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Jacobian consistency property analysis for generalized complementarity problems

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Abstract. In this paper, we present the Jacobian consistency property for generalized complementarity problems. The idea is based on a nonsmooth system of equations reformulation of the generalized complementarity problem and a smoothing approximation function for solving the nonsmooth system of equations.

Keywords: Jacobian consistency property; generalized complementarity problem; smoothing approximation function

1. Introduction

Let $F,G: R^n \to R^n$ are any two continuously differentiable functions. The generalized complementarity problems is to find a solution of the following problem

$$F(x) \geq 0, G(x) \geq 0, F(x)^T G(x) = 0, \tag{1.1}$$

In this paper, (1.1) is also denoted GCP(F, G) as denoted in [1-3]. When $F(x) = x$, GCP(F, G) reduces to the nonlinear complementarity problems(NCP), which has a large number of important applications and a lot of effort has been spent on the nonlinear complementarity problems, see [4-6].

The basic idea of most methods for solving (1.1) is to reformulate this problem as a nonsmooth system of equations. In this paper, we also concentrate ourself on the equation-based approach where (1.1) is written equivalently as

$$\phi(x) = 0$$

for a suitable equation-operator $\phi: R^n \to R^n$. The reformulation is based on the Fischer-Burmeister function $\varphi: R^2 \to R$ defined by

$$\varphi(a,b) = \sqrt{a^2 + b^2} - a - b.$$

Then, we can see that (1.1) is equivalent to the following problem

$$\phi(x) = \begin{pmatrix} \varphi(F_1, G_1) \\ \vdots \\ \varphi(F_n, G_n) \end{pmatrix} = 0. \tag{1.2}$$

From the property of Fischer-Burmeister function, x solves (1.1) if and only if x solves (1.2). Because ϕ is nonsmooth, instead of solving (1.2) by the classical Newton method, we can apply a nonsmooth Newton method to solving it. This methods are locally superlinearly or quadratically convergent, but can not be globalized in a simple way for general function ϕ (for $\phi^T \phi$ is usually not continuously differentiable). The way to deal with the nonsmoothness of ϕ is to approximate the function by a smooth function $\phi_\mu : R^n \to R^n$, where $\mu > 0$ denotes the smoothing parameter. Having this in mind, in this paper, we give Jacobian consistency property analysis for GCP(F, G).

The organization of this paper is as follows. The preliminaryresults and the Jacobian consistency property are discussed in the following section. In the last section, we conclude this paper with some remarks.

Notation. Throughout this paper, if $H: R^n \to R^m$ is continuously differentiable, then $H'(x) \in R^{m\times n}$ denotes the Jacobian of H at a point x. Its ith component function is denoted by H_i, the gradient of H_i at x, denoted by $\nabla_x H_i(x)$, is a column vector whose j th component is $\frac{\partial H_j(x)}{\partial x_j}$.

2. Preliminaries and Jacobian Consistency Property

Note that ϕ in (1.2) is locally Lipschitz continuous. By Rademacher theorem, ϕ is differentiable almost everywhere. The B-differential of ϕ is defined by

$$\partial_B \phi(x) = \left\{ \lim_{x_k \to x, x_k \in D_\phi} \phi'(x_k) \right\},$$

where D_ϕ is the differentiable set of ϕ. The Clarke Jacobian of ϕ is defined by

$$\partial \phi(x) = conv \partial_B \phi(x).$$

Denote

$$\Omega = \left\{ x \in R^n \mid (F_i(x))^2 + (G_i(x))^2 > 0, i = 1, \cdots, n \right\}.$$

We know that ϕ is locally Lipschitz on R^n and F-differentiable on the set Ω. In the remainder of this paper, we use a kind of generalized Jacobian, denoted by $\partial_C \phi(x)$ for ϕ and defined as the following

$$\partial_C \phi(x) = D^F(x) \nabla F(x)^T + D^G(x) \nabla G(x)^T,$$

where D^F and D^G are sets of $n \times n$ diagonal matrices. Each pair

$$(D^F(x), D^G(x)) = (diag\{D_1^F, \cdots, D_n^F\}, diag\{D_1^G, \cdots, D_n^G\}),$$

for $i = 1, \cdots, n$, satisfied the following conditions

$$(D_i^F + 1)^2 + (D_i^G + 1)^2 \le 1.$$

If $(F_i(x))^2 + (G_i(x))^2 > 0$, we have

$$D_i^F = \frac{F_i(x)}{\sqrt{(F_i(x))^2 + (G_i(x))^2}} - 1, D_i^G = \frac{G_i(x)}{\sqrt{(F_i(x))^2 + (G_i(x))^2}} - 1.$$

If $(F_i(x), G_i(x)) = (0,0)$, we can choose $D_i^F = \varsigma_i - 1, D_i^G = \rho_i - 1$, where $(\varsigma_i, \rho_i) \in R^2$ and satisfies $\|(\varsigma_i, \rho_i)\| \le 1$. By the definition of Clarke Jacobian, we have $\partial \phi(x) \subseteq \partial_C \phi(x)$. This definition is also discussed in [2].

We are now able to give the following definition, which are borrowed from [3].

Definition 2.1. Let ϕ be a Lipschitz continuous function in R^n. We call $\tilde{f} : R^{n+1} \to R^n$ a smoothing approximation function of ϕ if $\tilde{f}$ is continuously differentiable with respect to the first variable and there is a constant $\mu > 0$ such that for any $x \in R^n$ and $\varepsilon \in R_{++}$,

$$\left\| \tilde{f}(x, \varepsilon) - \phi(x) \right\| \le \mu \varepsilon.$$

Furthermore, if for any $x \in R^n$,

$$\lim_{\varepsilon \downarrow 0} dist((\nabla_x \tilde{f}(x, \varepsilon))^T, \partial_C \phi(x)) = 0,$$

then we say $\tilde{f}$ satisfies the Jacobian consistency property.

In this paper, the corresponding smoothing approximation function ϕ_μ for function ϕ in (1.2) is defined by

$$\phi_\mu(x) = \begin{pmatrix} \psi_\mu(F_1, G_1) \\ \vdots \\ \psi_\mu(F_n, G_n) \end{pmatrix}, \tag{2.1}$$

where $\psi_\mu = \sqrt{a^2 + b^2 + 2\mu} - a - b, \mu > 0$, (defined by Kanzow in [5]).

Proposition 2.1. The function ϕ_μ defined in (2.1) satisfies the following inequality

$$\left\|\phi_{\mu_1}(x)-\phi_{\mu_2}\right\| \le \sqrt{2n}\,|\sqrt{\mu_1}-\sqrt{\mu_2}\,|$$

for $\forall x \in R^n$ and $\mu_1, \mu_2 \ge 0$. And, we also can get

$$\left\|\phi_{\mu}(x)-\phi(x)\right\| \le \sqrt{2n}\sqrt{\mu}$$

for $\forall x \in R^n$ and $\mu \ge 0$.

In the following, we give the main result of this paper, which isdenoted by Jacobian consistency property.

Theorem 2.1. Let $x \in R^n$ be arbitrary but fixed. Then we have

$$\lim_{\mu\downarrow 0} dist(\phi'_\mu(x), \partial_C \phi(x)) = 0.$$

Proof. By the definition of ϕ_μ, we can get

$$\phi'_\mu(x) = diag(\frac{F_i(x)}{\sqrt{F_i^2(x)+G_i^2(x)+2\mu}}-1)F'(x) + diag(\frac{G_i(x)}{\sqrt{F_i^2(x)+G_i^2(x)+2\mu}})G'(x).$$

Now, we consider the distance of the columns of the transposed Jacobian. Denote $\Theta = \{i \mid F_i(x) = G_i(x) = 0\}$ and the i th component function of ϕ_μ by $\phi_{\mu,i}$, we can get

$$\lim_{\mu\downarrow 0}\nabla\phi_{\mu,i}(x) = (\frac{F_i(x)}{\sqrt{F_i^2(x)+G_i^2(x)}}-1)\nabla F_i(x) + (\frac{G_i(x)}{\sqrt{F_i^2(x)+G_i^2(x)}}-1)\nabla G_i(x), i \in \Omega$$

$$\lim_{\mu\downarrow 0}\nabla\phi_{\mu,i}(x) = -\nabla F_i(x) - \nabla G_i(x), i \in \Theta.$$

So, we get the theorem by the definition of $\partial_C \phi(x)$ with $(\varsigma_i, \rho_i) = (0,0)$ when $i \in \Theta$.

Discussion 2.1. By the above theorem, we can know that for every fixed $\delta > 0$ and a parameter $\overline{\mu} = \overline{\mu}(x,\delta) > 0$ such that $dist(\phi'_\mu(x), \partial_C \phi(x)) \le \delta, 0 < \mu < \overline{\mu}.$

Discussion 2.2. We can replace the generalized Newton equation

$$V_k d = -\phi(x_k), V_k \in \partial_C \phi(x),$$

by the linear equation

$$\phi'_{\mu_k}(x_k)d = -\phi(x_k)$$

for solving (1.2). By Theorem 2.1, we also can guarantee local fast convergence of the above iteration.

3. Final Remarks

In this section, we give some of the remarks of the function ϕ and some properties about the natural merit function $\Psi = \frac{1}{2}\phi(x)^T\phi(x).$

Remark 3.1. Let $\{x_k\}$ is any convergent sequence and $\lim_{k\to+\infty} x_k = x^*$. Then ϕ is semismooth and

$$\left\|\phi(x_k)-\phi(x^*)-V_k(x_k-x^*)\right\| = o(\left\|x_k-x^*\right\|)$$

for $V_k \in \partial_C \phi(x_k)$.

Remark 3.2. The natural merit function Ψ is continuously differentiable and $\nabla\Psi(x) = V^T\phi(x)$, where $V \in \partial_C \phi(x).$

Remark 3.3. Let $\{x_k\}$ and $\{\mu_k\}$ are two any convergent sequences and $\lim_{k\to+\infty} x_k = x^*$, $\{\mu_k\}\downarrow 0$. Then we get the following two equations

$$\lim_{k\to\infty}\nabla\Psi_{\mu_k}(x_k) = \nabla\Psi(x^*),$$

$$\lim_{k\to\infty}\nabla\phi'_{\mu_k}(x_k)^T\phi(x_k) = \nabla\Psi(x^*).$$

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4. References

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