ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 7, No. 4, 2012, *pp*. 267-271

A nonsmooth Levenberg-Marquardt method for generalized complementarity problem

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(*Received May 18, 2012, accepted October 10, 2012*)

Abstract. A new method for the solution of the generalized complementarily problem is introduced. The method is based on a no smooth equation reformulation of the generalized complementarily problem and on a no smooth Levenberg-Marquardt method for its solution. The method is shown to be globally convergent. Numerical results are also given.

Keywords: Nonsmooth Levenberg-Marquardt method; generalized complementarity problem; global convergence.

1. Introduction

Complementarity theory is a branch of the mathematical sciences with a wide range of applications in industry, physical, regional, and engineering sciences. In this paper, we consider the following generalized complementarity problem

$$F(x) \geq 0, G(x) \geq 0, F(x)^T G(x) = 0, \tag{1}$$

where $F, G : R^n \to R^n$ are any two continuously differentiable functions. This problem is denoted GCP(F, G) .(see[1-2].) Several problems arising in different fields, such as game theory, mathematical programming, mechanics and geometry, have the same mathematical form which may be stated as (1.1). And (1.1) covers some related problems, such as if $F(x) = x$, then (1.1) reduces to the nonlinear complementarity problem. In the past years, several investigators have been concerned with both the theoretical and computational aspects of the above problem. Several important results have been established(see [1-10]).

In this paper, we consider a nonsmooth Levenberg-Marquardt method with Goldstein line search for generalized complementarity problem. This paper is organized as follows. In the next section, we introduce the nonsmooth Levenberg-Marquardt method and the global convergence of the method. Finally, numerical experimental results are presented.

2. New Levenberg-Marquardt method and its convergence

In this section, we describe a nonsmooth Levenberg-Marquardt method for generalized complementarity problem. In paper [1], C.Kanzow, M.Fukushima have studied an unconstrained minimization reformulation of (1.1). The merit function is based on the function

$$\varphi(a,b) = \sqrt{a^2 + b^2} - a - b.$$

The approach presented in this paper is similar, but we use a different merit function, which is based on the following function

$$\min\{a, b\}$$

where "min" denotes the componentwise minimum operator. When x satisfied

$$\begin{aligned} &\min\{F_1(x), G_1(x)\} = 0, \\ &\vdots \\ &\min\{F_n(x), G_n(x)\} = 0, \end{aligned} \tag{2}$$

x solves (1). Throughout this section, we denote

$$h_i(x) = \min\{F_i(x), G_i(x)\}, x \in R^n, i = 1,2,\cdots,n,$$

$$H(x) = (h_1(x),\cdots,h_n(x))^T, x \in R^n,$$

Thus, the equations (2) can be briefly rewritten as

$$H(x) = (h_1(x),\cdots,h_n(x))^T = 0, \tag{3}$$

which is nonsmooth equations. For solving the systems of equations, we take ∂_* as a tool instead of the Clarke generalized Jacobian, B-differential and b-differential. We give the following $\partial_* H(x)$ for H in (3)

$$\partial_* H(x) = \{(\nabla h_1(x),\cdots,\nabla h_n(x))^T, x \in R^n\}, \tag{4}$$

where $\nabla h_i(x) = \nabla F_i(x),$ if $F_i(x) < G_i(x)$, $\nabla h_i(x) = \nabla F_i(x)$ or $\nabla h_i(x) = \nabla G_i(x),$ if $F_i(x) = G_i(x)$, $\nabla h_i(x) = \nabla G_i(x),$ if $F_i(x) > G_i(x)$. In what follows, we use (4) as a tool instead of the Clarke generalized Jacobian and b-differential in nonsmooth Levenberg-Marquardt method.

Proposition 2.1 Suppose that $H(x)$ and $\partial_* H(x)$ are defined by (3) and by (4), and all $V \in \partial_* H(x)$ are nonsingular. Then there exists a scalar $\xi > 0$ such that

$$\|V^{-1}\| \le \xi, \ \forall V \in \partial_* H(x).$$

$$\|V\| \le \vartheta, \ \forall V \in \partial_* H(x), x \in N(x,\varepsilon),$$

holds for some constants $\vartheta > 0, \varepsilon > 0$ and $N(x,\varepsilon)$ is a neighbor of x.

By the continuously differentiable property of F and G in (1.1), the above Proposition 2.1 can be easily obtained.

Denote the corresponding merit function as

$$\psi(x) = \frac{1}{2}\|H(x)\|^2.$$

We assume that the above merit function is continuously differentiable. Now, we give the following nonsmooth Levenberg-Marquardt method with Goldstein line search for generalized complementarity problem (1.1).

Nonsmooth Levenberg-Marquardt method with Goldstein line search

Step 0. Given a staring vector $x_0 \in R^n, \rho > 0, p > 2, \sigma \in (0, \frac{1}{2}), \varepsilon \ge 0.$

Step 1. If $\psi(x_k) \le \varepsilon,$ stop.

Step 2. Select an element $V_k \in \partial_* H(x_k)$, find an approximate solution $d_k \in R^n$ of the system

$$((V_k)^T V_k + \lambda_k I)d = -(V_k)^T H(x_k), \tag{5}$$

where $\lambda_k \ge 0$ is Levenberg-Marquardt parameter. If the condition

$$\nabla \psi(x_k)^T d_k \le -\rho \|d_k\|^p \tag{6}$$

is not satisfied, set $d_k = -(V_k)^T H(x_k)$.

Step 3. Find α_k by Goldstein line search

$$\psi(x_k) + (1-\sigma)\alpha_k \nabla \psi(x_k)^T d_k \le \psi(x_k + \alpha_k d_k), \tag{7}$$

$$\psi(x_k) + \sigma\alpha_k \nabla \psi(x_k)^T d_k \ge \psi(x_k + \alpha_k d_k). \tag{8}$$

Set $x_{k+1} = x_k + \alpha_k d_k$, let $k := k+1$, and go to Step 1.

In what follows, as usual in analyzing the behavior of algorithms, we assume that the above method produces an infinite sequence of points. Based upon the above method, we give the following global convergence result about nonsmooth Levenberg-Marquardt method with Goldstein line search for solving generalized complementarity problem (1). The main proof of the following theorem is similar to Theorem 12 in [4]. But the $\partial_* H(x)$ in (5) and the line search (7), (8), which is used for the solution of α_k is differing to Theorem 12 in [4].

Theorem 2.1 Suppose that the sequence $\{\lambda_k\}$ is bounded. Then each accumulation point of the sequence x_k generated by the above method is a stationary point of ψ.

Proof Assume that $\{x_k\}_K \to x^*$. If there are infinitely many $k \in K$ such that $d_k = -\nabla \psi(x_k)$, then the assertion follows immediately from Proposition 1.9 and Proposition 1.16 in [10]. Hence we can assume without loss of generality that if $\{x_k\}_K$ is a convergent subsequence of $\{x_k\}$, then d_k is always given by (5). We show that for every convergent subsequence $\{x_k\}_K$ for which

$$\lim_{k\in K, k\to\infty} \nabla \psi(x_k) \neq 0, \tag{9}$$

there holds

$$\limsup_{k\in K, k\to\infty} \|d_k\| < \infty, \tag{10}$$

and

$$\limsup_{k\in K, k\to\infty} \left|\nabla \psi(x_k)^T d_k\right| > 0. \tag{11}$$

In the following, we assume that $x_k \to x^*$. Suppose that x^* is not a stationary point of ψ. By (5), we have

$$\|\nabla \psi(x_k)\| = \left\|((V_k)^T V_k + \lambda_k I) d_k\right\| \le \left\|(V_k)^T V_k + \lambda_k I\right\| \cdot \|d_k\|, \tag{12}$$

So

$$\|d_k\| \ge \frac{\|\nabla \psi(x_k)\|}{\left\|(V_k)^T V_k + \lambda_k I\right\|}.$$

Note that the denominator in the above inequality is nonzero, otherwise by (2.11), we have $\|\nabla \psi(x_k)\| = 0$. x_k would be a stationary point and the algorithm would have stopped. By assumption $\|\lambda_k I\| \le M < +\infty$ and Proposition 2.1, there exists a constant $k_1 > 0$ such that

$$\left\|(V_k)^T V_k + \lambda_k I\right\| \le k_1$$

from the above inequality, we get

$$\|d_k\| \ge \frac{1}{k_1} \|\nabla \psi(x_k)\| \tag{13}$$

Formula (10) now readily follows from the fact that we are assuming that the direction satisfies (6) with $p > 2$, while the gradient $\nabla \psi(x_k)$ is bounded on the convergent sequence $\{x_k\}$. If (11) is not satisfied there exists a subsequence $\{x_k\}_{K'}$ of $\{x_k\}_K$,

$$\lim_{k\in K', k\to\infty} \left|\nabla \psi(x_k)^T d_k\right| = 0.$$

This implies, by (6), that

$$\lim_{k\in K', k\to\infty} \|d_k\| = 0.$$

Together with (13) implies

$$\lim_{k\in K', k\to\infty} \|\nabla \psi(x_k)\| = 0.$$

contradicting (9). The sequence $\{d_k\}$ is uniformly gradient related to $\{x_k\}$ according to the definition given in [10] and the assertion of the theorem also follows from Proposition 10 and Proposition 17 in [10]. We complete the proof.

1. **Remark 2.1** Suppose that (1) has a nonempty solution set, $x^* \in R^n$ is a solution of (1) if and only if $\psi(x^*) = 0$.

3. Numerical results

In this section, in order to show the performance of the above nonsmooth Levenberg-Marquardt method with Goldstein line search, we present some numerical results for the nonsmooth Levenberg-Marquardt method with Goldstein line search. The results indicate that the method work quit well in practice. We coded the algorithms in Matlab 7.0.

Example 3.1 We consider the generalized complementarity problem (1.1), where the functions

$$F(x_1, x_2) = (x_1^2, 2x_1^2 + 3x_2^2)^T,\ G(x_1, x_2) = (4x_1^2 + 10, x_1^2)^T$$

Both F and G are $R^2 \to R^2$ continuously differentiable functions.

We use nonsmooth Levenberg-Marquardt method with Goldstein line search to compute Example 3.1. Results for Example 3.1 with initial point $x_0 = (100,10)^T$ are presented in Table 3.1.

Table 3.1

Step	$\psi(x)$
2	6.250003125000194e+006
3	3.906259765629028e+005
4	2.441431884812087e+004
5	1.525943756584212e+003
6	95.38369227246184
7	5.96453298026289
8	0.37354670617196
9	0.02353785886130
10	0.01209032248827
11	0.00758723940013
12	0.00352828184957
13	0.00212747182949
14	9.409438865392813e-004
15	5.573881839067386e-004
16	3.748295466710731e-004
17	2.715241235025840e-004
18	2.067091366900774e-004
19	1.631051000343907e-004
20	1.322442231610374e-004
21	1.095398238136108e-004
22	9.231658839608631e-005

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