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On General Binary Relation Based Rough Set

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Abstract. To keep the key idea of rough set and the representation of information in rough set theory, empty representation is processed properly and three new forms of rough approximation sets are defined as a generation of general binary relation based rough set model. Moreover, the properties of approximation operators in these new rough sets are discussed. The relations among them are studied in this paper. In addition, examples are arranged to interpret what we studied in this paper.

Keywords: General binary relation, Approximation operator, Rough set, Empty description, Duality

1. Introduction

Rough set theory was first proposed by Pawlak Z in 1980's. It is a mathematical tool to process information with uncertainty and vagueness. And it is also a useful soft computing tool in intelligence computing. The rough set theory can yet be regarded as a kind of more effective method to handle complex systems in data mining (DM) and knowledge discovery in database (KDD) [1,3,4,5]. Rough set theory, probability theory, fuzzy set theory and evidence theory are all tools to deal with uncertainty. Compared with other theories, the most significant difference is that no more prior information is needed but the specified information system for problems in rough set theory. Much better affections may come about in practical problems by combining rough set and other methods. Noise is a factor which can't be avoided in practice. Influenced by noise in data and limited by the requests of practical problems, the original rough set proposed by Pawlak is confined in practical applications. Many generalized rough set models have been proposed and studied systematically. The popularizations such as variable precision rough set, dominance-based rough set, tolerance-based rough set, fuzzy rough set, rough set based on covering, rough set based on evidence theory, probabilistic rough set, etc.[1,2,8,9,10,11,12,14], makes that rough set theory affects in more areas and fields. More useful information is being discovered and more values are being produced in real world. As researches on rough set are expanding in depth and further, the rough set theory is now being more and more abundant, theorized and systematic. Successful applications have been applied in many areas and fields, such as in subjects medical science, chemistry, materials science, geographical science, management, finance, conflicts resolutions, and so on. Excellent effects have been succeeded in many areas. Requirements in applications and the propelling of achievements are promoting rough set theory to be one of the most active research areas in information science and several interdisciplines [12,14,15].

Studying on general binary relation based rough set can make rough set theory more adaptable for generalized relations and produce more values in practice. Theories on general binary relation based rough set, which is denoted by GBRS, have been placed to some extent [6,7,14]. On the basis of Pawlak rough set, equivalence relations are popularized to general binary relations. And generalizations of GBRS are studied by constructive method, axiomatic method and the key idea of rough approximating in this paper. Properties of the corresponding approximation operators are discussed and proved. Moreover, examples are employed to help understand what we study in this paper.

2. Pawlak Rough Set

The classical rough set proposed by Pawlak is on the basis of equivalence relation on universe [3,4,5].

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Objects carry some unique information under relations and they can be classified by equivalence relations. Objects classified as the same class carry the information which is totally the same. If a concept can be presented by the union of some classes, that is the concept equals to the combination of some classes, then the concept carry all the information represented by the objects in these classes. Else, if a concept can't equal to the union of some classes, a pair of sets, which are constructed by the union of some classes, are employed to represent the concept approximately. One of the two sets is consisted of objects with totally confirm information that the concept carries. The selection of objects in this form relies on the precisely inclusion of classes to the concept. The other set is consisted of objects with information that possibly support the concept. The selection of these objects considers the nonempty joint between classes and the concept. The pair of sets is called, respectively, lower approximation set and upper approximation set [3,4,5,9,11,12,13]. Some basic definitions in Pawlak rough set will be illustrated for use in this paper.

Definition 2.1([3,14]) Let U be a nonempty finite set consisted of objects and called universe. For any $x \in U$, x is called an object. $R \subseteq U \times U$ is a binary relation on universe. x has the relation R with y if and only if $(x,y) \in R$, that is $xRy \Leftrightarrow (x,y) \in R$. If R satisfies

(1) Reflective: $\forall x \in U, xRx$;

(2) Symmetric: $xRy \Rightarrow yRx$;

(3) Transitive: $xRy, yRz \Rightarrow xRz$;

then R is an equivalence relation on the universe.

Definition 2.2([3,14]) Let U be the universe. $R \subseteq U \times U$ is an equivalence relation on the universe. Then (U,R) is called Pawlak approximation space. For any $X \subseteq U$, X is called a concept on the universe. For any $x \in U$, $[x]_R = \{y \in U | (x,y) \in R\}$ is called the equivalence class of x with respect to R. $U/R = \{[x]_R | x \in U\}$ is called the partition induced by R to U.

Definition 2.3([4,14]) Let U be the universe. $R \subseteq U \times U$ is an equivalence relation on the universe. For any $X \subseteq U$, the lower approximation and upper approximation of X with respect to Pawlak approximation space (U,R) are defined, respectively, as

$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\},$$
$$\overline{R}(X) = \{x \in U | [x]_R \cap X \neq \emptyset\}.$$

X is definable with respect to R if and only if $\underline{R}(X) = \overline{R}(X)$. Else, X is rough with respect to R if and only if $\underline{R}(X) \neq \overline{R}(X)$. $\underline{R}$ and $\overline{R}$ are called, respectively, the lower approximation operator and upper approximation operator with respect to R.

This model is Pawlak rough set model. If $[x]_R \subseteq X$, we usually say that x is precisely supporting the concept X with respect to R. Correspondingly, if $[x]_R \cap X \neq \emptyset$, it is said that x is possibly supporting the concept X with respect to R.

Theorem 2.1([3,14,15]**)** Let U be the universe. $R \subseteq U \times U$ is an equivalence relation on the universe. For any $X, Y \subseteq U$, the following properties of lower and upper approximation operators hold.

(1) $\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$;
(2a) $\underline{R}(\sim X) = \sim \overline{R}(X)$;
(2b) $\overline{R}(\sim X) = \sim \underline{R}(X)$;
(3a) $\underline{R}(\emptyset) = \overline{R}(\emptyset) = \emptyset$;
(3b) $\underline{R}(U) = \overline{R}(U) = U$;
(4a) $X \subseteq Y \Rightarrow \underline{R}(X) \subseteq \underline{R}(Y)$;
(4b) $X \subseteq Y \Rightarrow \overline{R}(X) \subseteq \overline{R}(Y)$;
(5a) $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$;
(5b) $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$;
(6a) $\underline{R}(X \cup Y) \subseteq \underline{R}(X) \cup \underline{R}(Y)$;
(6b) $\overline{R}(X \cap Y) \supseteq \overline{R}(X) \cap \overline{R}(Y)$.

Classes are the useful descriptions to characterize concepts in rough set theory. Possible descriptions with more uncertainty information should be not less than compatible descriptions with precisely support. Thus, the key idea to represent concepts by a pair of approximation sets should satisfy $\underline{R}(X)\subseteq\overline{R}(X)$ for any $X\subseteq U$ in rough set theory. As an important property in rough set theory, the duality should also be considered in the generalized models. Properties on generalized approximation operators should take those in the above theorem as reference and model.

Uncertainty in rough set theory is due to the existence of boundary region. Boundary is a set consisted of objects with descriptions support neither X nor $\sim X$ with respect to the approximation space. The uncertainty of concept can be reflected and measured by roughness and accuracy. And the detailed forms of these uncertainty measures won't be arranged in this paper. More about rough set theory can be referred in references [3,4,5,12,13,14,15]. Readers who need can look back into relative references and study in further.

3. Construction of General Binary Relation Based Rough Set

Since the equivalence relation limits the application of Pawlak rough set, generalized rough set models have been proposed and studied. Equivalence relations can be replaced by general binary relations and general binary relation based rough set has been constructed. The general binary relation based rough set (denoted shortly by GBRS) which was proposed by the constructive method will be introduced in this section.

Definition 3.1([14,15]) Let U be the universe. $R\subseteq U\times U$ is a general binary relation on the universe. For any $x\in U$, denote $R_s(x)=\{y\in U|xRy\}$, then $R_s(x)$ is called the successor neighborhood of x with respect to R. Any $y\in R_s(x)$ is called the successor of x with respect to R. For any $y\in U$, denote $R_p(y)=\{x\in U|xRy\}$, then $R_p(y)$ is called the processor neighborhood of y with respect to R. Any $x\in R_p(y)$ is called the processor of y with respect to R.

From the above definition, the successor neighborhood $R_s(x)$ is consisted of all objects y which satisfies xRy with x and the processor neighborhood $R_p(y)$ is the set of all objects x satisfying xRy with y. They can both be treated as classes induced by the general binary relation R. According to the constructive method of rough set theory, approximation operators in the following form can be defined.

Definition 3.2([14,15]) Let U be the universe. $R\subseteq U\times U$ is a general binary relation on the universe. Denote $A=(U,R)$ and it is call a generalized approximation space. For any $X\subseteq U$, the lower approximation set and upper approximation set of X with respect to A can be defined as follows.

$$\text{(I)}\quad \underline{apr}^{\mathrm{I}}_A(X)=\{x\in U|R_s(x)\subseteq X\},$$
$$\overline{apr}^{\mathrm{I}}_A(X)=\{x\in U|R_s(x)\cap X\neq\varnothing\}.$$

X is definable with respect to A if and only if $\underline{apr}^{\mathrm{I}}_A(X)=\overline{apr}^{\mathrm{I}}_A(X)$. Else, X is rough with respect to A if and only if $\underline{apr}^{\mathrm{I}}_A(X)\neq\overline{apr}^{\mathrm{I}}_A(X)$.

For convenience to study in this paper, GBRS showed in this form is called the first type of GBRS and denoted by GBRS(I).

The approximation sets in GBRS(I) is defined by successor neighborhood and they can also be defined by processor neighborhood similarly. We just need to replace $R_s(x)$ by $R_p(x)$ in the above definition. The approximations defined in these two forms are totally similar. Definitions and properties of approximation operators are not affected in discussion. The form of GBRS(I) defined by processor neighborhood will be not intend to be discussed in this paper.

Studies on GBRS(I) can be reviewed in references [14,15]. We just present the properties of approximation operators and more details can be looked back into these references to study further.

Theorem 3.1([14,15]) Let U be the universe. $R\subseteq U\times U$ is a general binary relation on the universe. For any $X,Y\subseteq U$, the following properties of lower and upper approximation operators hold in GBRS(I).

$$(1a)\ \underline{apr}^{I}_{A}(\sim X)=\sim\overline{apr}^{I}_{A}(X);$$
$$(1b)\ \overline{apr}^{I}_{A}(\sim X)=\sim\underline{apr}^{I}_{A}(X);$$
$$(2a)\ \overline{apr}^{I}_{A}(U)\subseteq\underline{apr}^{I}_{A}(U)=U;$$
$$(2b)\ \overline{apr}^{I}_{A}(\varnothing)=\varnothing\subseteq\underline{apr}^{I}_{A}(\varnothing);$$
$$(3a)\ X\subseteq Y\Rightarrow\underline{apr}^{I}_{A}(X)\subseteq\underline{apr}^{I}_{A}(Y);$$
$$(3b)\ X\subseteq Y\Rightarrow\overline{apr}^{I}_{A}(X)\subseteq\overline{apr}^{I}_{A}(Y);$$
$$(4a)\ \underline{apr}^{I}_{A}(X\cap Y)=\underline{apr}^{I}_{A}(X)\cap\underline{apr}^{I}_{A}(Y);$$
$$(4b)\ \overline{apr}^{I}_{A}(X\cup Y)=\overline{apr}^{I}_{A}(X)\cup\overline{apr}^{I}_{A}(Y);$$
$$(5a)\ \underline{apr}^{I}_{A}(X\cup Y)\supseteq\underline{apr}^{I}_{A}(X)\cup\underline{apr}^{I}_{A}(Y);$$
$$(5b)\ \overline{apr}^{I}_{A}(X\cap Y)\subseteq\overline{apr}^{I}_{A}(X)\cap\overline{apr}^{I}_{A}(Y).$$

From the descriptions, which are the representation forms of information and used in form of classes in rough set theory, while $R_s(x)=\varnothing$, x is considered carrying empty information and the successor neighborhood of x with respect to R is an empty description. Empty information is also an information form and empty descriptions may need to be considered in concept characterizing. Empty descriptions are considered to support any concept precisely in the above Definition 3.1. That is, if $R_s(x)=\varnothing$, x is regard as an object which support any concept precisely with respect to R since $R_s(x)=\varnothing\subseteq X$ hold for any $X\subseteq U$. Thus, any x such that $R_s(x)=\varnothing$ is included in the lower approximation set of arbitrary concept. But in any upper approximation sets, all x satisfy $R_s(x)=\varnothing$ are not included for $\forall x\in\overline{appr}^{I}_{A}(X)\Leftrightarrow R_s(x)\cap X\neq\varnothing$. That is, for any $x\in\overline{appr}^{I}_{A}(X)$, $R_s(x)\neq\varnothing$. The empty descriptions are considered in the lower approximation but they are left out in upper approximation. Hence, we have that the property $\underline{apr}^{I}_{A}(X)\subseteq\overline{apr}^{I}_{A}(X)$ does not hold for an arbitrary general binary relation in GBRS(I) . To understand GBRS(I) in the key idea of Pawlak rough approximations, objects precisely support any concept don't have the capability to possibly support all concepts in GBRS(I). This conclusion is illogical and unreasonable in applications. However, GBRS(I) is still a very useful rough set model for being construct directly from Pawlak rough set. Its strong suit is that the approximation operators still satisfy duality. So GBRS(I) can possess better generalization ability for binary relations satisfy particular properties such as reflexive and serial.

Example may be vivid to express the properties analyzed above and the following example is employed to interpret GBRS(I) practicality.

Example 3.1 Let $U=\{x_1,x_2,\cdots,x_5\}$ be the universe. $R\subseteq U\times U$ is a general binary relation on the universe. $R=\{(x_1,x_2),(x_1,x_4),(x_2,x_3),(x_2,x_5),(x_3,x_3),(x_3,x_4),(x_5,x_4),(x_5,x_5)\}$. $A=(U,R)$ is the generalized approximation space with respect to R on the universe. $X_1=\{x_1,x_2,x_4\}, X_2=\{x_3,x_4,x_5\}$. Calculate the lower and upper approximations of X_1, X_2 using GBRS(I).

According to Definition 3.1, we have that

$$R_s(x_1)=\{x_2,x_4\};$$
$$R_s(x_2)=\{x_3,x_5\};$$
$$R_s(x_3)=\{x_3,x_4\};$$
$$R_s(x_4)=\varnothing;$$
$$R_s(x_5)=\{x_4,x_5\}.$$

From Definition 3.2, the lower and upper approximations of X_1, X_2 can be obtained and listed in the following.

$$\underline{apr}^{\mathrm{I}}_A(X_1)=\{x_1,x_4\};$$
$$\overline{apr}^{\mathrm{I}}_A(X_1)=\{x_1,x_3,x_5\};$$
$$\underline{apr}^{\mathrm{I}}_A(X_2)=\{x_2,x_3,x_4,x_5\};$$
$$\overline{apr}^{\mathrm{I}}_A(X_2)=\{x_1,x_2,x_3,x_4,x_5\}.$$

Then we have that $\underline{apr}^{\mathrm{I}}_A(X)\subseteq\overline{apr}^{\mathrm{I}}_A(X)$ don't hold in GBRS(I). Moreover, properties can be verified from this example and we don't illustrate them in detail.

4. GBRS with Empty Descriptions Left Out

According to the analysis of GBRS(I), we generalize general binary relation based rough set and discuss the properties of the new approximation operators in the following sections. From Section 3, the empty information or empty descriptions can be disposed such that the approximation operators adapt the rough set theory logically. Then, general binary relation based rough set can be generalized by empty descriptions being regarded as supporting none concepts in rough set theory. The empty descriptions should be left out via this treatment to the empty information. Therefore, the lower and upper approximation sets are both consisted of objects bear nonempty information with respect to general binary relation. That is, $\forall x\in\underline{apr}_A(X)\Rightarrow R_s(x)\neq\varnothing$ and $\forall x\in\overline{apr}_A(X)\Rightarrow R_s(x)\neq\varnothing$ hold. Whereupon, we have the following new definition of general binary relation based rough set with respect to the generalized approximation space $A=(U,R)$.

Definition 4.1 Let U be the universe. $R\subseteq U\times U$ is a general binary relation on the universe. For any $X\subseteq U$, the lower and upper approximations of X with respect to the generalized approximation space are defined, respectively, as

$$\text{(II)}\quad \underline{apr}^{\mathrm{II}}_A(X)=\{x\in U\,|\,(R_s(x)\subseteq X)\wedge(R_s(x)\neq\varnothing)\},$$
$$\overline{apr}^{\mathrm{II}}_A(X)=\{x\in U\,|\,R_s(x)\cap X\neq\varnothing\}.$$

X is definable with respect to A if and only if $\underline{apr}^{\mathrm{II}}_A(X)=\overline{apr}^{\mathrm{II}}_A(X)$. Else, X is rough with respect to A if and only if $\underline{apr}^{\mathrm{II}}_A(X)\neq\overline{apr}^{\mathrm{II}}_A(X)$. This form of rough set is called the second type of general binary relation based rough set and denoted by GBRS(II).

Corresponding to studies in the front section, the properties of approximation operators in GBRS(II) can be discussed and studied further in the following theorem.

Theorem 4.1 Let U be the universe. $R\subseteq U\times U$ is a general binary relation on the universe. For any $X,Y\subseteq U$, the following properties of lower approximation operator $\underline{apr}^{\mathrm{II}}_A$ and upper approximation operator $\overline{apr}^{\mathrm{II}}_A$ hold with respect to generalized approximation space A.

$$\text{(1a)}\quad \underline{apr}^{\mathrm{II}}_A(X)\subseteq\overline{apr}^{\mathrm{II}}_A(X);$$
$$\text{(1b)}\quad \bigcup_{x\in\underline{apr}_A(X)} R_s(x)\subseteq X\subseteq \bigcup_{x\in\overline{apr}_A(X)} R_s(x);$$
$$\text{(2a)}\quad \underline{apr}^{\mathrm{II}}_A(\varnothing)=\overline{apr}^{\mathrm{II}}_A(\varnothing)=\varnothing;$$
$$\text{(2b)}\quad \underline{apr}^{\mathrm{II}}_A(U)=\overline{apr}^{\mathrm{II}}_A(U)\subseteq U;$$
$$\text{(3a)}\quad X\subseteq Y\Rightarrow\underline{apr}^{\mathrm{II}}_A(X)\subseteq\underline{apr}^{\mathrm{II}}_A(Y);$$
$$\text{(3b)}\quad X\subseteq Y\Rightarrow\overline{apr}^{\mathrm{II}}_A(X)\subseteq\overline{apr}^{\mathrm{II}}_A(Y);$$
$$\text{(4a)}\quad \underline{apr}^{\mathrm{II}}_A(X\cap Y)=\underline{apr}^{\mathrm{II}}_A(X)\cap\underline{apr}^{\mathrm{II}}_A(Y);$$
$$\text{(4b)}\quad \overline{apr}^{\mathrm{II}}_A(X\cup Y)=\overline{apr}_A(X)\cup\overline{apr}^{\mathrm{II}}_A(Y);$$
$$\text{(5a)}\quad \underline{apr}^{\mathrm{II}}_A(X\cup Y)\supseteq\underline{apr}^{\mathrm{II}}_A(X)\cup\underline{apr}^{\mathrm{II}}_A(Y);$$
$$\text{(5b)}\quad \overline{apr}^{\mathrm{II}}_A(X\cap Y)\subseteq\overline{apr}^{\mathrm{II}}_A(X)\cap\overline{apr}^{\mathrm{II}}_A(Y).$$

Proof.

(1) While $\underline{apr}^{\mathrm{II}}_A(X)=\varnothing$, this item is obvious. Assume that $\underline{apr}^{\mathrm{II}}_A(X)\neq\varnothing$, then we have

$$\begin{aligned}\forall x\in \underline{apr}_A^{II}(X) &\Leftrightarrow (R_s(x)\subseteq X)\wedge(R_s(x)\neq\varnothing)\\ &\Rightarrow R_s(x)\cap X\neq\varnothing\\ &\Leftrightarrow x\in \overline{apr}_A^{II}(X).\end{aligned}$$

This item $\underline{apr}_A^{II}(X)\subseteq\overline{apr}_A^{II}(X)$ is proved. Item (1b) can be proved directly from Definition 4.1.

(2a) $\forall x\in U, R_s(x)\subseteq\varnothing\Leftrightarrow R_s(x)=\varnothing$ and $R_s(x)\cap\varnothing=\varnothing$. And then

$$\begin{aligned}\underline{apr}_A^{II}(\varnothing)&=\{x\in U|(R_s(x)\subseteq\varnothing)\wedge(R_s(x)\neq\varnothing)\}\\ &=\{x\in U|(R_s(x)=\varnothing)\wedge(R_s(x)\neq\varnothing)\}\\ &=\varnothing,\\ \overline{apr}_A^{II}(\varnothing)&=\{x\in U|R_s(x)\cap\varnothing\neq\varnothing\}=\varnothing.\end{aligned}$$

(2b) $\forall x\in U, R_s(x)\neq\varnothing\Leftrightarrow R_s(x)\cap U\neq\varnothing$ and $R_s(x)\subseteq U$. Then we have

$$\begin{aligned}\underline{apr}_A^{II}(U)&=\{x\in U|(R_s(x)\subseteq U)\wedge(R_s(x)\neq\varnothing)\}=\{x\in U|R_s(x)\neq\varnothing\},\\ \overline{apr}_A^{II}(U)&=\{x\in U|R_s(x)\cap U\neq\varnothing\}=\{x\in U|R_s(x)\neq\varnothing\}.\end{aligned}$$

Hence, $\underline{apr}_A^{II}(U)=\overline{apr}_A(U)$ is proved. Moreover, from $\{x\in U|R_s(x)\neq\varnothing\}\cup\{x\in U|R_s(x)=\varnothing\}=U$,we can easily obtain that $\underline{apr}_A^{II}(U)=\overline{apr}_A^{II}(U)\subseteq U$.

(3a) As $X\subseteq Y$, while $\underline{apr}_A^{II}(X)=\varnothing$, the item is obvious. Suppose that $\underline{apr}_A^{II}(X)\neq\varnothing$, then we have

$$\forall x\in\underline{apr}_A^{II}(X)\Leftrightarrow R_s(X)\subseteq X\Rightarrow R_s(X)\subseteq Y\Leftrightarrow x\in\underline{apr}_A^{II}(Y).$$

This item is proved.

(3b)As $X\subseteq Y$, while $\underline{apr}_A^{II}(X)=\varnothing$, this item is apparent. For $\underline{apr}_A^{II}(X)\neq\varnothing$, we have that

$$\forall x\in\overline{apr}_A^{II}(X), R_s(X)\cap X\neq\varnothing\Rightarrow R_s(X)\cap Y\neq\varnothing\Rightarrow x\in\overline{apr}_A^{II}(Y).$$

This item is proved.

(4a) From property (3a) in this theorem, one can easily have that $\underline{apr}_A^{II}(X\cap Y)\subseteq\underline{apr}_A^{II}(X)\cap\underline{apr}_A^{II}(Y)$. While $\underline{apr}_A^{II}(X)\cap\underline{apr}_A^{II}(Y)=\varnothing$, this property holds obviously. Assume that $\underline{apr}_A^{II}(X)\cap\underline{apr}_A^{II}(Y)\neq\varnothing$, then we have

$$\begin{aligned}&\forall x\in\underline{apr}_A^{II}(X)\cap\underline{apr}_A^{II}(Y)\\ &\Rightarrow(x\in\underline{apr}_A^{II}(X))\wedge(x\in\underline{apr}_A^{II}(Y))\\ &\Rightarrow[(R_s(x)\subseteq X)\wedge(R_s(x)\neq\varnothing)]\wedge[(R_s(x)\subseteq Y)\wedge(R_s(x)\neq\varnothing)]\\ &\Rightarrow(R_s(x)\subseteq X)\wedge(R_s(x)\neq\varnothing)\wedge(R_s(x)\subseteq Y)\\ &\Rightarrow(R_s(x)\subseteq X\cap Y)\wedge(R_s(x)\neq\varnothing)\\ &\Rightarrow x\in\underline{apr}_A^{II}(X\cap Y).\end{aligned}$$

The property $\underline{apr}_A^{II}(X\cap Y)\supseteq\underline{apr}_A^{II}(X)\cap\underline{apr}_A^{II}(Y)$ is obtained. Hence, this item is proved. And item (4b) can be proved similarly as item (4a) in this theorem.

Properties (5a) (5b) can be proved directly from items (3a)(3b) in this theorem. □

By the comparison of the properties above with those in Pawlak rough set, one can have that the duality doesn't hold in GBRS(II). The item (1) in Theorem 2.1 corresponds to items (1a) (1b) in Theorem 4.1 since Pawlak rough set can be defined by two equivalence forms. Considering the signification of Pawlak rough set, the approximation to concepts goes along in terms of the relation R. Objects in the approximation sets employ the information and descriptions they are bearing with respect to the relation R to characterize the approximated concepts. An object itself belongs to a concept or not relies on if it possesses the information, which precisely support the concept with respect to the relation R. That is, $R_s(x)\subseteq X\Leftrightarrow x\in\underline{appr}_A^{II}(X)$ hold but $y\in R_s(x)\subseteq X\Rightarrow y\in\underline{appr}_A^{II}(X)$ doesn't hold in GBRS(II). So, there exists no affirmative inclusion between the lower approximation set and the approximated concept in general binary relation based rough set. Similarly, there is no affirmative inclusion between the upper approximation set and the approximated concept in GBRS. We can only have that the property lower approximation set is included in upper approximation set holds in GBRS(II). And this comports as item (1a) in Theorem 4.1. Objects in classes undertake the information which are used to depict concepts directly in problems consulting with rough set. Among these vectors, ones precisely support a concept can't be more than those in the depicted concept itself with respect to the relation R. This comports as item (1b) in Theorem 4.1.

Based on these analysis, GBRS(II) has the capability to represent concepts approximately by means of ignoring empty descriptions. It can make the approximation to concepts feasible in sense of logic and practice. At the same time, a fly in the ointment is that the approximation operators in GBRS(II) dissatisfy the duality. Approximation operators which meet duality transform synchronously. The transformation is antithetical and obeys duality. While the lower approximation decrease, the upper approximation increases and acts in accordance with the lower approximation's transformation. While the lower approximation increases, the upper approximation decreases correspondingly to the transform of the lower approximation. Contrarily, the dual synchronous change of lower approximation holds while the upper approximation transforms. Regrettably, the dual synchronous change of lower and upper approximations can't hold while concepts are depicted with respect to arbitrary general binary relations in GBRS(II). However, GBRS(II) is useful theoretically and can affect much in applications.

We still employ Example 3.1 to explain GBRS(II) in this section and the results are presented in the following.

Example 4.1 (Continued from Example 3.1) From Example 3.1, calculate the lower and upper approximations and clear that the duality doesn't hold in GBRS(II).

According Definition 4.1 and Example 3.1, the approximations are listed as follows.

$$\underline{apr}_A^{\mathrm{II}}(X_1)=\{x_1\};$$
$$\overline{apr}_A^{\mathrm{II}}(X_1)=\{x_1,x_3,x_5\};$$
$$\underline{apr}_A^{\mathrm{II}}(X_2)=\{x_2,x_3,x_5\};$$
$$\overline{apr}_A^{\mathrm{II}}(X_2)=\{x_1,x_2,x_3,x_4,x_5\}.$$

And $\square X_1=\{x_3,x_5\}, \square X_2=\{x_1,x_2\}$. Furthermore, we have that

$$\underline{apr}_A^{\mathrm{II}}(\square X_1)=\{x_2\};$$
$$\overline{apr}_A^{\mathrm{II}}(\square X_1)=\{x_2,x_3,x_5\};$$
$$\underline{apr}_A^{\mathrm{II}}(\square X_2)=\varnothing;$$
$$\overline{apr}_A^{\mathrm{II}}(\square X_2)=\{x_1\}.$$

Then, we can see that the duality doesn't hold in GBRS(III). Other properties in Theorem 4.1 can be verified and they are not arranged in this paper.

5. GBRS with Empty Descriptions Taken into Account

Taking GBRS(I) as a foundation, if objects with empty descriptions are all considered to support the any concepts precisely with respect to the relation R, then they should possibly support the concepts with respect to R according to the meaning and key idea of Pawlak rough approximating. Therefore, GBRS(I) can be generalized with empty descriptions taken into account in both lower and upper approximations. Then GBRS in the following form can be defined and discussed.

Definition 5.1 Let U be the universe. $R\subseteq U\times U$ is a general binary relation on the universe. For any $X\subseteq U$, the lower and upper approximations of X with respect to generalized approximation space A are defined, respectively, as

$$\text{(III)}\quad \underline{apr}_A^{\mathrm{III}}(X)=\{x\in U|R_s(x)\subseteq X\},$$
$$\overline{apr}_A^{\mathrm{III}}(X)=\{x\in U|(R_s(x)\cap X\neq\varnothing)\vee(R_s(x)=\varnothing)\}.$$

X is definable with respect to A if and only if $\underline{apr}_A^{\mathrm{III}}(X)=\overline{apr}_A^{\mathrm{III}}(X)$. Else, X is rough with respect to A if and only if $\underline{apr}_A^{\mathrm{III}}(X)\neq\overline{apr}_A^{\mathrm{III}}(X)$. This form of rough set is called the third type of general binary relation based rough set and denoted by GBRS(III).

Similarly as the above discussions, properties of approximation operators in GBRS(III) can be studied and presented in the following theorem.

Theorem 5.1 Let U be the universe. $R \subseteq U \times U$ is a general binary relation on the universe. For any $X, Y \subseteq U$, the following properties of lower and upper approximation operators with respect to generalized approximation space A hold in GBRS(III).

(1a) $\underline{apr}_A^{III}(X) \subseteq \overline{apr}_A^{III}(X)$;

(1b) $\bigcup_{x \in \underline{apr}_A^{III}(X)} R_s(x) \subseteq X \subseteq \bigcup_{x \in \overline{apr}_A^{III}(X)} R_s(x)$;

(1c) $x \in U, R_s(x) = \varnothing \Rightarrow \forall X \subseteq U, x \in \underline{apr}_A^{III}(X)$;

(1d) $\bigcup_{x \in U} \{x | R_s(x) = \varnothing\} \subseteq \bigcap_{k=1}^{m} \underline{apr}_A^{III}(X_k), (\forall X_k \subseteq U)$;

(1e) $\bigcup_{k=1}^{m} X_k = U \Rightarrow \bigcup \{x \in U | R_s(x) = \varnothing\} = \bigcap_{k=1}^{m} \underline{apr}_A^{III}(X_k), (X_k \subseteq U)$;

(2a) $\varnothing \subseteq \underline{apr}_A^{III}(\varnothing) = \overline{apr}_A^{III}(\varnothing)$;

(2b) $\underline{apr}_A^{III}(U) = \overline{apr}_A^{III}(U) = U$;

(3a) $X \subseteq Y \Rightarrow \underline{apr}_A^{III}(X) \subseteq \underline{apr}_A^{III}(Y)$;

(3b) $X \subseteq Y \Rightarrow \overline{apr}_A^{III}(X) \subseteq \overline{apr}_A^{III}(Y)$;

(4a) $\underline{apr}_A^{III}(X \cap Y) = \underline{apr}_A^{III}(X) \cap \underline{apr}_A^{III}(Y)$;

(4b) $\overline{apr}_A^{III}(X \cup Y) = \overline{apr}_A^{III}(X) \cup \overline{apr}_A^{III}(Y)$;

(5a) $\underline{apr}_A^{III}(X \cup Y) \supseteq \underline{apr}_A^{III}(X) \cup \underline{apr}_A^{III}(Y)$;

(5b) $\overline{apr}_A^{III}(X \cap Y) \subseteq \overline{apr}_A^{III}(X) \cap \overline{apr}_A^{III}(Y)$.

Proof.

(1a) For any $x \in \underline{apr}_A^{III}(X)$, while $R_s(x) = \varnothing$, $x \in \overline{apr}_A^{III}$ is obvious and this item holds. Assume that $R_s(x) \neq \varnothing$, then we have that

$$x \in \underline{apr}_A^{III}(X) \Leftrightarrow R_s(x) \subseteq X \Rightarrow R_s(x) \cap X \neq \varnothing \Leftrightarrow x \in \overline{apr}_A^{III}(X).$$

Hence, $\underline{apr}_A^{III}(X) \subseteq \overline{apr}_A^{III}(X)$ is proved.

Item (1b) (1c) can be proved directly from Definition 5.1.

(1d) For any $x \in \bigcup_{x \in U} \{x | R_s(x) = \varnothing\}$, $R_s(x) = \varnothing$ holds. Then, one can have that

$$R_s(x) = \varnothing \subseteq \underline{apr}_A^{III}(X_k), (\forall X_k \subseteq U) \Rightarrow x \in \bigcap_{k=1}^{m} \underline{apr}_A^{III}(X_k), (\forall X_k \subseteq U).$$

Hence, this property is proved.

(2a) According to Definition 5.1, we can easily obtain that

$$\underline{apr}_A^{III}(\varnothing) = \{x \in U | R_s(x) \subseteq \varnothing\} = \{x \in U | R_s(x) = \varnothing\},$$

$$\begin{aligned}\overline{apr}_A^{III}(\varnothing) &= \{x \in U | (R_s(x) \cap \varnothing \neq \varnothing) \vee (R_s(x) = \varnothing)\} \\ &= \{x \in U | R_s(x) = \varnothing\},\end{aligned}$$

Thus, we have $\underline{apr}_A^{III}(\varnothing) = \overline{apr}_A^{III}(\varnothing)$ hold. If there exists any $x \in U$ such that $R_s(x) = \varnothing$, then we have that $\underline{apr}_A^{III}(\varnothing) \neq \varnothing$. Else $\underline{apr}_A^{III}(\varnothing) = \varnothing$. Hence, this item $\varnothing \subseteq \underline{apr}_A^{III}(\varnothing) = \overline{apr}_A^{III}(\varnothing)$ is proved.

(2b) For any $x \in U$, $R_s(x) \subseteq U$ holds. Furthermore, the following processes hold.

$$\begin{aligned}\underline{apr}_A^{III}(U) &= \{x \in U | R_s(x) \subseteq U\} = U, \\ \overline{apr}_A^{III}(U) &= \{x \in U | (R_s(x) \cap U \neq \varnothing) \vee (R_s(x) = \varnothing)\} \\ &= \{x \in U | R_s(x) \subseteq U\},\end{aligned}$$

Then, this item is proved.

(3a) Since $X \subseteq Y$, for any $x \in \underline{apr}_A^{III}(X)$, we have that $R_s(x) \subseteq X \subseteq Y$. Obviously, $x \in \in \underline{apr}_A^{III}(Y)$ hold and this item is proved.

(3b) For any $x \in \overline{apr}_A^{III}(X)$, while $R_s(x) = \varnothing$, $x \in \overline{apr}_A^{III}(Y)$ is obvious. Suppose $R_s(x) \neq \varnothing$, then we have $R_s(x) \cap X \neq \varnothing$. As $X \subseteq Y$ has been known , one can acquire that $R_s(x) \cap Y \neq \varnothing$. That is, $x \in \overline{apr}_A^{III}(Y)$ holds. Hence, This item is proved.

(4a) We can easily have that

$$
\begin{aligned}
\forall x \in \underline{apr}_A^{\mathrm{III}}(X \cap Y) &\Leftrightarrow R_s(x) \subseteq X \cap Y \\
&\Leftrightarrow (R_s(x) \subseteq X) \wedge (R_s(x) \subseteq Y) \\
&\Leftrightarrow (x \in \underline{apr}_A^{\mathrm{III}}(X)) \wedge (x \in \underline{apr}_A^{\mathrm{III}}(Y)) \\
&\Leftrightarrow x \in \underline{apr}_A^{\mathrm{III}}(X) \cap \underline{apr}_A^{\mathrm{III}}(Y).
\end{aligned}
$$

Thus, $\underline{apr}_A^{\mathrm{III}}(X \cap Y) = \underline{apr}_A^{\mathrm{III}}(X) \cap \underline{apr}_A^{\mathrm{III}}(Y)$ is proved.

(4b) Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, the formula $\overline{apr}_A^{\mathrm{III}}(X) \cup \overline{apr}_A^{\mathrm{III}}(Y) \subseteq \overline{apr}_A^{\mathrm{III}}(X \cup Y)$ is obvious from item (3b) in this theorem. Moreover, we have that

$$
\begin{aligned}
\forall x \in \overline{apr}_A^{\mathrm{III}}(X \cup Y) &\Rightarrow (R_s(x) \subseteq X \cup Y) \vee (R_s(x) = \varnothing) \\
&\Rightarrow (R_s(x) \subseteq X) \vee (R_s(x) = \varnothing) \vee (R_s(x) \subseteq Y) \\
&\Rightarrow [(R_s(x) \subseteq X) \vee (R_s(x) = \varnothing)] \vee [(R_s(x) \subseteq Y) \vee (R_s(x) = \varnothing)] \\
&\Rightarrow (x \in \overline{apr}_A^{\mathrm{III}}(X)) \vee (x \in \overline{apr}_A^{\mathrm{III}}(Y)) \\
&\Rightarrow x \in \overline{apr}_A^{\mathrm{III}}(X) \cup \overline{apr}_A^{\mathrm{III}}(Y).
\end{aligned}
$$

Hence, $\overline{apr}_A^{\mathrm{III}}(X \cup Y) \subseteq \overline{apr}_A^{\mathrm{III}}(X) \cup \overline{apr}_A^{\mathrm{III}}(Y)$ holds. This item is proved.

Items (5a) (5b) can be proved directly and respectively by items (3a) (3b) in this theorem. □

From Definition 5.1 and Theorem 5.1, it can be known that the greatest strengths of GBRS(III) is that objects precisely support concepts are ones carrying possibly support information with respect to the relation. That is to say $\underline{apr}_A^{\mathrm{III}}(X) \subseteq \overline{apr}_A^{\mathrm{III}}(X)$. It accords with the key idea and the meaning of rough approximating to concepts in rough set theory.

Empty descriptions are considered and disposed as supporting any concept precisely and possibly supporting all concepts with respect to the relation. The depictions to concepts are more comprehensive and integrated by this disposal of empty information. But the lower and upper approximation operators in GBRS still can’t satisfy duality and the dual synchronous change of lower and upper approximations can't hold with respect to arbitrary general binary relations in GBRS(III). Though the duality is not satisfied, GBRS(III) has the ability to represent concepts approximately in rough set theory.

The example developed in section 3 will be still employed in this section to illustrate GBRS(III) as follows.

Example 5.1 (Continued from Example 3.1) From Example 3.1, calculate the lower and upper approximations and clear that the duality doesn’t hold in GBRS(III).

According to Definition 5.1 and Example 3.1, the approximations are calculated and listed in the following.

$$
\begin{aligned}
&\underline{apr}_A^{\mathrm{III}}(X_1) = \{x_1, x_4\}; \\
&\overline{apr}_A^{\mathrm{III}}(X_1) = \{x_1, x_3, x_4, x_5\}; \\
&\underline{apr}_A^{\mathrm{III}}(X_2) = \{x_2, x_3, x_4, x_5\}; \\
&\overline{apr}_A^{\mathrm{III}}(X_2) = \{x_1, x_2, x_3, x_4, x_5\}.
\end{aligned}
$$

Furthermore, $\sim X_1 = \{x_3, x_5\}, \sim X_2 = \{x_1, x_2\}$. Moreover, we have that

$$
\begin{aligned}
&\underline{apr}_A^{\mathrm{III}}(\sim X_1) = \{x_2, x_4\}; \\
&\overline{apr}_A^{\mathrm{III}}(\sim X_1) = \{x_2, x_3, x_4, x_5\}; \\
&\underline{apr}_A^{\mathrm{III}}(\sim X_2) = \{x_4\}; \\
&\overline{apr}_A^{\mathrm{III}}(\sim X_2) = \{x_1, x_4\}.
\end{aligned}
$$

Then, we can easily verify that the duality doesn’t hold in GBRS(III), either. Other properties will be not verified in this section.

6. Axiomatic GBRS

From the above approaches, we can see that the lower and upper approximation operators in GBRS(I) satisfy duality but those in GBRS(II) and GBRS(III) don't satisfy the duality. Empty descriptions are only employed to depict concepts precisely but not considered as possibly support descriptions in GBRS(I). Unlike GBRS(I), empty descriptions are ignored totally in GBRS(II) and are employed to characterize concepts both precisely and possibly in GBRS(III). GBRS(I) is generalized and obtained by constructive method in rough set theory. Both the lower and upper approximation operators of GBRS(I) keep identical with those of Pawlak rough set in the form and expression. The upper approximation ensures the correction of descriptions possibly support concepts. But the lower approximation imports extra objects carrying empty information with respect to general binary relations. These objects are included in lower approximation set but not included in upper approximation set. With empty descriptions being ignored and left out, GBRS(II) can depict concepts more feasible and reasonable corresponding to the rough approximation idea in rough set theory. As empty descriptions being considered in both lower and upper approximations, GBRS(III) has the capability to depict concepts with more comprehensive information. But the duality fails in GBRS(II) and GBRS(III). The approximation operators can't act dually.

According to the axiomatic method in rough set theory and the above three forms of GBRS, lower and upper approximation operators satisfy duality can be advanced and discussed further. A new form of GBRS can be defined and studied in the following.

Definition 6.1 Let U be the universe. $R \subseteq U \times U$ is a general binary relation on the universe. For any $X \subseteq U$, the lower and upper approximations of X with respect to the generalized approximation space A are defined, respectively, as

$$\text{(IV)}\ \underline{apr}_A^{\mathrm{IV}}(X) = \{x \in U \mid (R_s(x) \subseteq X) \wedge (R_s(x) \neq \varnothing)\},$$
$$\overline{apr}_A^{\mathrm{IV}}(X) = \{x \in U \mid (R_s(x) \cap X \neq \varnothing) \vee (R_s(x) = \varnothing)\}.$$

X is definable with respect to A if and only if $\underline{apr}_A^{\mathrm{IV}}(X) = \overline{apr}_A^{\mathrm{IV}}(X)$. Else, X is rough with respect to A if and only if $\underline{apr}_A^{\mathrm{IV}}(X) \neq \overline{apr}_A^{\mathrm{IV}}(X)$. This form of rough set is called the forth type of general binary relation based rough set and denoted by GBRS(IV).

Similarly as the above approaches, properties of lower and upper approximation operators in GBRS(IV) will be discussed and studied in the following theorem.

Theorem 6.1 Let U be the universe. $R \subseteq U \times U$ is a general binary relation on the universe. For any $X, Y \subseteq U$, the following properties of lower and upper approximation operators with respect to generalized approximation space A hold in GBRS(IV).

(1a) $\underline{apr}_A^{\mathrm{IV}}(X) \subseteq \overline{apr}_A^{\mathrm{IV}}(X)$;

(1b) $\bigcup_{x \in \underline{apr}_A^{\mathrm{IV}}(X)} R_s(x) \subseteq X$, $\bigcup_{x \in \underline{apr}_A^{\mathrm{IV}}(X)} R_s(x) = \varnothing$ *is ordered while* $\underline{apr}_A^{\mathrm{IV}}(X) = \varnothing$;

(1c) $X \subseteq \bigcup_{x \in \overline{apr}_A^{\mathrm{IV}}(X)} R_s(x)$, *while* $\bigcup_{x \in U} R_s(x) = U$;

(2a) $\underline{apr}_A^{\mathrm{IV}}(\sim X) = \sim \overline{apr}_A^{\mathrm{IV}}(X)$;

(2b) $\overline{apr}_A^{\mathrm{IV}}(\sim X) = \sim \underline{apr}_A^{\mathrm{IV}}(X)$;

(3a) $\varnothing = \underline{apr}_A^{\mathrm{IV}}(\varnothing) \subseteq \overline{apr}_A^{\mathrm{IV}}(\varnothing)$;

(3b) $\underline{apr}_A^{\mathrm{IV}}(U) \subseteq \overline{apr}_A^{\mathrm{IV}}(U) = U$;

(4a) $X \subseteq Y \Rightarrow \underline{apr}_A^{\mathrm{IV}}(X) \subseteq \underline{apr}_A^{\mathrm{IV}}(Y)$;

(4b) $X \subseteq Y \Rightarrow \overline{apr}_A^{\mathrm{IV}}(X) \subseteq \overline{apr}_A^{\mathrm{IV}}(Y)$;

(5a) $\underline{apr}_A^{\mathrm{IV}}(X \cap Y) = \underline{apr}_A^{\mathrm{IV}}(X) \cap \underline{apr}_A^{\mathrm{IV}}(Y)$;

(5b) $\overline{apr}_A^{\mathrm{IV}}(X \cup Y) = \overline{apr}_A^{\mathrm{IV}}(X) \cup \overline{apr}_A^{\mathrm{IV}}(Y)$;

(6a) $\underline{apr}_A^{\mathrm{IV}}(X \cup Y) \supseteq \underline{apr}_A^{\mathrm{IV}}(X) \cup \underline{apr}_A^{\mathrm{IV}}(Y)$;

(6b) $\overline{apr}_A^{\mathrm{IV}}(X \cap Y) \subseteq \overline{apr}_A^{\mathrm{IV}}(X) \cap \overline{apr}_A^{\mathrm{IV}}(Y)$.

Proof.

(1a)From Definition 6.1, we can easily have that

$$\begin{aligned} \forall x \in \underline{apr}_A^{\mathrm{IV}}(X) &\Leftrightarrow (R_s(x) \subseteq X) \wedge (R_s(x) \neq \varnothing) \\ &\Rightarrow R_s(x) \cap X = R_s(x) \neq \varnothing \\ &\Rightarrow x \in \overline{apr}_A^{\mathrm{IV}}(X). \end{aligned}$$

Hence, $\underline{apr}_A^{\mathrm{IV}}(X) \subseteq \overline{apr}_A^{\mathrm{IV}}(X)$ is proved.

(1b) While $\underline{apr}_A^{\mathrm{IV}}(X) = \varnothing$, $\bigcup_{x \in \underline{apr}_A^{\mathrm{IV}}(X)} R_s(x) = \varnothing$ and this item is obvious. Assume that $\underline{apr}_A^{\mathrm{IV}}(X) \neq \varnothing$. For any

$x \in \underline{apr}_A^{IV}(X)$, one can obtain that $(R_s(x) \subseteq X) \wedge (R_s(x) \neq \varnothing)$. Then, we have that $\varnothing \neq \bigcup_{x \in \underline{apr}_A^{IV}(X)} R_s(x) \subseteq X$. Hence, the property is proved.

(1c) From $\bigcup_{x \in U} R_s(x) = U$, one can have that $\{R_s(x) | x \in U\}$ constructs a covering of the universe. Thus,

$$\begin{aligned} \forall y \in X &\Rightarrow \exists x \in U \ s.t.\ y \in R_s(x) \neq \varnothing \\ &\Rightarrow y \in R_s(x) \bigcap X \neq \varnothing \\ &\Rightarrow X \subseteq \cup \{R_s(x) | R_s(x) \bigcap X \neq \varnothing\}. \end{aligned}$$

Moreover, from $x \in \overline{apr}_A^{IV}(X) \Leftrightarrow (R_s(x) \bigcap X \neq \varnothing) \vee (R_s(x) = \varnothing)$, one can obtain that

$$\bigcup_{x \in \overline{apr}_A^{IV}(X)} R_s(x) = \{\cup \{R_s(x) | R_s(x) \bigcap X \neq \varnothing\}\} \bigcup \{\cup \{R_s(x) | R_s(x) = \varnothing\}\}.$$

Hence,

$$X \subseteq \bigcup_{x \in U} \{R_s(x) | R_s(x) \bigcap X \neq \varnothing\} \subseteq \bigcup_{x \in \overline{apr}_A^{IV}(X)} R_s(x).$$

This item is proved.

(2a) As is known $R_s(x) \subseteq \sim X \Leftrightarrow R_s(x) \bigcap X = \varnothing$， this item can be proved from Definition 6.1 as follows.

$$\begin{aligned} \sim \underline{apr}_A^{IV}(\sim X) &= \sim \{x \in U | (R_s(x) \subseteq \sim X) \wedge (R_s(x) \neq \varnothing)\} \\ &= \sim \{x \in U | (R_s(x) \bigcap X = \varnothing) \wedge (R_s(x) \neq \varnothing)\} \\ &= \{x \in U | (R_s(x) \bigcap X \neq \varnothing) \vee (R_s(x) = \varnothing)\} \\ &= \overline{apr}_A^{IV}(X). \end{aligned}$$

(2b) This item can be proved similarly as item (2a) in this theorem.

$$\begin{aligned} \sim \overline{apr}_A^{IV}(\sim X) &= \sim \{x \in U | (R_s(x) \bigcap \sim X \neq \varnothing) \vee (R_s(x) = \varnothing)\} \\ &= \{x \in U | (R_s(x) \bigcap \sim X = \varnothing) \wedge (R_s(x) \neq \varnothing)\} \\ &= \{x \in U | (R_s(x) \subseteq X) \wedge (R_s(x) \neq \varnothing)\} \\ &= \underline{apr}_A^{IV}(X). \end{aligned}$$

(3a) From the property of cantor set, there exists no object satisfying $(R_s(x) \subseteq \varnothing) \wedge (R_s(x) \neq \varnothing)$, i.e.,

$$\underline{apr}_A^{IV}(\varnothing) = \{x \in U | (R_s(x) \subseteq \varnothing) \wedge (R_s(x) \neq \varnothing)\} = \varnothing.$$

Moreover, one can easily obtain that

$$\overline{apr}_A^{IV}(\varnothing) = \{x \in U | (R_s(x) \bigcap \varnothing \neq \varnothing) \vee (R_s(x) = \varnothing)\} = \{x \in U | R_s(x) = \varnothing\}.$$

Hence, this item is proved.

Item (3b) can be proved by the duality or proved similarly as item (3a) in this theorem.

(4a) Since $X \subseteq Y$, we have that

$$\begin{aligned} \forall x \in \underline{apr}_A^{IV}(X) &\Leftrightarrow (R_s(x) \subseteq X) \wedge (R_s(x) \neq \varnothing) \\ &\Rightarrow (R_s(x) \subseteq Y) \wedge (R_s(x) \neq \varnothing) \\ &\Leftrightarrow x \in \underline{apr}_A^{IV}(Y). \end{aligned}$$

Thus, $\underline{apr}_A^{IV}(X) \subseteq \underline{apr}_A^{IV}(Y)$. This item is proved.

Item (4b) can be proved similarly as item (4a) in this theorem.

Items (5a)(5b)(6a)(6b) can be proved as those in the above sections. □

GBRS(IV) is a more feasible model based on general binary relation. The lower and upper approximation operators satisfy duality and hold the property lower approximation is included in the upper approximation. Empty descriptions are considered as precisely support none concepts but possibly support arbitrary one with respect to general binary relation in rough set theory. This disposal of empty information can promote concept description and representation more reasonable and logical. Knowledge acquisition can be more precise and integrated in problems consulting with general binary relation based rough set.

We still employ Example 3.1 to illustrate GBRS(IV) as follows in this section.

Example 6.1(Continued from Example 3.1) From Example 3.1, calculate the lower and upper approximations and clear that the duality doesn't hold in GBRS(IV).

According to Definition 6.1 and Example 3.1, the approximations are calculated and listed in the following.

$$\underline{apr}_A^{\mathrm{III}}(X_1)=\{x_1\};$$
$$\overline{apr}_A^{\mathrm{III}}(X_1)=\{x_1,x_3,x_4,x_5\};$$
$$\underline{apr}_A^{\mathrm{III}}(X_2)=\{x_2,x_3,x_5\};$$
$$\overline{apr}_A^{\mathrm{III}}(X_2)=\{x_1,x_2,x_3,x_4,x_5\}.$$

Furthermore, $\square X_1=\{x_3,x_5\},\square X_2=\{x_1,x_2\}$. Moreover, we have that

$$\underline{apr}_A^{\mathrm{III}}(\square X_1)=\{x_2\};$$
$$\overline{apr}_A^{\mathrm{III}}(\square X_1)=\{x_2,x_3,x_4,x_5\};$$
$$\underline{apr}_A^{\mathrm{III}}(\square X_2)=\varnothing;$$
$$\overline{apr}_A^{\mathrm{III}}(\square X_2)=\{x_1,x_4\}.$$

Then, we can easily verify the properties in Theorem 6.1 and they will not be arranged in this section for being obvious.

7. Comparison and Relations among Four Types of GBRS

As defined and discussed separately in the above sections, four types of GBRS can be related and compared. The following theorems can be concluded and studied.

Theorem 7.1 While the general binary relation is serial, the four types of GBRS defined by successor neighborhood are equivalent.

Theorem 7.2 While the general binary relation is inverse serial, the four types of GBRS defined by processor neighborhood are equivalent.

Theorem 7.3 While the general binary relation is reflexive, lower and upper approximation operators in the four types of GBRS all satisfy $\underline{apr}_A^{*}(X)\subseteq X\subseteq\overline{apr}_A^{*}$, where * represents the type of GBRS (I), (II), (III), (IV), respectively.

Theorem 7.4 While the relations are equivalence relations, then four types of GBRS are all Pawlak rough set.

Theorem 7.5 Let U be the universe. $R\subseteq U\times U$ is a general binary relation on the universe. For any $X\subseteq U$, the following relations among four types of GBRS hold.

(1) $\underline{apr}_A^{\mathrm{IV}}(X)=\underline{apr}_A^{\mathrm{II}}(X)\subseteq\underline{apr}_A^{\mathrm{III}}(X)=\underline{apr}_A^{\mathrm{I}}(X);$

(2) $\overline{apr}_A^{\mathrm{I}}(X)=\overline{apr}_A^{\mathrm{II}}(X)\subseteq\overline{apr}_A^{\mathrm{III}}(X)=\overline{apr}_A^{\mathrm{IV}}(X);$

(3) $\underline{apr}_A^{\mathrm{II}}(X)\subseteq\overline{apr}_A^{\mathrm{I}}(X);$

(4) $\underline{apr}_A^{\mathrm{I}}(X)\subseteq\overline{apr}_A^{\mathrm{III}}(X).$

Proof. These theorems are obvious and can be proved directly by the corresponding definitions and properties.□

Other special general binary relations can be considered and we don't discuss them any more in this paper. In addition, empty descriptions and empty information are usually existed in many rough set models and they can be treated according to requirements similarly as one of the four forms presented in this paper.

8. Conclusions

According to the theoretical constructive method and axiomatic method in rough set theory, general binary relation based rough set was discussed and studied in this paper. The key idea of approximative representation to concepts and the form of information representation were considered. Empty descriptions and empty information were disposed in different ways. By analyzing of the general binary relation based rough set constructed from Pawlak rough set, three new types of general binary relation based rough set were defined and studied. Moreover, important properties of different lower and upper approximation operators were discussed and approached in further. The four types of general binary relation based rough set have valuable significance under different perspectives and requirements. And they can adapt the approximating to concepts better in applications and researches employing general binary relation based rough set. Properties of different lower and upper approximation operators can promote the corresponding type of general binary relation based rough set more effective and succinct whenever being used. The four types of general binary relation based rough set can be associated with each other to improve the generalization ability and continuability of rough set in dealing with more practical problems.

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10.References

[1] I. Masahiro and T. Tetsuzo. Generalized rough sets and rule extraction. *in Rough Sets and Current Trends in Computing*. Berlin: Springer Berlin/Heidelberg. 2002, **2475**: 105-112.

[2] K. Michiro. On the structure of generalized rough sets. *Information Sciences.* 2006, **176**: 589-600.

[3] Z. Pawlak and A. Skowron. Rudiments of rough sets. *Information Sciences.* 2007, **177**: 3-27.

[4] Z. Pawlak and A. Skowron. Rough sets: Some Extensions. *Information Sciences.* 2007, **177**: 28-40.

[5] Z. Pawlak and A. Skowron. Rough sets and Boolean Reasoning. *Information Sciences.* 2007, **177**: 41-73.

[6] Z. William and F.Y. Wang. Binary Relation Based Rough Sets. *FSKD, LNAI.* 2006, **4223**: 276-285.

[7] Z. William. Relationship between generalized rough sets based on binary relation and covering. *Information Sciences.* 2009, **179**: 210-225.

[8] Z. William. Generalized rough sets based on relations. *Information Sciences*. 2007, **177**: 4997-5011.

[9] W.H. Xu and W.X. Zhang. Measuring roughness of generalized rough sets induced by a covering. *Fuzzy Sets and Systems*. 2007, **158**: 2443-2455.

[10] W.H. Xu and W.X. Zhang. Methods for knowledge reduction in inconsistent ordered information systems. *Journal of Applied Mathematics and Computing*. 2008, **26**: 313-323.

[11] W.H. Xu, X.Y. Zhang, J.M Zhong and W.X. Zhang. Attribute reduction in ordered information systems based on evidence theory. *Knowledge and Information Systems.* 2010, **2**5: 69-184.

[12] Y.Y. Yao. Generalized rough set models. *in Rough Sets in Knowledge Discovery*. Polkowski and Skowron, Eds. Heidelberg: Physica-Verlag. 1998, pp.286-318.

[13] Y.Y. Yao and T.Y. Lin. Generalization of rough sets using modal logics. *Intelligent Automation and Soft Computing.* 1996, **2**: 103-120.

[14] W.X. Zhang, W.Z. Wu, J.Y. Liang and D.Y. Li. *Rough Set Theory and Methods*. Beijing China: Science Press, 2003.

[15] W.X. Zhang, Y. Liang and W.Z. Wu. *Information System and Knowledge Discovery*. Beijing, China: Science Press, 2003.