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A Sinc-Collocation Method for Second-Order Boundary Value Problems of Nonlinear Integro-Differential Equation

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Abstract. The sinc-collocation method is presented for solving second-order boundary value problems of nonlinear integro-differential equation. The method is effective for approximation in the case of the presenceof end-point singularities. Some properties of the sinc-collocation method required for our subsequent development are given and are utilized to reduce the computation of solution of the second-order boundary value problems of nonlinear integro-differential equation to some algebraic equations. Some numerical results are also given to demonstrate the validity and applicability of the presented technique.

Keywords: Sinc function, Collocation method, Boundary value problems, Second-order, Nonlinear integro-differential equation.

1. Introduction

Boundary value problems for integro-differential equations are important because they have many applications in the study of physical, biological and chemical phenomena [1]. Liz and Nieto [2], study a two point boundary value problem for a nonlinear second order integro-differential equation of Fredholm type by using upper and lower solutions. In [1], an iterative method is presented to solve a class of boundary value problems for second-order integro-differential equation in the reproducing kernel space. For linear and nonlinear second order Fredholm integro-differential equations, semiorthogonal spline wavelets was developed in [3] and Chebyshev finite difference method was discussed in [4]. Also in [5], Saadatmandi and Dehghan applied the Legendre polynomials for the solution of the linear Fredholm integro-differential-difference equation of high order.

In this paper, a sinc-collocation procedure is developed for the numerical solution second-order boundary value problems of nonlinear integro-differential equation of the form:

$$u''(x)+p(x)u'(x)+q(x)u(x)+\lambda_1\int_a^x k_1(x,t)u(t)dt+\lambda_2\int_\Gamma k_2(x,t)u(t)dt=f\big(x,u(x)\big), \qquad (1)$$

$$x,t\in\Gamma=[a,b], \quad u(a)=\alpha, \qquad u(b)=\beta,$$

where the parameters λ_1, λ_2, the kernels $k_1(x,t)$, $k_2(x,t)$, the functions $p(x),q(x)$ are given and $f(x,u(x))$ is nonlinear in $u(x)$, where $u(x)$ is the unknown function to be determined. There has been a great deal of research work on the existence of solutions for boundary value problems, for instance see [6, 7, 8].

Sinc methods have increasingly been recognized as powerful tools for problems in applied physics and engineering [9, 10]. The sinc-collocation method is a simple method with high accuracy for solving a large variety of nonlinear problems. In Reference [11], the sinc-collocation method is presented for solving boundary value problems for nonlinear third-order differential equations. Authors of [12], used the sinc-collocation method for solving a nonlinear system of second-order boundary value problems. Mohsen and El-Gamel [13], used the sinc-collocation method for solving the linear integro-differential equations of the Fredholm type. Also in [14], the sinc-collocation is presented for solving linear and nonlinear Volterra integral and integro-differential equations. In [15, 16], the sinc-collocation is used for the numerical solution Fredholm and Volterra integro-differential equations. Also sinc-collocation method is used for solving of a system of nonlinear second-order integro-differential equations with boundary conditions of the Fredholm and Volterra types [17]. We also refer the interested reader to [18, 19, 20, 21, 22, 23] for more research works on sinc methods.

The main purpose of the present paper is to develop methods for numerical solution of the second-order boundary value problems of nonlinear integro-differential equation (1). Our method consists of reducing the solution of (1) to a set of algebraic equations. The properties of sinc function are then utilized to evaluate the unknown coefficients. The organization of the rest of this article is as follows. In Section 2, we review some of the main properties of sinc function that are necessary for our subsequent development. In Section 3, we illustrate how the sinc method may be used to replace Eq. (1) by an explicit system of nonlinear algebraic equations. Section 4, presents appropriate techniques to treat no homogeneous boundary conditions. In Section 5, some numerical results are given to clarify the method.

2. Sinc function properties

Sinc function properties are discussed thoroughly in [9, 10]. In this section an overview of the formulation of the sinc function required for our subsequent development is presented. The sinc function is defined on the whole real line, $-\infty < x < \infty$, by

$$Sinc(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \quad (2)$$

For any $h > 0$, the translated sinc functions with evenly spaced nodes are given by

$$S(j,h)(x) = Sinc\left(\frac{x - jh}{h}\right) = \begin{cases} \dfrac{\sin\left[\frac{\pi}{h}(x - jh)\right]}{\frac{\pi}{h}(x - jh)}, & x \neq jh \\ 1, & x = jh \end{cases} \quad (3)$$

which are called the jth sinc functions. The sinc function form for the interpolating point $x_k = kh$ is given by

$$S(j,h)(kh) = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases} \quad (4)$$

If u is defined on the real line, then for $h > 0$ the series

$$C(u,h)(x) = \sum_{j=-\infty}^{\infty} u(jh) Sinc\left(\frac{x - jh}{h}\right), \quad (5)$$

is called the Wittaker cardinal expansion of u, whenever this series converges [9,10]. But in practice we need to use some specific numbers of terms in the above series, such as $j = -N, \ldots, N$, where N is the number of sinc grid points. They are based in the infinite strip D_S in the complex plane

$$D_S = \left\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \right\}. \quad (6)$$

To construct an approximation on the interval (a, b), we consider the conformal map

$$\phi(z) = Ln\left(\frac{z - a}{b - z}\right). \quad (7)$$

The map carries the eye-shaped region

$$D_E = \left\{ z \in \mathbb{C} : \left| arg\left(\frac{z - a}{b - z}\right) \right| < d \leq \frac{\pi}{2} \right\}. \quad (8)$$

For the sinc method, the basis functions on the interval (a, b) for $z \in D_E$ are derived from the composite translated sinc functions,

$$S(j,h)o\phi(z) = Sinc\left(\frac{\phi(z) - jh}{h}\right). \quad (9)$$

The function

$$z = \phi^{-1}(w) = \frac{a + be^{w}}{1 + e^{w}}, \quad (10)$$

is an inverse mapping of $w = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \{\phi^{-1}(u) \in D_E: -\infty < u < \infty\}. \quad (11)$$

The sinc grid points $z_j \in (a, b)$ in D_E will be denoted by x_j because they are real. For the evenly spaced nodes $\{jh\}_{j=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_j = \phi^{-1}(jh) = \frac{a + be^{jh}}{1 + e^{jh}}, \qquad j = 0, \pm 1, \pm 2, \ldots \quad (12)$$

For further explanation of the procedure, the important class of functions is denoted by $L_\alpha(D_E)$. The properties of functions in $L_\alpha(D_E)$ and detailed discussions are given in [9, 10]. We recall the following definitions and theorems for our purpose.

Definition 1. Let $L_\alpha(D_E)$ be the set of all analytic functions, for which there exists a constant, C, such that

$$|u(z)| \le C\frac{|\rho(z)|^{\alpha}}{[1 + |\rho(z)|]^{2\alpha}}, \quad z \in D_E, \qquad 0 < \alpha \le 1, \quad (13)$$

where $\rho(z) = e^{\phi(z)}$.

Theorem 1. Let $u \in L_\alpha(D_E)$, let N be a positive integer, and let h be selected by the formula

$$h = \left(\frac{\pi d}{\alpha N}\right)^{1/2}, \quad (14)$$

then there exists positive constant c_1, independent of N, such that

$$\sup_{z\in\Gamma}\left|u(z) - \sum_{j=-N}^{N} u(z_j)S(j,h)o\phi(z)\right| \le c_1 e^{-(\pi d\alpha N)^{1/2}}. \quad (15)$$

Theorem 2. Let $\frac{u}{\phi'} \in L_\alpha(D_E)$, let N be a positive integer and let h be selected by the formula (14), then there exist positive constant c_2, independent of N, such that

$$\left|\int_{\Gamma} u(z)dz - h\sum_{k=-N}^{N}\frac{u(z_k)}{\phi'(z_k)}\right| \le c_2 e^{-(\pi d\alpha N)^{1/2}}. \quad (16)$$

Theorem 3. Let $\frac{u}{\phi'} \in L_\alpha(D_E)$, with $\alpha > 0$, and $d > 0$, let $\delta_{kj}^{(-1)}$ be defined as

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \int_0^{k-j}\frac{\sin(\pi t)}{\pi t}dt,$$

and let $h = \left(\frac{\pi d}{\alpha N}\right)^{1/2}$. Then there exists a constant c_3, which is independent of N, such

$$\left|\int_a^{z_k} u(t)dt - h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{u(z_j)}{\phi'(z_j)}\right| \le c_3 e^{-(\pi d\alpha N)^{1/2}}. \quad (17)$$

We also require derivatives of composite sinc functions evaluated at the nodes. The nth derivative $u(x)$ at some points x_j can be approximated using a finite number of terms as

$$u^{(n)}(x_j) \simeq h^{-n}\sum_{i=-N}^{N}\delta_{ij}^{(n)}u(x_j), \quad (18)$$

where

$$\delta_{ij}^{(n)} = h^n\frac{d^n}{d\phi^n}[S(i,h)o\phi(x)]\Big|_{x=x_j}. \quad (19)$$

In particular

$$\delta_{ij}^{(0)} = [S(i,h)o\phi(x)]\Big|_{x=x_j} = \begin{cases} 1, & i=j, \\ 0, & i\neq j. \end{cases} \tag{20}$$

$$\delta_{ij}^{(1)} = h\frac{d}{d\phi}[S(i,h)o\phi(x)]\Big|_{x=x_j} = \begin{cases} 0\quad, & i=j, \\ \dfrac{(-1)^{j-i}}{j-i}, & i\neq j. \end{cases} \tag{21}$$

and

$$\delta_{ij}^{(2)} = h\frac{d^2}{d\phi^2}[S(i,h)o\phi(x)]\Big|_{x=x_j} = \begin{cases} \dfrac{-\pi^2}{3}, & i=j, \\ \dfrac{-2(-1)^{j-i}}{(j-i)^2}, & i\neq j. \end{cases} \tag{22}$$

3. The sinc-collocation method

Let us consider the nonlinear equation (1), with homogeneous boundary conditions. We assume $u(x)$ to be the exact solution of the boundary value problem (1) and let $u \in L_\alpha(D_E)$, We consider the Whittaker cardinal expansion (5). The series in relation (5) contains an infinite number of terms. Let N be a positive integer, then function $u(x)$ defined over the interval $[a,b]$ is approximated by using a finite number of terms in (5) as

$$u(x) \simeq \sum_{i=-N}^{N} u_i S(i,h)o\phi(x), \tag{23}$$

where $u_i = u(x_i)$ and $\phi(x)$ is defined by (7). We consider the equation (1), and let

$$g(x) = f\big(x,u(x)\big),$$

then

$$u''(x) + p(x)u'(x) + q(x)u(x) + \lambda_1\int_a^x k_1(x,t)u(t)dt + \lambda_2\int_\Gamma k_2(x,t)u(t)dt = g(x), \tag{24}$$

By using Eq. (23) we have

$$u'(x) \simeq \sum_{i=-N}^{N} u_i\frac{d}{dx}[S(i,h)o\phi(x)],$$

$$u''(x) \simeq \sum_{i=-N}^{N} u_i\frac{d^2}{dx^2}[S(i,h)o\phi(x)].$$

Note that

$$\frac{d}{dx}[S(i,h)o\phi(x)] = \phi'(x)\frac{d}{d\phi}[S(i,h)o\phi(x)],$$

and

$$\frac{d^2}{dx^2}[S(i,h)o\phi(x)] = \phi''(x)\frac{d}{d\phi}[S(i,h)o\phi(x)] + \big(\phi'(x)\big)^2\frac{d^2}{d\phi^2}[S(i,h)o\phi(x)].$$

Having substituted $x = x_i$ for $j = -N,\dots,N,$ where x_i are sinc grid points given in (12), and by using relations (4), (19), we have

$$u'(x_j) \simeq \sum_{i=-N}^{N} u_i\phi'_j h^{-1}\delta_{ij}^{(1)}, \tag{25}$$

and

$$u''(x_j) \simeq \sum_{i=-N}^{N} u_i \left(\phi''_j h^{-1}\delta_{ij}^{(1)} + (\phi'_j)^2 h^{-2}\delta_{ij}^{(2)}\right), \tag{26}$$

where $\phi'_i = \phi'(x_i)$, $\phi''_i = \phi''(x_i)$.

We suppose that $\frac{k_i(x,.)}{\phi'} u \in L_\alpha(D_E)$, $i = 1, 2$, by applying Theorems 2, 3 and setting $x = x_j$, we obtain

$$\int_a^{x_j} k_1(x_j,t)u(t)dt \simeq h \sum_{i=-N}^{N} \frac{k_{1j,i}}{\phi'_i} \delta_{ji}^{(-1)} u_i, \tag{27}$$

and

$$\int_\Gamma \mathrm{k}_2(x_j,t)u(t)dt \simeq h \sum_{i=-N}^{N} \frac{k_{2j,i}}{\phi'_i} u_i, \tag{28}$$

where $k_{1j,i} = k_1(x_j,t_i), k_{2j,i} = k_2(x_j,t_i)$.

By using relations (25-28), and substituting $x = x_j$, $j = -N, \dots, N$, we can rewrite (24) as

$$\sum_{i=-N}^{N} \left(\phi''_j \frac{\delta_{ij}^{(1)}}{h} + (\phi'_j)^2 \frac{\delta_{ij}^{(2)}}{h^2}\right) u_i + p_j \sum_{i=-N}^{N} \phi'_j \frac{\delta_{ij}^{(1)}}{h} u_i + q_j u_j + \lambda_1 h \sum_{i=-N}^{N} \frac{k_{1j,i}}{\phi'_i} \delta_{ji}^{(-1)} u_i + \lambda_2 h \sum_{i=-N}^{N} \frac{k_{2j,i}}{\phi'_i} u_i = g_j, \tag{29}$$

where $g_j = f(x_j, u_j)$, for $j = -N, \dots, N$, with ordering the up formula, we have

$$\frac{1}{h} \sum_{i=-N}^{N} (\phi''_j + p_j \phi'_j) \delta_{ij}^{(1)} u_i + \frac{1}{h^2} \sum_{i=-N}^{N} (\phi'_j)^2 \delta_{ij}^{(2)} u_i + q_j u_j + \lambda_1 h \sum_{i=-N}^{N} \frac{k_{1j,i}}{\phi'_i} \delta_{ji}^{(-1)} u_i + \lambda_2 h \sum_{i=-N}^{N} \frac{k_{2j,i}}{\phi'_i} u_i = g_j, \tag{30}$$

where $j = -N, \dots, N$. We now rewrite this system which is the nonlinear system of equations in matrix form. Corresponding to a given function $u(x)$ defined on Γ, we use the notation $D(u) = diag(u(x_{-N}), \dots, u(x_N))$, $K_l = [k_l(x_j, t_i)]$, $l = 1, 2$ and $j, i = -N, \dots, N$.

We set $I^{(m)} = [\delta_{ij}^{(m)}]$, $m = -1, 1, 2$, where $\delta_{ij}^{(m)}$ denotes the (i, j)th element of the matrix $I^{(m)}$, and since

$$\delta_{ij}^{(0)} = \delta_{ji}^{(0)}, \delta_{ij}^{(1)} = -\delta_{ji}^{(1)}, \delta_{ij}^{(2)} = \delta_{ji}^{(2)},$$

we can simplify the system (30) in the matrix $AU = G$, where

$$A = -\frac{1}{h}[D(\phi'') + D(p)D(\phi')]I^{(1)} + \frac{1}{h^2} D(\phi')^2 I^{(2)} + D(q) + \lambda_1 h \left[\left(I^{(-1)} D\left(\frac{1}{\phi'}\right)\right) o K_1\right] + \lambda_2 h \left[K_2 D\left(\frac{1}{\phi'}\right)\right],$$

$$U = [u(x_{-N}), u(x_{-N+1}), \dots, u(x_{N-1}), u(x_N)]^T,$$
$$G = [g(x_{-N}), g(x_{-N+1}), \dots, g(x_{N-1}), g(x_N)]^T.$$

The notation "o" denotes the Hadamard matrix multiplication. The above nonlinear system consists of $2N + 1$ equations with $2N + 1$ unknown coefficients $\{u_i\}_{i=-N}^{N}$. Solving this nonlinear system by the well known Newton's method. Consequently $u(x)$ given in (23) can be calculated.

3. Treatment of boundary conditions

In the previous section the development of the sinc-collocation technique for homogeneous boundary conditions provided a practical approach, since the sinc functions composed with the various conformal maps, $S(i,h)o\phi(x)$, are zero at the endpoints of the interval. If the boundary conditions are nonhomogeneous, then these conditions need be converted to homogeneous conditions via an interpolation by a known function. Using the transformation

$$y(x) = u(x) - \frac{b-x}{b-a}\alpha - \frac{x-a}{b-a}\beta, \tag{31}$$

to the problem (1), yields the equation

$$y''(x) + p(x)y'(x) + q(x)y(x) + \lambda_1 \int_a^x k_1(x,t)y(t)dt + \lambda_2 \int_\Gamma k_2(x,t)y(t)dt = \hat{f}(x, y(x)), \tag{32}$$

$x \in \Gamma = [a,b], \quad y(a) = 0, \quad y(b) = 0$,

where

$$\hat{f}(x,y(x)) = f\left(x, y(x) + \frac{b-x}{b-a}\alpha + \frac{x-a}{b-a}\beta\right) - p(x)\left(\frac{\beta-\alpha}{b-a}\right) - q(x)\left(\frac{b-x}{b-a}\alpha + \frac{x-a}{b-a}\beta\right)$$
$$- \lambda_1 \int_a^x k_1(x,t)\left(\frac{b-t}{b-a}\alpha + \frac{t-a}{b-a}\beta\right)dt - \lambda_2 \int_\Gamma k_2(x,t)\left(\frac{b-t}{b-a}\alpha + \frac{t-a}{b-a}\beta\right)dt.$$

Table 1: Comparison absolute error $u(x)$ for Example 1

x	Exact solution	Method of [1]	Present method N = 20	Present method N = 25
0.08	0.00588172	4.6259×10^{-5}	5.3868×10^{-6}	2.8293×10^{-6}
0.16	0.0214124	4.81776×10^{-5}	1.8045×10^{-5}	8.9730×10^{-6}
0.32	0.0684497	4.16744×10^{-5}	7.6048×10^{-6}	7.3063×10^{-6}
0.48	0.11526	3.48007×10^{-5}	1.6378×10^{-5}	1.2008×10^{-5}
0.64	0.137594	2.96533×10^{-5}	3.1392×10^{-6}	8.7603×10^{-6}
0.80	0.114777	2.69251×10^{-5}	7.9223×10^{-6}	4.9231×10^{-6}
0.96	0.031457	2.39306×10^{-5}	3.7492×10^{-7}	1.0128×10^{-6}

3. Illustrative Examples

We applied the method presented in this paper and solved some examples. We also compare our method with introduced in [1, 24]. It is shown that the sinc-collocation method yields better results. The solutions of the given examples are obtained for $\alpha = \frac{1}{2}$, $d = \frac{\pi}{2}$ and for different values of N. Let $u(x)$, $u'(x)$ denote the exact solutions of the given examples, and let $u_N(x)$, $u'_N(x)$ be the computed solutions by our method. Let $\Gamma = [a,b]$ and ϕ a conformal map onto D_E, where $\phi(x)$ is defined by (7). We use the absolute errors, defined as

$$E_N(x) = |u_N(x) - u(x)|, \qquad E'_N(x) = |u'_N(x) - u'(x)|, \qquad a < x < b.$$

So the numerical technique described in previous sections was applied to the following examples:

Example 1: Consider the singular boundary value problem [1],

$$u''(x) + \frac{1}{\sqrt{x}}u'(x) + \frac{1}{x}u(x) + \int_0^x (t+x)u(t)dt + \int_0^1 txu(t)dt - \frac{1}{1+sin(u^2(x))}$$
$$-e^{u^9(x)} + u^{11}(x) = f(x), \qquad 0 < x < 1, \tag{33}$$

where $u(0) = u(1) = 0$ and

$$f(x) = \frac{1}{\sqrt{x}}\left(\left(2\sqrt{x} + x - 4x\sqrt{x} - x^2\right)\cos x + \left(1 - \sqrt{x} - 2x - 2x\sqrt{x} + x^2\sqrt{x}\right)\sin x\right) + 2\left(1 - 4x - x^2 + x^3\right)\cos x + \left(6 + 3x - 5x^2\right)\sin x - (2 + 4\cos 1 - 5\sin 1)x - \frac{1}{1 + \sin\left(\left((x - x^2)\sin x\right)^2\right)} - e^{\left((x-x^2)\sin x\right)^9} + \left((x - x^2)\sin x\right)^{11} + 2(x - 1), \quad (34)$$

for which the exact solution is $u(x) = (x - x^2)sinx$.

Table 2: Comparison absolute error $u'(x)$ for Example 1

x	Exact solution	Method of [1]	Present method	
			N = 25	N = 30
0.08	0.140493	1.26577×10^{-5}	1.2953×10^{-4}	3.0105×10^{-6}
0.16	0.24102	6.05795×10^{-5}	7.7563×10^{-5}	6.0314×10^{-5}
0.32	0.319798	4.45588×10^{-5}	8.1110×10^{-5}	2.7979×10^{-5}
0.48	0.239865	3.8886×10^{-5}	6.3109×10^{-5}	1.7409×10^{-5}
0.64	0.0175881	2.52869×10^{-5}	4.6633×10^{-5}	3.9654×10^{-6}
0.80	−0.318941	7.07368×10^{-5}	7.6312×10^{-5}	3.4064×10^{-5}
0.96	−0.731633	4.60855×10^{-5}	2.6826×10^{-5}	3.2563×10^{-5}

Table 3: Comparison absolute error $u(x)$ for Example 2

x	Method of [24]	Present method	
		N = 10	N = 15
0.1	3.95622×10^{-3}	1.0641×10^{-2}	1.3982×10^{-3}
0.2	6.01293×10^{-3}	9.6044×10^{-3}	2.0176×10^{-3}
0.3	6.00105×10^{-3}	3.2104×10^{-3}	1.5292×10^{-3}
0.4	6.35575×10^{-3}	1.1736×10^{-2}	1.0231×10^{-3}
0.5	7.02651×10^{-3}	1.2435×10^{-2}	1.3113×10^{-3}
0.6	6.70261×10^{-3}	1.1736×10^{-2}	1.0231×10^{-3}
0.7	6.44357×10^{-3}	3.2104×10^{-3}	1.5292×10^{-3}
0.8	6.40303×10^{-3}	9.6044×10^{-3}	2.0176×10^{-3}
0.9	4.33703×10^{-3}	1.0641×10^{-2}	1.3982×10^{-3}

Table 1, presents the absolute error $u(x)$, for $N = 20$ and $N = 25$, using the present method at the same points as [1], together with the results given in [1]. Also Table 2, presents the absolute error $u'(x)$, for $N = 25$ and $N = 30$, using the present method at the same points as [1], together with the results obtained by given in [1].

Example 2: In this example we consider the nonlinear second-order differential equation [24],

$$u''(x) - u^2(x) = 2\pi^2 \cos(2\pi x) - sin^4(\pi x), \quad 0 \le x \le 1$$

with the boundary conditions

$$u(0) = 0, \quad u(1) = 0.$$

The exact solution of this problem is $u(x) = sin^2(\pi x)$. Table 3, presents the absolute values of errors for $N = 10$ and $N = 15$, by using the present method at the same points as [24].

Example 3: We consider the second-order boundary value problem of volterra integro- differential equation

$$u''(x) + \frac{1}{1 + u^2(x)} + xe^{u(x)} + \int_0^x xtu(t)dt = f(x), \qquad 0 \le x \le 1,$$

$$u(0) = 1, \quad u(1) = 2,$$

where

$$f(x) = 2 + \frac{1}{1 + (1 + x^2)^2} + xe^{1+x^2} + \frac{x^3}{4}(2 + x^2).$$

The true solution is $u(x) = 1 + x^2$.

We solve this problem, for $N = 15$ and $N = 30$. The absolute errors are tabulated in Table 4.

Table 4: Results for Example 3

x	E_{15}	E_{30}
0.05	1.9141×10^{-5}	8.0853×10^{-8}
0.15	3.4870×10^{-6}	4.1152×10^{-8}
0.25	1.8463×10^{-5}	4.0552×10^{-10}
0.35	6.5224×10^{-6}	5.0559×10^{-8}
0.45	1.6167×10^{-6}	2.6244×10^{-8}
0.55	1.9012×10^{-6}	2.5612×10^{-8}
0.65	7.2555×10^{-6}	5.1438×10^{-8}
0.75	1.9338×10^{-5}	1.3473×10^{-10}
0.85	2.7679×10^{-6}	4.1907×10^{-8}
0.95	1.9517×10^{-5}	8.1172×10^{-8}

Example 4: Consider the second-order boundary value problem of fredholm integro- differential equation

$$u''(x) + \frac{1}{\sqrt{1+x}}u'(x) + \frac{1}{x-1}u(x) + \int_{-1}^{1} e^{(x+t)}u(t)dt = u^2(x) + 3e^{-u(x)} + f(x),$$

with the boundary conditions $u(-1) = 0, \quad u(1) = 0,$

where $f(x) = 2 + \frac{2x}{\sqrt{x+1}} + x + 1 - 4e^{x-1} - (x^2-1)^2 - 3e^{-x^2+1},$

with exact solution $u(x) = x^2 - 1$. We solve equation for $N = 15$ and $N = 25$. The absolute errors are tabulated in Table 5.

3. Conclusion

The sinc-collocation method is used to solve the second-order boundary value problemsof nonlinear integro-differential equation. Properties of the sinc function are utilized to reduce the computation of this problem to some algebraic equations. The method is computationally attractive and applications are demonstrated through illustrative examples. The results of the present method for this type of problem clearly indicate that our methods is accurate even when singularity occurs at the boundary.

Table 5: Results for Example 4

x	E_{15}	E_{25}
-0.95	4.6482×10^{-5}	1.2347×10^{-6}
-0.75	7.4734×10^{-5}	5.4515×10^{-7}
-0.55	2.0712×10^{-5}	8.5048×10^{-7}
-0.35	3.5332×10^{-5}	1.6112×10^{-7}
-0.15	6.3807×10^{-5}	1.5565×10^{-6}
0.05	3.6695×10^{-5}	9.5408×10^{-7}
0.25	5.5576×10^{-6}	3.7185×10^{-7}
0.45	6.7451×10^{-5}	1.5479×10^{-6}
0.65	1.9001×10^{-5}	1.3849×10^{-6}
0.85	5.9382×10^{-5}	9.9596×10^{-8}

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