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Computational Method for Nonlinear Singularly Perturbed Singular Boundary Value Problems using Nonpolynomial Spline

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Abstract. We report a fourth order accurate numerical technique via nonpolynomial spline for singularly perturbed singular two point boundary value problems of the form

$$-\varepsilon u''(r)+f(r,u,u')=0,\ \ u(a)=A,\ u(b)=B.$$

The numerical scheme is developed for problems arising in the various fields of science and engineering. The scheme is three point nonlinear systems of equations. The method is applied to a few test examples to illustrate the accuracy and the implementation of the method.

Keywords: Non polynomial spline, Singular perturbation, Singular equation, Boundary layer, Taylor's series, Root mean square errors.

1. Introduction

Consider the following nonlinear singular perturbation problems (SPP)

$$-\varepsilon u''(r)+f(r,u,u')=0,\ u(a)=A,\ u(b)=B,\ a\le r\le b \tag{1}$$

where $0<\varepsilon<<1$, A and B are finite constants and assuming that f is bounded and smooth function satisfying

$$\frac{\partial f}{\partial u}\ge 0,\ \frac{\partial f}{\partial v}\le 0,\ \frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}\ge\lambda>0$$

$$\lim_{v\to\infty} f(r,u,v)=\mathrm{O}\left(|v|^2\right),\ a\le r\le b \text{ and } u,v\in R$$

Howes [1], suggested that under the above conditions, the problem (1) posses a unique solution. SPP occur in many branches of science and engineering such as heat transport problems with large Peclet numbers, Navier-Strokes flows with large Reynolds numbers, convection-diffusion process, gas porous electrodes theory, fluid dynamics, chemical kinetics, modeling of steady and unsteady viscous flow problems. The solution of SPP exhibits a multi-scale character. There are many methods based on finite difference, boundary element, collocations method etc. available for solving linear SPP[2-10]. Recently, Tirmizi [11], have proposed a nonpolynomial spline method for linear singular perturbation problems which has second and fourth order of convergence depending upon the choice of free parameters. Kadalbajoo and Patidar [12] has considered second order convergent spline in compression technique for the nonlinear singular perturbation problems. However, their methods are only applicable to non-singular problems. Difficulties were experienced in the past for the numerical solution of singularly perturbed singular two point boundary value problems in polar coordinates. The solution usually deteriorates in the vicinity of singularity. The aim of this paper is to design a computationally efficient numerical technique based on nonpolynomial spline and finite difference approximations in such a way that fourth order convergence is retained for smaller values of ε and restriction on grid size can be avoided in case of singularity.

In this paper, we are concerned with the problem of applying nonpolynomial spline functions to develop numerical schemes for obtaining approximate solution for the nonlinear singular two point boundary value

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problems. The C^{∞} - differentiability of the trigonometric part of nonpolynomial spline basis compensates for the loss of smoothness inherited by polynomial splines. The resulting nonpolynomial spline three point difference schemes are of fourth order accuracy. The importance of our work is that the proposed methods are applicable to problems both in rectangular and polar coordinates.

The paper is organized as follows: In section 2, we give a brief description of the mathematical method. In section 3, we design difference schemes of class of singular equation in operator compact form. Some nonlinear singular and nonsingular examples are illustrated to justify the accuracy and efficiency of the proposed method in section 4. The numerical results exhibit oscillation free solution for $0 < \varepsilon << 1$, even in the vicinity of the singularity.

2. Nonpolynomial Spline Finite Difference Method

For the numerical approximation of problems (1), we divided the domain Ω = [*a*, *b*] into a set of nodes with interval of $h = 1/(N+1), N$ being a positive integer. The nonpolynomial spline approximations is obtained on Ω that consists of the central point $r_k = a + kh$ and two neighbouring grids $r_{k\pm1}$. The approximate solution of this equation is sought in the form of the function $S_k(r)$ which interpolates $f(r, u, u')$ at r_k defined as follows

$$S_k(r) = \alpha_k \sin\tau(r - r_k) + \beta_k \cos\tau(r - r_k) + \gamma_k(r - r_k) + \delta_k, \ k = 0(1)N \tag{2}$$

where α_k, β_k, γ_k, and δ_k are constants and τ is the frequency of the trigonometric functions. Thus, the cubic nonpolynomial spline is defined by the relations:

$$\text{(i)} \ S(r) \in C^{\infty}(\Omega) \tag{3}$$

$$\text{(ii)} \ S''(r_k) = M_k, \ S(r_k) = u_k, \ k = 0(1)N+1$$

We obtain via algebraic calculations the following expressions

$$\alpha_k = \frac{h^2}{\theta^2 \sin\theta}(M_k \cos\theta - M_{k+1}), \quad \beta_k = -\frac{h^2}{\theta^2} M_k$$

$$\gamma_k = \frac{1}{h}(u_{k+1} - u_k) + \frac{h}{\theta^2}(M_{k+1} - M_k), \quad \delta_k = u_k + \frac{h^2}{\theta^2} M_k$$

where $\theta = h\tau$.

Following, Islam and Tirmizi [11, 13] and , Rashidinina *et. al.* [14], we obtain

$$u_{k-1} - 2u_k + u_{k+1} - h^2(\alpha M_{k-1} + 2\beta M_k + \alpha M_{k+1}) = 0, \ k = 1(1)N \tag{4}$$

where, $\alpha = \dfrac{\theta - \sin\theta}{\theta^2 \sin\theta}$, $\beta = \dfrac{\sin\theta - \theta\cos\theta}{\theta^2 \sin\theta}$

Consider the following approximations

$$\hat{u}'_k = \frac{u_{k+1} - u_{k-1}}{2h}, \quad \hat{u}'_{k\pm1} = \frac{\pm 3u_{k\pm1} \mp 4u_k \pm u_{k\mp1}}{2h}, \ \hat{H}_{k\pm1} = f(r_{k\pm1}, u_{k\pm1}, \hat{u}'_{k\pm1})$$

$$\hat{\hat{u}}'_k = \hat{u}'_k + h\omega(\hat{H}_{k+1} - \hat{H}_{k-1}) \tag{5}$$

The above nonpolynomial spline finite difference approximation for $\alpha = \dfrac{1}{12}$, $\beta = \dfrac{5}{12}$, have local truncation errors of $-\dfrac{h^4}{12}(1 + 20\varepsilon\omega)u'''(r_k) + O(h^6)$.

3. Application to Singular Problems

We discuss the application of method (4), for the numerical solution of model problems of various classes of singular perturbation problems. Consider the following singularly perturbed model problem

$$-\varepsilon u'' + a(r)u' + b(r)u + \delta_1 e^u + \delta_2 u^2 + \delta_3 (u')^2 + \delta_4 uu' + g(r) = 0, \quad r \in \Omega \tag{6}$$

where $a(r)=\frac{\lambda}{r}$ and $b(r)=-\frac{\lambda}{r^2}$. For $\lambda = 1$ and 2 the above equation represents in cylindrical and spherical symmetry respectively. The model problem is considered in such a way that self-adjoint singularly perturbed problem and general linear singularly perturbed two point boundary value problems are the particular cases of the equation (6).

Now, we discuss the application of nonpolynomial spline formula (4) and finite difference approximations (5) to the nonlinear singularly perturbed singular equation (6), we obtain

$$\begin{aligned}
\varepsilon\delta_r^2 u_k &= 2h^2\omega\beta a_k a_{k-1}\left(\delta_r^2 - \mu_r\delta_r\right)u_k + 2h^2\omega\beta a_k a_{k+1}\left(\delta_r^2 + \mu_r\delta_r\right)u_k \\
&+ h^3\omega\beta a_k b_{k-1}\left(2\mu_r\delta_r - \delta_r^2 - 2\right)u_k + h^3\omega\beta a_k b_{k+1}\left(2\mu_r\delta_r + \delta_r^2 + 2\right)u_k \\
&+ 2h^3\omega\beta a_k\left(g_{k+1} - g_{k-1}\right) + 2h^3\omega\beta a_k\theta_2\left(2+\delta_r^2\right)u_k\left(2\mu_r\delta_r\right)u_k \\
&+ 2h^3\omega\beta a_k\theta_1\left(e^{\mu_r\delta_r u_k} - e^{-\mu_r\delta_r u_k}\right)e^{\left(1+\frac{1}{2}\delta_r^2\right)u_k} \\
&+ h^2\omega\beta\theta_4 a_k\left(u_k\delta_r^2 u_k + \left(2\mu_r\delta_r u_k\right)^2 + 2\delta_r^4 u_k\right) + h\beta a_k\left(4\omega\theta_3\delta_r^2 + 1\right)u_k\left(2\mu_r\delta_r\right)u_k \\
&+ h\alpha a_{k-1}\left(\mu_r\delta_r - \delta_r^2\right)u_k + h\alpha a_{k+1}\left(\mu_r\delta_r + \delta_r^2\right)u_k + 2h^2\beta b_k u_k \\
&+ h^2\alpha b_{k-1}\left(1 - \mu_r\delta_r + \frac{1}{2}\delta_r^2\right)u_k + h^2\alpha b_{k+1}\left(1 + \mu_r\delta_r + \frac{1}{2}\delta_r^2\right)u_k \\
&+ h^2\alpha\left(g_{k+1} + g_{k-1}\right) + 2h^2\beta g_k + 2\alpha\theta_3\delta_r^4 u_k \\
&+ h^2\alpha\theta_2\left(2u_k\delta_r^2 u_k + \frac{1}{2}\left(2\mu_r\delta_r u_k\right)^2 + \frac{1}{2}\delta_r^4 u_k + 2u_k^2\right) \\
&+ h\theta_4\left(\alpha\left(1+\frac{3}{2}\delta_r^2\right) + \beta\right)u_k\left(2\mu_r\delta_r\right)u_k + \frac{1}{2}\theta_3(\alpha+\beta)\left(2\mu_r\delta_r u_k\right)^2 \\
&+ h^2\alpha\theta_1\left(e^{\mu_r\delta_r u_k} + e^{-\mu_r\delta_r u_k}\right)e^{\left(1+\frac{1}{2}\delta_r^2\right)u_k} + 2h^2\beta\left(\theta_1 e^{u_k} + \theta_2 u_k^2\right) + O\left(h^6\right)
\end{aligned} \tag{7}$$

The nonpolynomial spline finite difference method (7) is of order four for $\omega = -\frac{1}{20\varepsilon}$ (see Bawa [2]). However, the method fails when the coefficients $a(r), b(r)$ and $g(r)$ contains singularities and the solutions are to be determined at $k = \pm 1$. We overcome this difficulty by modifying the scheme (7) in such a way that solutions retain the order and accuracy even in the vicinity of the singularity. We use following Taylor's approximation

$$a_{k\pm1} = a_k \pm h a_k' + \frac{h^2}{2}a_k'' \pm \frac{h^3}{6}a_k''' + \frac{h^4}{24}a_k'''' \pm \frac{h^5}{120}a_k''''' + O\left(h^6\right) \tag{8}$$

Using the Taylor's approximation for $a_{k\pm1}$, $b_{k\pm1}$ and $g_{k\pm1}$ in equation (7) and neglecting $O\left(h^6\right)$ terms, we obtain the following nonpolynomial spline schemes in operator compact form

$$\begin{aligned}
\Delta_k \equiv\ & 2h^3\beta\omega\theta_1 a_k e^{u_k}\left(2\mu_r\delta_r\right)u_k + h^2\theta_1 e^{u_k}\left(\frac{1}{4}\alpha\left(2\mu_r\delta_r u_k\right)^2 + 2\beta + 2\alpha + \alpha\delta_r^2 u_k\right) \\
&+ h^4\left(\alpha g_k'' + \alpha b_k'' u_k + 4\beta\omega a_k b_k' u_k + 4\beta\omega a_k g_k'\right) \\
&+ h^3\left(\alpha b_k' + 4\beta\omega\theta_2 a_k u_k + \frac{1}{2}\alpha a_k'' + 2\beta\omega a_k\left(b_k + a_k'\right)\right)\left(2\mu_r\delta_r\right)u_k
\end{aligned}$$

$$+h^2\left(\beta\omega\theta_4 a_k+\frac{1}{2}\alpha\theta_2\right)\left(2\mu_r\delta_r u_k\right)^2$$
$$+h^2\left(2\alpha\theta_2 u_k+4\beta\omega\theta_4 a_k u_k+4\beta\omega a_k^2+2\alpha a_k'+\alpha b_k\right)\delta_r^2 u_k$$
$$+h^2\left(2\alpha b_k u_k+2\alpha\theta_2 u_k^2+2\beta\theta_2 u_k^2+2\beta b_k u_k+2(\alpha+\beta)g_k\right)$$
$$+h\left(\left(\frac{3}{2}\alpha\theta_4+4\beta\omega\theta_3 a_k\right)\delta_r^2 u_k+\alpha\theta_4 u_k+\alpha a_k+\beta\theta_4 u_k+\beta a_k\right)\left(2\mu_r\delta_r\right)u_k$$
$$+\frac{\theta_3}{2}(\alpha+\beta)\left(2\mu_r\delta_r u_k\right)^2+2\alpha\theta_3\left(\delta_r^2 u_k\right)^2-\varepsilon\delta_r^2 u_k \qquad (9)$$

Note that the nonpolynomial spline difference scheme (9) is of fourth order accurate and free from the terms $1/(k\pm 1)$ and hence, easily solved for $k=1(1)N$.

4. Computational Results

Table 1: Root Mean Square Errors for Example 1.

N \ ε	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
$\theta_1=0, \theta_2=0, \theta_3=1, \theta_4=0$							
16	4.17e-06	2.09e-06	1.04e-06	5.21e-07	2.61e-07	1.30e-07	6.52e-08
32	1.09e-06	5.45e-07	2.72e-07	1.36e-07	6.81e-08	3.41e-08	1.70e-08
64	2.79e-07	1.39e-07	6.97e-08	3.48e-08	1.74e-08	8.71e-09	4.35e-09
128	7.05e-08	3.52e-08	1.76e-08	8.81e-09	4.41e-09	2.20e-09	1.10e-09
256	1.77e-08	8.86e-09	4.43e-09	2.22e-09	1.11e-09	5.54e-10	2.77e-10
512	4.44e-09	2.22e-09	1.11e-09	5.56e-10	2.78e-10	1.39e-10	6.94e-11
1024	1.11e-09	5.56e-10	2.78e-10	1.39e-10	6.95e-11	3.48e-11	1.74e-11
$\theta_1=1, \theta_2=0, \theta_3=0, \theta_4=0$							
16	1.07e-08	4.51e-09	1.51e-09	7.41e-10	9.29e-10	2.29e-10	8.83e-11
32	7.46e-10	3.13e-10	1.04e-10	5.14e-11	6.92e-11	1.82e-11	8.95e-12
64	4.92e-11	2.07e-11	6.89e-12	3.39e-12	4.58e-12	1.21e-12	6.30e-13
128	3.16e-12	1.33e-12	4.42e-13	2.18e-13	2.94e-13	7.82e-14	4.09e-14
256	2.00e-13	8.40e-14	2.80e-14	1.38e-14	1.86e-14	4.80e-15	2.56e-15
512	1.26e-14	5.29e-15	1.75e-15	8.57e-16	1.22e-15	3.72e-16	2.40e-16
1024	7.23e-16	3.25e-16	8.73e-17	2.94e-17	1.04e-16	4.87e-17	1.79e-16
$\theta_1=0, \theta_2=1, \theta_3=0, \theta_4=0$							
16	6.44e-09	3.22e-09	1.61e-09	8.05e-10	4.03e-10	2.01e-10	1.01e-10
32	4.47e-10	2.24e-10	1.12e-10	5.59e-11	2.79e-11	1.40e-11	6.99e-12
64	2.95e-11	1.47e-11	7.37e-12	3.68e-12	1.84e-12	9.21e-13	4.61e-13
128	1.89e-12	9.46e-13	4.73e-13	2.37e-13	1.18e-13	5.92e-14	2.96e-14
256	1.20e-13	6.00e-14	3.00e-14	1.50e-14	7.49e-15	3.75e-15	1.87e-15
512	7.55e-15	3.77e-15	1.89e-15	9.43e-16	4.72e-16	2.36e-16	1.18e-16
1024	4.71e-16	2.35e-16	1.18e-16	5.88e-17	2.94e-17	1.47e-17	7.35e-18
$\theta_1=0, \theta_2=0, \theta_3=0, \theta_4=1$							
16	1.20e-06	6.00e-07	3.00e-07	1.50e-07	7.50e-08	3.75e-08	1.87e-08
32	3.13e-07	1.57e-07	7.83e-08	3.92e-08	1.96e-08	9.79e-09	4.89e-09
64	8.01e-08	4.00e-08	2.00e-08	1.00e-08	5.01e-09	2.50e-09	1.25e-09
128	2.03e-08	1.01e-08	5.06e-09	2.53e-09	1.27e-09	6.33e-10	3.16e-10
256	5.09e-09	2.55e-09	1.27e-09	6.37e-10	3.18e-10	1.59e-10	7.96e-11
512	1.28e-09	6.39e-10	3.19e-10	1.60e-10	7.98e-11	3.99e-11	2.00e-11
1024	3.20e-10	1.60e-10	7.99e-11	4.00e-11	2.00e-11	9.99e-12	5.00e-12

In order to illustrate the performance of the nonpolynomial spline finite difference technique in solving boundary value problems for singularly perturbed singular and non-singular problems (Ascher *et. al.* [15], Chang *et. al.* [16]) and the efficiency of the method, the following examples are considered. The right hand side function and boundary conditions may be obtained using the exact solution $u(r) = \varepsilon \sinh(r)$ as a test procedure. The examples have been solved by the presented method with different values of N and ε. We have implemented Newton's method using five inner iterations as an standard procedure and computed the root mean square errors(Hageman and Young [18]), defined as

$$E_{u_k} = \sqrt{\sum_{k=1}^{N} \frac{\left|u_k - U(r_k)\right|^2}{N}}$$

All programs are written in C and computations were carried out using Linux environment. Table 1-2 exhibit the root mean square errors.

Example 1: Consider the following nonlinear non-singular problem

$$\varepsilon u'' + \theta_1 e^u + \theta_2 u^2 + \theta_3 (u')^2 + \theta_4 u u' = g(r), \quad r \in \Omega$$

The root mean square errors for different values of $\theta_k, k = 1(1)4$ are tabulated in Table 1.

Example 2: Consider the following nonlinear singular problem

$$\varepsilon u'' + \frac{\lambda}{r} u' - \frac{\lambda}{r^2} u + e^u + u^2 + (u')^2 + u u' = g(r), \quad r \in \Omega$$

The root mean square errors for different values of λ are tabulated in Table 2.

Table 2: Root Mean Square Errors for Example 2.

N \ ε	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
$\lambda = 1$							
16	1.65e-06	2.77e-06	5.69e-06	1.34e-05	4.12e-05	2.97e-03	7.44e-05
32	2.67e-07	2.74e-07	5.43e-07	1.21e-06	2.98e-06	8.75e-06	7.48e-05
64	6.18e-08	2.97e-08	5.04e-08	1.11e-07	2.59e-07	6.42e-07	1.79e-06
128	1.55e-08	4.83e-09	4.68e-09	9.98e-09	2.31e-08	5.52e-08	1.36e-07
256	3.91e-09	1.11e-09	4.97e-10	8.95e-10	2.06e-09	4.88e-09	1.17e-08
512	9.80e-10	2.74e-10	8.12e-11	8.15e-11	1.83e-10	4.33e-10	1.03e-09
1024	2.45e-10	6.86e-11	1.85e-11	8.47e-12	1.63e-11	3.83e-11	9.13e-11
$\lambda = 2$							
16	5.53e-06	1.12e-05	2.55e-05	6.66e-05	2.51e-04	8.65e-04	1.62e-04
32	5.47e-07	1.08e-06	2.39e-06	5.70e-06	1.48e-05	5.08e-05	1.19e-05
64	5.79e-08	1.01e-07	2.20e-07	5.12e-07	1.24e-06	3.17e-06	9.73e-06
128	9.02e-09	9.31e-09	1.99e-08	4.61e-08	1.09e-07	2.65e-07	6.67e-07
256	2.03e-09	9.67e-10	1.79e-09	4.12e-09	9.74e-09	2.33e-08	5.62e-08
512	5.03e-10	1.51e-10	1.62e-10	3.66e-10	8.65e-10	2.06e-09	4.93e-09
1024	1.26e-10	3.40e-11	1.65e-11	3.25e-11	7.67e-11	1.82e-10	4.35e-10

5. Conclusions

The nonpolynomial cubic spline finite difference method can solve general singular perturbation problems with singularity. The method is fourth order convergent and can be easily implemented. It has been observed that root mean square errors confirm the order and accuracy of the proposed method. Extension of the method to higher dimensions is an open problem.

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