The Prove of a Class of Variational Inequalities

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Abstract. If $f(x)$ is a differentiable convex function and its Heissen matrix is positive semi-definite,we can prove the inequality $(x-x^*)^T \nabla f(x) \geq -\frac{1}{2}(x-\overset{\square}{x})^T (\nabla f(x) - \nabla f(\overset{\square}{x}))$.

Meet the above inequality from the general convex function of the convergence card sequence of functions on the measurable set.

Keyword. Convex function, positive semidifinite, cauchy sequences, convergence

1. Introduction

let $f(x)$ be a differentiable convex function，and $\nabla f(x)$ be the gradient of the function. x^* is the only optimal value point，which makes $\nabla f(x^*) = 0$.Then to any x and $\overset{\square}{x}$, if H is positive semi-definite,

$$(x-x^*)^T \nabla f(x) \geq -\frac{1}{2}(x-\overset{\square}{x})^T (\nabla f(x) - \nabla f(\overset{\square}{x})) \qquad (1)$$

(see [11],[12],[13],[14],[15],[16],[17])，but if $f(x)$ only be a differentiable convex function we can get weaker inequality(see [18],[19]),can we get the inequality to any convex function?

2. Some properties

Conclusion 1：Let H be positive semi-definite.Then we have

$$a^T Hb \geq -\frac{1}{2}(a-b)^T H(a-b)$$

Proof：$(a-b)^T H(a-b)$

$= (a-b, H(a-b))$

$= (a, Ha) - (b, Ha) - (a, Hb) + (b, Hb)$

since H is positive semi-definite and $(a, Hb) = (b, Ha)$,

we have $(a, Hb) \geq -\frac{1}{2}(a-b, H(a-b))$ 。

Conclusion 2: assume that $f(x)$ is differentiable on R^n, then $f(x)$ convex if and only if

$$f(y) - f(x) \geq \nabla f(x)^T (y-x) 。$$

Proof：if $f(x)$ is convex，then

$$f((1-\theta)x + \theta y) \leq (1-\theta)f(x) + \theta f(y)，\quad \theta \in [0,1], x, y, \in R^n$$

So we have $f(y) - f(x) \geq \dfrac{f(x + \theta(y-x)) - f(x)}{\theta}$

letting $\theta \to 0_+$，we get $f(y) - f(x) \geq (y-x)^T \nabla f(x)$.

Conversely if $f(x) = \frac{1}{2}x^T Hx + c^T x, H^T = H$ and H is semi-definite，the conclusion is obviously.

Theorem 1. Let $f(x) = \frac{1}{2}x^T Hx + c^T x, H^T = H$ and H be positive semi-ddefinite，Then we have

$$(x - x^*)^T \nabla f(x) \geq -\frac{1}{2}(x - \tilde{x})^T (\nabla f(x) - \nabla f(\tilde{x})) \qquad (2)$$

Proof: note that in this case $\nabla f(x) = Hx + c$ and thus the equivalent of（2）is

$$(x - x^*)^T (Hx + c) \geq -\frac{1}{2}(x - \tilde{x})^T H(x - \tilde{x}) \qquad (3)$$

By using $Hx^* + c = 0$, we have $(x - x^*)^T\{(Hx + c) - H(x - x^*)\} = 0$ and consequently $(x - x^*)^T (Hx + c) = (x - x^*)^T H(x - x^*)$

Since H is positive semi-definite ,we can get the conclusion .

For a general convex function ，weaker than (1) the conclusion is clearly established.

Theorem 2.Let $f(x) : R^n \to R$ be convex and differentiable. Then we have

$$(x - x^*)^T \nabla f(x) \geq -(x - \tilde{x})^T (\nabla f(x) - \nabla f(\tilde{x})). \text{ (see [15],[16])} \qquad (4)$$

Proof：since $f(x)$ is convex and differentiable,we have

$$f(x^*) \geq f(x) + (x^* - x)^T \nabla f(x). \qquad (5)$$

Since x^* is the minimum point，$f(\tilde{x}) \geq f(x^*).$ Therefore,it follows from (5)that

$$(x - x^*)^T \nabla f(x) \geq f(x) - f(\tilde{x}) \qquad (6)$$

Since f is convex，$f(x) - f(\tilde{x}) \geq \nabla f(\tilde{x})^T (x - \tilde{x})$ Using (5) (6) we have:

$$(x - \tilde{x})^T \nabla f(x) \geq (x - \tilde{x})^T \nabla f(\tilde{x}) \qquad (7)$$

Adding $(\tilde{x} - x)^T \nabla f(x)$ to the both sides of (7),we get

$$(\tilde{x} - x^*)^T \nabla f(x) \geq (x - \tilde{x})^T (\nabla f(\tilde{x}) - \nabla f(x)) .$$

For general convex function ,can conclude that (1).

3. The main result

Conclusion 3: $L^p(\mu)$ is a complete metric space，that is to say any cauchy sequence of the $L^p(\mu)$ converges to an element of the $L^p(\mu)$.

Theorem 3: If $1 \leq p \leq \infty$, and if $\{f_n\}$ is a Cauchy sequence in $L^p(\mu)$，with limit f ,then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to $f(x)$.

According to Theorem 3(see [20]),for a differentiable convex function $f(x)$, $x \in R^n$,can always find a Cauchy sequence $\{f_n\}$ of $L^p(\mu)$ .By the conclusion of Theorem 3 shows that a subsequence $\{f_{n_i}\}$ can always be found in the above Cauchy sequence ,which converge to $f(x)$. f_{n_i} is a differentiable convex function and meets $f_{n_i}(x) = \frac{1}{2}x^T H_{n_i} x + c^T x, H_{n_i} = H_{n_i}^T$, among which, H_{n_i} is positive semi-definite.

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