assume that;

$$\langle I_\lambda'(u_n), u_n\rangle \to 0.$$

Then by Lemma 2.5 $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$ and we may assume, without loss of generality, that $u_n \rightharpoonup u_0$ in $W^{1,p}(\Omega)$ and $u_n \to u_0$ in $L^m(\Omega)$ for $1 \le m < p^*$ and $u_n(x) \to u_0(x)$, a.e.

By Corollaries 3.4 and 3.5 for $u_0 \in W^{1,p}\backslash\{0\}$, there exists t_0 such that $t_0 u_0 \in N_\lambda^+$ and so $\phi'_{u_0}(t_0) = 0$. Now we show that $u_n \to u_0$ in $W^{1,p}(\Omega)$. Suppose that this is false, then

$$M(u_0) < \lim_{n\to\infty} \inf M(u_n). \tag{33}$$

Also by (10) we have

$$\phi'_{u_n}(t) = t^{p-1}M(u_n) - \lambda\int_\Omega f(x, t|u_n|)|u_n|dx + t^{q-1}G(u_n) - \int_{\partial\Omega} h\,(x, t|u_n|)|u_n|dx, \tag{34}$$

and

$$\phi'_{u_0}(t) = t^{p-1}M(u_0) - \lambda\int_\Omega f(x, t|u_0|)|u_0|dx + t^{q-1}G(u_0) - \int_{\partial\Omega} h\,(x, t|u_0|)|u_0|dx. \tag{35}$$

So from (33)–(35) and Remark 4.1, $\phi'_{u_n}(t_0) > \phi'_{u_0}(t_0) = 0$ for n sufficiently large. Since $\{u_n\} \subseteq N_\lambda^+(\Omega)$, by considering the possible fibering maps it is easy to see that $\phi'_{u_n}(t) < 0$ for $0 < t < 1$ and $\phi'_{u_n}(1) = 0$ for all *n*. Hence we must have $t_0 > 1$, but $t_0 u_0 \in N_\lambda^+$ and so by (10)

$$I_\lambda(t_0u_0) = \phi_{u_0}(t_0) < \phi_{u_0}(1) < \lim_{n\to\infty} \phi_{u_n}(1) = \lim_{n\to\infty} I_\lambda(u_n) = inf_{u\in N_\lambda^+} I_\lambda(u_n),$$

which is a contradiction. Therefore, $u_n \to u_0$ in $W^{1,p}(\Omega)$ and so

$$I_\lambda(u_0) = \lim_{n\to\infty} I_\lambda(u_n) = inf_{u\in N_\lambda^+} I_\lambda(u)$$

thus, u_0 is a minimizer for I_λ on $N_\lambda^+(\Omega)$.

In the next theorem, we will establish the existence of a local minimum for I_λ on $N_\lambda^-(\Omega)$.

Theorem 4.3. If $0 < \lambda < min\{\lambda_0, \lambda_2, \lambda_3, \lambda_4\}$, then there exists a minimizer of I_λ on $N_\lambda^-(\Omega)$.

Proof. By Corollary 3.2, there exists $\varepsilon > 0$ such that $I_\lambda(u) \ge \varepsilon > 0$ for all $u \in N_\lambda^-(\Omega)$, i.e.

$$inf_{u\in N_\lambda^-} I_\lambda(u) > 0,$$

hence there exists a minimizing sequence $\{u_n\} \subset N_\lambda^-(\Omega)$ such that

$$\lim_{n\to\infty} I_\lambda(u_n) = inf_{u\in N_\lambda^-} I_\lambda(u) > 0. \tag{36}$$

Similary as in the proof of the Theorem 4.2 we find that, $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$ and so

$$\begin{cases} u_n \rightharpoonup u_0 & \textit{weakly in } \mathrm{W}^{1,p}(\Omega), \\ u_n \to u_0 & \textit{strongly in } \mathrm{L}^\alpha(\Omega), \quad 1\le\alpha<P^*. \end{cases} \tag{37}$$

Since $u_n \epsilon N_\lambda^-$ so by (11) $\phi''_{u_o}(1) < 0$, letting $n \to \infty$, by (10) , (37) and Remark 4.1 we see that

$$\phi''_{u_o}(1) = (p-1)M(u_0) - \lambda\int_\Omega f_u(x, |u_0|)u_0^2dx + (q-1)G(u_0) - \int_{\partial\Omega} h_u(x, |u_0|)u_0^2dx \le 0. \tag{38}$$

On the other hand for $u_n \epsilon N_\lambda^-$, by lemma 2.6 we have $\int_{\partial\Omega} h_u(x, |u_n|)u_n^2dx > 0$, letting $n \to \infty$, we see that $\int_{\partial\Omega} h_u(x, |u_0|)u_0^2dx \ge 0$, if $\int_{\partial\Omega} h_u(x, |u_0|)u_0^2dx = 0$, then by (f2), (6), (8) and (38) we have

$$(p-1)M(u_0) \le \lambda \int_\Omega f_u\,(x, |u_0|u_0^2dx \le \lambda C_2\, S_2^2 M(u_0),$$

which is a contradiction for $\lambda < \lambda_2$. So $\int_{\partial\Omega} h_u(x, |u_0|)u_0^2dx > > 0$ and by Corollary 3.5 there exists $t_0 > 0$ such that $t_0u_0 \in N_\lambda^-(\Omega)$. We claim that $u_n \to u_0$ in $W^{1,p}(\Omega)$, Suppose that this is false, so

$$M(u_0) < \lim_{n\to\infty} \inf M(u_n). \tag{39}$$

But $u_n \in N_\lambda^-$ and so $I_\lambda(u_n) \ge I_\lambda(tu_n)$ for all $t \ge 0$, now by using (36)–(39) and Remark 4.1, we can write

$$I_\lambda(t_0u_0) = \frac{1}{p}t_0^p M(u_0) - \lambda\int_\Omega F(x, t_0|u_0|)dx + \frac{1}{q}t_0^q G(u_0) - \int_{\partial\Omega} H(x, t_0|u_0|)dx$$
$$< \lim_{n\to\infty}\left(\frac{1}{p}t_0^p M(u_n) - \lambda\int_\Omega F(x, t_0|u_n|)dx + \frac{1}{q}t_0^q G(u_n) - \int_{\partial\Omega} H(x, t_0|u_n|)dx\right)$$
$$= \lim_{n\to\infty} I_\lambda(t_0u_n) \le \lim_{n\to\infty} I_\lambda(u_n) = inf_{u\epsilon N_\lambda^-} I_\lambda(u),$$

which is a contradiction. Therefore, $u_n \to u_0$ in $W^{1,p}(\Omega)$ and so the proof is complete.

Corollary 4.4. For $0 < \lambda < min\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, equation (1) has at least two positive solutions.

Proof. By Theorems 4.2 and 4.3 there exist two solutions $u_0^+ \epsilon N_\lambda^+(\Omega)$ and $u_0^- \epsilon N_\lambda^-(\Omega)$ such that $I_\lambda(u_0^+) = inf_{u\epsilon N_\lambda^+} I_\lambda(u)$, $I_\lambda(u_0^-) = in\, f_{u\epsilon N_\lambda^-} I_\lambda(u)$, $u^\pm \neq 0$ and by Lemmas 2.2 and 2.3 u_0^+ and u_0^- are critical points of I_λ on $W^{1,p}$ and hence are weak solutions of problem (1). On the other hand $I_\lambda(u) = I_\lambda(|u|)$, so we may assume u_0^+ and u_0^- are positive solution. It remains to show that this solutions are distinct. Since $N_\lambda^+ \cap N_\lambda^- = \emptyset$, this implies that u_0^+ and u_0^- are distinct and the proof is complete.

5. References

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