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Rational solitary wave solutions for some nonlinear differential difference equations

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Abstract. In this article, we put a direct method to construct the rational solitary wave solutions for some nonlinear differential difference equations in mathematical physics which may be called the rational solitary wave difference method. We use the proposed method to construct the rational solitary exact solutions for some nonlinear differential difference equations via the lattice equation, the discrete nonlinear Klein Gordon equation. The proposed method is more effective and powerful to obtain many rational solitary exact solutions for nonlinear differential difference equations.

Keywords: Solitary wave solutions, Traveling wave solutions, The lattice equation, The discrete Klein Gordon equation.

1. Introduction

It is well known that the investigation of differential difference equations (DDEs) which describe many important phenomena and dynamical processes in many different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains a many others and so on, has played an important role in the study of modern physics. Unlike difference equations which are fully discredited, DDEs are semi- discredited with some (or all) of their special variables discredited while time is usually kept continuous. DDEs also play an important role in numerical simulations of nonlinear partial differential equations (NLPDEs), queuing problems, and discretization in solid state and quantum physics.

Since the work of Fermi, Pasta, and Ulam in the 1960s [1], DDEs have been the focus of many nonlinear studies. On the other hand, a considerable number of well-known analytic methods are successfully extended to nonlinear DDEs by researchers [2–17]. However, no method obeys the strength and the flexibility for finding all solutions to all types of nonlinear DDEs. Zhang etal. [18] and Aslan [19] used the (G'/G)-expansion method to some physically important nonlinear DDEs. Qiong etal. [12] constructed the Jacobi elliptic solutions for nonlinear DDEs. Recently Zhang etal [20] and Gepreel [29,30] have used the Jacobi elliptic function method for constructing new and more general Jacobi elliptic function solutions of some nonlinear difference differential equations. The main objective of this paper, is to modify the rational solitary wave method which discussed by Xie [31] to solve the nonlinear differential difference equations instead of solving the nonlinear partial differential equations which my be called rational solitary wave difference method. We use the proposed method to calculate the rational solitary wave solutions for some nonlinear DDEs in mathematical physics via the lattice equation and the discrete nonlinear Klein Gordon equation.

2. Description of the rational solitary wave difference method

In this section, we would like to outline on algorithm for using the rational solitary wave difference method to solve nonlinear DDEs. For a given nonlinear DDEs

$$\begin{aligned}&\Delta(u_{n+p_1}(x),\dots,u_{n+p_k}(x),u'_{n+p_1}(x),\dots,u'_{n+p_k}(x),\dots,u^{(r)}_{n+p_1}(x),\dots,u^{(r)}_{n+p_k}(x),\\&v_{n+p_1}(x),\dots,v_{n+p_k}(x),v'_{n+p_1}(x),\dots,v'_{n+p_k}(x),\dots,v^{(r)}_{n+p_1}(x),\dots,v^{(r)}_{n+p_k}(x),\dots)=0,\end{aligned}\qquad(1)$$

where $\Delta = (\Delta_1,...,\Delta_g), x = (x_1, x_2,...., x_m), n = (n_1,..., n_Q)$ and $g, m, Q, p_1,..., p_k$ are integers, $u_i^{(r)}, v_i^{(r)}$ denotes the set of all r th order derivatives of u_i, v_i with respect x.

The main steps of the algorithm for the rational solitary wave difference method to solve nonlinear DDEs are outlined as follows:

Step 1. We take the traveling wave solutions of the following form:

$$u_n(x) = U(\xi_n), \quad v_n(x) = V(\xi_n), \; ... \quad , \tag{2}$$

where

$$\xi_n = \sum_{i=1}^{Q} d_i n_i - \sum_{j=1}^{m} c_j x_j + \xi_0, \tag{3}$$

and $d_i (i = 1,.., Q)$, $c_j, (j = 1,..., m)$, the phase ξ_0 are constants to be determined later. The transformations (2) is reduced Eqs.(1) to the following nonlinear differential difference equations

$$\begin{aligned} &\Omega(U(\xi_{n+p_1}),...,U(\xi_{n+p_k}), U'(\xi_{n+p_1}),...,U'(\xi_{n+p_k}),...,U^{(r)}(\xi_{n+p_1}),...,U^{(r)}(\xi_{n+p_k}),\\ &V(\xi_{n+p_1}),...,V(\xi_{n+p_k}), V'(\xi_{n+p_1}),...,V'(\xi_{n+p_k}),...,V^{(r)}_{n+p_1}(\xi_{n+p_1}),...,V^{(r)}_{n+p_k}(\xi_{n+p_k}),...) = 0, \end{aligned} \tag{4}$$

where $\Omega = (\Omega_1,...,\Omega_g)$.

Step 2. We suppose the rational solitary wave series expansion solutions of Eqs (4) in the following form:

$$\begin{aligned} U(\xi_n) &= \sum_{i=0}^{N} a_i [g(\xi_n)]^i + \sum_{j=1}^{N} b_j [g(\xi_n)]^{j-1} f(\xi_n),\\ V(\xi_n) &= \sum_{i=0}^{L} \alpha_i [g(\xi_n)]^i + \sum_{j=1}^{L} \beta_j [g(\xi_n)]^{j-1} f(\xi_n),...., \end{aligned} \tag{5}$$

with

$$f(\xi_n) = \frac{1}{A\tanh(\xi_n) + B\sec h(\xi_n)}, \quad g(\xi_n) = \frac{\sec h(\xi_n)}{A\tanh(\xi_n) + B\sec h(\xi_n)}, \tag{6}$$

which satisfy

$$\begin{aligned} &f'(\xi_n) = -Ag^2(\xi_n) + \frac{Bg(\xi_n)}{A}[1 - Bg(\xi_n)], \qquad g'(\xi_n) = -Af(\xi_n)g(\xi_n),\\ &f^2(\xi_n) = g^2(\xi_n) + \frac{1}{A^2}[1 - Bg(\xi_n)]^2,\\ &f(\xi_n \pm d) = \frac{A^2 f(d) f(\xi_n) \pm [1 - Bg(\xi_n)][1 - Bg(d)]}{A^2 f(d)[1 - Bg(\xi_n)] \pm A^2 f(\xi_n)[1 - Bg(d)] + BA^2 g(\xi_n)g(d)},\\ &g(\xi_n \pm d) = \frac{g(\xi_n)g(d)}{f(d)[1 - Bg(\xi_n)] \pm f(\xi_n)[1 - Bg(d)] + Bg(\xi_n)g(d)}, \end{aligned} \tag{7}$$

where $a_i, \alpha_i, b_j, \beta_j, A, B$ are constants to be determined.

Also, we can assume that

$$f(\xi_n) = \frac{1}{A\tan(\xi_n) + B\sec(\xi_n)}, \quad g(\xi_n) = \frac{\sec(\xi_n)}{A\tan(\xi_n) + B\sec(\xi_n)} \tag{8}$$

which satisfy

$$
\begin{aligned}
& f'(\xi_n) = -Ag^2(\xi_n) - \frac{Bg(\xi_n)}{A}[1-Bg(\xi_n)], \quad g'(\xi_n) = -Af(\xi_n)g(\xi_n), \\
& f^2(\xi_n) = g^2(\xi_n) - \frac{1}{A^2}[1-Bg(\xi_n)]^2, \\
& f(\xi_n \pm d) = \frac{A^2 f(d) f(\xi_n) \mp [1-Bg(\xi_n)][1-Bg(d)]}{A^2 f(d)[1-Bg(\xi_n)] \pm A^2 f(\xi_n)[1-Bg(d)] + BA^2 g(\xi_n)g(d)}, \\
& g(\xi_n \pm d) = \frac{g(\xi_n)g(d)}{f(d)[1-Bg(\xi_n)] \pm f(\xi_n)[1-Bg(d)] + Bg(\xi_n)g(d)}.
\end{aligned}
\tag{9}
$$

Equations (7) and (9) can be written into unified form

$$
\begin{aligned}
& f'(\xi_n) = -Ag^2(\xi_n) + \delta\frac{Bg(\xi_n)}{A}[1-Bg(\xi_n)], \quad g'(\xi_n) = -Af(\xi_n)g(\xi_n), \\
& f^2(\xi_n) = g^2(\xi_n) + \delta\frac{1}{A^2}[1-Bg(\xi_n)]^2, \\
& f(\xi_n \pm d) = \frac{A^2 f(d) f(\xi_n) \pm \delta[1-Bg(\xi_n)][1-Bg(d)]}{A^2 f(d)[1-Bg(\xi_n)] \pm A^2 f(\xi_n)[1-Bg(d)] + BA^2 g(\xi_n)g(d)}, \\
& g(\xi_n \pm d) = \frac{g(\xi_n)g(d)}{f(d)[1-Bg(\xi_n)] \pm f(\xi_n)[1-Bg(d)] + Bg(\xi_n)g(d)}.
\end{aligned}
\tag{10}
$$

Step 4. Determine the degree $N, L, ...$ of Eqs. (4) by balancing the nonlinear term(s) and the highest order derivatives of $U(\xi_n), V(\xi_n), ...$ in Eqs. (4). It should be noted that the leading terms $U(\xi_{n\pm p}), V(\xi_{n\pm p}), ..., p \neq 0$ will not affect the balance because we are interested in balancing the terms of $f(\xi_n)$ and $g(\xi_n)$.

Step 5. Substituting Eqs. (5) and (10) the given values of $K, L, ...$ into Eqs.(4). Cleaning the denominator and collecting all terms with the same degree of $f(\xi_n)$ and $g(\xi_n)$ together, the left hand side of Eq. (4) is converted into a polynomial in $f(\xi_n)$ and $g(\xi_n)$. Setting each coefficients $f^i(\xi_n), g^j(\xi_n) (i = 0,1, j = 0,1,2,...)$ of these polynomials to be zero, we derive a set of algebraic equations for $a_i, \alpha_i, b_j, \beta_j, C_i\ A, B$.

Step 6. Solving the over determined system of nonlinear algebraic equations by using Maple or Mathematica software package. We end up with explicit expressions for $a_i, \alpha_i, b_j, \beta_j, C_i\ A, B$

Step7. Substituting $a_i, \alpha_i, b_j, \beta_j, C_i\ A, B$ into Eq.(5) along with (6) and (8), we can finally obtain the rational solitary wave solutions for nonlinear difference differential equations (1).

3. Applications

In this section, we apply the proposed rational solitary wave difference method to construct the traveling wave solutions for some nonlinear DDEs via the the lattice equation, the discrete nonlinear Klein Gordon equation, which are very important in the mathematical physics, modern physics and have been paid attention by many researchers.

3.1. Example 1. The lattice equation

In this section, we study the lattice equation which take the following form [30,32]

$$\frac{du_n(t)}{dt} = (\alpha + \beta u_n + \gamma u_n^2)(u_{n+1} - u_{n-1}), \tag{20}$$

where α, β, γ is an arbitrary constant. The lattice equation contains hybrid lattice equation, mKdV lattice equation, modified Volterra lattice equation, and Langmuir chain equation for some special values α, β, γ. According to the above steps, to seek traveling wave solutions of Eq. (20), we construct the traveling wave transformation

$$u_n(t) = U(\xi_n), \qquad \xi_n = d\,n - c_1 t + \xi_0, \tag{21}$$

where d, c_1 and ξ_0 are constants. The transformation (21) permits us converting Eq. (20) into the following form:

$$-c_1 U'(\xi_n) = (\alpha + \beta U(\xi_n) + \gamma U^2(\xi_n))[U(\xi_n + d) - U(\xi_n - d)], \tag{22}$$

where $' = d/d\xi_n$. Considering the homogeneous balance between the highest order derivative and the nonlinear term in (22), we get $N = 1$. Thus the solution of Eq. (22) has the following form:

$$U(\xi_n) = a_0 + a_1 f(\xi_n) + b_1 g(\xi_n), \tag{23}$$

where a_0, a_1 and b_1 are constants to be determined later. With the aid of Maple, substituting Eq.(23) and Eqs.(10) into Eq.(22) and collecting all terms with the same power in $f^i(\xi_n), g^j(\xi_n) (i = 0,1,\ j = 0,1,2,...)$. Setting the coefficients of these terms $f^i(\xi_n), g^j(\xi_n) (i = 0,1,\ j = 0,1,2,...)$ to be zero yields a set of algebraic equations which have the following solutions:
When $\delta = 1$

Case 1.

$$a_0 = -\frac{\beta}{2\gamma}, \qquad a_1 = \frac{\pm A a_2}{\sqrt{A^2 + B^2}}, \qquad c_1 = \frac{4\gamma a_2^2(\cosh(d) + 1)}{\sinh(d)(A^2 + B^2)},$$
$$\alpha = \frac{[\beta^2(A^2 + B^2)(\cosh(d) - 1) - 4\gamma a_2^2(\cosh(d) + 1)]}{4\gamma(A^2 + B^2)(\cosh(d) - 1)}, \tag{24}$$

where A, B, a_2, γ, β are arbitrary constants. In this case the rational hyperbolic solitary wave solution for the nonlinear lattice equation takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} \pm \frac{A a_2}{\sqrt{A^2 + B^2}\,[A\tanh(\xi_n) + B\,\mathrm{sech}(\xi_n)]} + \frac{a_2\,\mathrm{sech}(\xi_n)}{[A\tanh(\xi_n) + B\,\mathrm{sech}(\xi_n)]}, \tag{25}$$

where

$$\xi_n = d\,n - \frac{4\gamma a_2^2(\cosh(d) + 1)}{\sinh(d)(A^2 + B^2)} t + \xi_0. \tag{26}$$

When $\delta = -1$

Case 2.

$$a_0 = -\frac{\beta}{2\gamma}, \qquad a_1 = \frac{\pm A a_2}{\sqrt{A^2 - B^2}}, \qquad c_1 = \frac{4\gamma a_2^2(\cos(d)+1)}{\sin(d)(A^2 - B^2)},$$
$$\alpha = \frac{[\beta^2(A^2 - B^2)(\cos(d)-1) + 4\gamma a_2^2(\cos(d)+1)]}{4\gamma(A^2 - B^2)(\cos(d)-1)}, \tag{27}$$

In this case the rational trigonometric solitary wave solution for the nonlinear lattice equation have the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} \pm \frac{A a_2}{\sqrt{A^2 - B^2}[A\tan(\xi_n) + B\sec(\xi_n)]} + \frac{a_2\sec(\xi_n)}{[A\tan(\xi_n) + B\sec(\xi_n)]}, \tag{28}$$

where

$$\xi_n = d\,n - \frac{4\gamma a_2^2(\cos(d)+1)}{\sin(d)(A^2 - B^2)}t + \xi_0. \tag{29}$$

Note that, there are other cases which are omitted here for convenience.

3.2 Example 2. The discrete nonlinear Klein Gordon equation

In this section, we consider the following discrete nonlinear Klein Gordon equation [29,33]:

$$\frac{d^2u_n(t)}{dt^2} = g(u_n)(u_{n+1} + u_{n-1} - 2su_n) \tag{30}$$

The non-constant (in contrast to the standard models of harmonic coupling and linear dispersion [34]) function $g(u_n)$ ensures the presence of nonlinear dispersion , which is critical for the existence of compactly supported solutions and s can take values in the interval [-1,1] . Kevrekidis etal [33] have obtained some exact compaction solutions and claim that this DDE does not have the traveling compact solution. If we set $g(u_n) = \alpha - u_n^2$ as similar in [33] and take the traveling transformation

$$u_n = U(\xi_n), \qquad \xi_n = dn - c_1 t + \xi_0, \tag{31}$$

where d , c_1, s and ξ_0 are constants. The transformation (31) permits us converting Eq. (30) into the following form:

$$c_1^2 U''(\xi_n) = (\alpha - U^2(\xi_n))[U(\xi_n + d) + U(\xi_n - d) - 2sU(\xi_n)], \tag{32}$$

where $' = d/d\xi_n$. Considering the homogeneous balance between the highest order derivative and the nonlinear term in (32), we get $N = 1$. Thus the solution of Eq. (32) has the following form:

$$U(\xi_n) = a_0 + a_1 f(\xi_n) + b_1 g(\xi_n), \tag{33}$$

where a_0, a_1 and b_1 are constants to be determined later .With the aid of Maple, substituting Eqs.(33) and (10) into Eq.(32) and collecting all terms with the same power in $f^i(\xi_n), g^j(\xi_n)(i = 0,1,\ j = 0,1,2,...)$. Setting the coefficients of these terms $f^i(\xi_n), g^j(\xi_n)(i = 0,1,\ j = 0,1,2,...)$ to be zero yields a set of algebraic equations which have the following solutions:When δ =1

Case 3.

$$a_0 = 0, \quad a_1 = \frac{\pm A a_2}{\sqrt{A^2 + B^2}}, \quad c_1 = \frac{\pm 2a_2}{\sqrt{A^2 + B^2}}, \quad \alpha = \frac{a_2^2[\cosh(d)+1]}{(A^2 + B^2)(\cosh(d)-1)}, \; s = 1 \quad (34)$$

where A, B, a_2, d are arbitrary constants. In this case the rational hyperbolic solitary wave solution for the discrete nonlinear Klein Gordon equation have the following form:

$$U(\xi_n) = \frac{\pm A a_2}{\sqrt{A^2 + B^2}\,[A\tanh(\xi_n) + B\sec h(\xi_n)]} \pm \frac{a_2 \sec h(\xi_n)}{[A\tanh(\xi_n) + B\sec h(\xi_n)]}, \quad (35)$$

where

$$\xi_n = d\,n \mp \frac{2a_2}{\sqrt{A^2 + B^2}} t + \xi_0, \quad (36)$$

When $\delta = -1$

Case 4.

$$a_0 = 0, \quad a_1 = \frac{\pm A a_2}{\sqrt{A^2 - B^2}}, \quad c_1 = \frac{\pm 2a_2}{\sqrt{A^2 - B^2}}, \quad \alpha = \frac{a_2^2[\cos(d)+1]}{(A^2 - B^2)(\cos(d)-1)}, \; s = 1 \quad (37)$$

where a_2, A, B are arbitrary constants. In this case the rational trigonometric solitary wave solution for the discrete nonlinear Klein Gordon equation have the following form:

$$U(\xi_n) = \frac{\pm A a_2}{\sqrt{A^2 - B^2}\,[A\tan(\xi_n) + B\sec(\xi_n)]} \pm \frac{a_2 \sec(\xi_n)}{[A\tan(\xi_n) + B\sec(\xi_n)]}, \quad (35)$$

where

$$\xi_n = d\,n \mp \frac{2a_2}{\sqrt{A^2 - B^2}} t + \xi_0, \quad (36)$$

Note that, there are other cases which are omitted here for convenience.

4. Conclusion

In this paper, we put a direct method to calculate the rational solitary wave solutions some nonlinear difference differential equations via the the lattice equation, the discrete nonlinear Klein Gordon equation. As a result, many new and more rational solitary wave solutions are obtained, from which hyperbolic function solutions and trigonometric function solutions.

5. References

[1]. Fermi E, Pasta J, Ulam S. Collected papers of Enrico Fermi II. Chicago, IL: University of Chicago Press; 1965.
[2]. W.P. Su, J.R. Schrieffer, A.J. Heege, Solitons in polyacetylene, Phys. Rev. Lett. 42 (1979) 1698–1701.
[3]. A.S. Davydov, The theory of contraction of proteins under their excitation, J. Theor. Biol. 38 (1973) 559–569.
[4]. P. Marquii, J.M. Bilbault, M. Rernoissnet, Observation of nonlinear localized modes in an electrical lattice, Phys. Rev. E 51 (1995) 6127–6133.
[5]. M. Toda, Theory of Nonlinear Lattices, Springer, Berlin, 1989.
[6]. M. Wadati, Transformation theories for nonlinear discrete systems, Prog. Suppl. Theor. Phys. 59 (1976) 36–63.
[7]. Y. Ohta, R. Hirota, A discrete KdV equation and its Casorati determinant solution, J. Phys. Soc. Jpn. 60 (1991) 2095.
[8]. M.J. Ablowitz, J. Ladik, Nonlinear differential–difference equations, J. Math. Phys. 16 (1975) 598–603.
[9]. X.B. Hu, W.X. Ma, Application of Hirota's bilinear formalism to the Toeplitz lattice some special soliton-like solutions, Phys. Lett. A 293 (2002) 161–165.
[10]. D. Baldwin, U. Goktas, W. Hereman, Symbolic computation of hyperbolic tangent solutions for nonlinear differential–difference equations, Comput.Phys. Commun. 162 (2004) 203–217.
[11]. S.K. Liu, Z.T. Fu, Z.G. Wang, S.D. Liu, Periodic solutions for a class of nonlinear differential–difference equations, Commun. Theor. Phys. 49 (2008) 1155-1158.

[12]. C.Qiong, L.Bin, Application of Jacobi elliptic function expansion method for nonlinear difference equations, Commun. Theor. Phys. 43 (2005) 385-388.
[13]. F. Xie, M. Jia, H. Zhao, Some solutions of discrete sine-Gordon equation, Chaos Solitons Fractals 33 (2007) 1791-1795.
[14]. S.D. Zhu, Exp-function method for the hybrid-lattice system, Int. J. Nonlinear Sci. 8 (2007) 461.
[15]. Aslan, A discrete generalization of the extended simplest equation method, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 1967–1973.
[16]. P. Yang, Y. Chen, Z.B. Li, ADM-Padé technique for the nonlinear lattice quations, Appl. Math. Comput. 210 (2009) 362–375.
[17]. S.D. Zhu, Y.M. Chu, S.L. Qiu, The homotopy perturbation method for discontinued problems arising in nanotechnology, Comput. Math. Appl. 58 (2009)
[18]. S. Zhang, L. Dong, J. Ba, Y. Sun, The (G'/G)-expansion method for nonlinear differential–difference equations, Phys. Lett. A 373 (2009) 905–910.
[19]. Aslan, The Ablowitz–Ladik lattice system by means of the extended (G'/G)-expansion method,Applied Math.Comput.216 (2010)2778-2782.
[20]. S.Zhang, Discrete Jacobi elliptic function expansion method for nonlinear difference equation, Phys.Scr. 80(2009) 045002-045010.
[21]. G. Wu and T.Xia, A new method for constructing soliton solutions to differential- difference equation with symbolic computation, Chaos, Solitons and Fractals, 39 (2009) 2245- 2248.
[22]. F. Xie and J. Wang, A new method for solving nonlinear differential- difference equation, Chaos, Solitons and Fractals, 27 (2006) 1067-1071.
[23]. C. Liu, Exponential function rational expansion method for nonlinear differential- difference equations, Chaos, Solitons and Fractals, 40 (2009) 708-716.
[24]. Q. Wang and Y.Yu, New rational formal for (1+1)- dimensional Toda equation and another Toda equation, Chaos, Solitons and Fractals, 29 (2006) 904-915.
[25]. Trombettoni, A.Smerzi, Discrete solitons and breathers with Dilute Bose Einstein condensates, Phys.Rev.Lett. 86 (2001) 2353-2356.
[26]. L. Hadzievski, A.Maluckov, M.Stepic and D.Kip, Power controlled soliton stability and steering in lattices with saturable nonlinearity , Phys.Rev.Lett. 93 (2004) 033901.
[27]. S.Gate and J.Herrmann, Soliton Propagation in materials with saturable nonlinearity, J. Opt. Soc. Amer. B 9 (1991) 2296-2302.
[28]. S. Gate and J. Herrmann, Soliton Propagation and soliton collision in double doped fibers with a non- Kerr- like nonlinear refractive- index change, Opt. Lett. 17 (1992) 484-486.
[29]. Khaled A.Gepreel, Rational Jacobi Elliptic Solutions for Nonlinear Difference Differential Equations, Nonlinear Sci. Lett. A, 2 (2011) 151-158.
[30]. Khaled A.Gepreel and A.R. Shehata, Rational Jacobi elliptic solutions for nonlinear differential–difference lattice equations, Applied Mathematics Letters 25 (2012) inpress.
[31]. F. Xie, New traveling wave solutions of the generalized coupled Hirota Satsuma KdV system, Chaos, Solitons and Fractals, 20 (2004) 1005-1012.
[32]. C.S.Liu, Expanential function rational expansion method for nonlinear differential difference equation, Chaos, Solitons and Fractals, 40 (2009) 708-716.
[33]. P.Kevrekidis and V.Konotop , Bright comoact breathers, Phys. Revi. 65 (2002) 066614.
[34]. P. Kevrekidis, V. Konotop and A. Bishop , Discrete comoactions: some exact results, J.Phys. A.Math.Theor. 35 (2002) L641-L652.