

The Computation of Common Infinity-norm Lyapunov Functions for Linear Switched Systems

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Abstract. This paper studies the problem of the computation of common infinity-norm Lyapunov functions. For a set of continuous-time LTI systems or discrete-time LTI systems whose system matrices are upper triangular form or lower triangular form, it is proved that there exist common infinity-norm Lyapunov functions for them. Then four algorithms of computing common infinity-norm Lyapunov functions are presented. Finally, several examples are listed.

Keywords: Common Lyapunov functions, infinity-norm, switched systems

1. Introduction

A switched systems is one that combines continuous (or discrete) dynamics with a logic-based switching mechanism that determines abrupt mode switches in the system, operation at various points in time^[1]. Most research has been devoted to the stability of switched systems^[2-4]. As we know, Lyapunov functions play an important role in the stability theory of control systems for some time. In view of this, a considerable amount of recent work has focused on applying similar ideas to switched systems. Most recently many authors have derived conditions for the stability of linear switched systems based on the existence of common Lyapunov functions for their constituent systems^[5-7]. For numerical and practical reasons, common quadratic Lyapunov functions are usually chosen^[8-10]. However, quadratic Lyapunov functions can be too conservation and efforts have been devoted to the development of other types of common Lyapunov functions. Common infinity-norm Lyapunov functions are important one which have been studied to considerable extent^[11-12]. How to compute common Lyapunov functions is of importance because this will provide some meaningful results of control systems. In this paper, we give algorithms of computing common infinity-norm Lyapunov functions.

2. Preliminaries

Throughout this note the following notation is used:

Let R^n denote real n dimensional space.

$R^{m \times n}$ denotes the set of $m \times n$ real matrices.

A^{-1} is the converse of $A \in R^{m \times n}$.

The l_p norms $\|x\|_p$, $1 \leq p \leq \infty$, are defined by

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p},$$

And

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$$\|x\|_{\infty} = \max_{1 \leq i \leq n} |x_i|.$$

The infinity norm of matrix $A \in R^{n \times n}$ is defined by

$$\|A\|_{\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$$

Consider a family of linear systems

$$\dot{x} = A_i x, \quad x \in R^n, \quad A_i \in R^{n \times n}, \quad i = 1, \dots, N. \quad (1)$$

Definition 1 A function

$$V(x) = \|Wx\|_{\infty}, \quad W \in R^{m \times n}$$

is said to be a common Lyapunov function of the linear systems (1) if there exist matrices

$$Q_i \in R^{m \times m}, \quad i = 1, \dots, N$$

such that

$$WA_i = Q_i W \quad (2)$$

and

$$q_{jj}^i + \sum_{\substack{k=1 \\ k \neq j}}^m |q_{jk}^i| < 0 \quad (3)$$

for all $1 \leq i \leq N$, $1 \leq j \leq m$. q_{jk}^i - entries of the matrix Q_i .

Given a set of stable discrete-time LTI systems described by the following equations

$$x(t+1) = A_i x(t), \quad x \in R^n, \quad A_i \in R^{n \times n}, \quad i = 1, \dots, N. \quad (4)$$

Definition 2 The function of the vector norm form

$$V(x) = \|Wx\|_{\infty}, \quad W \in R^{m \times n}, \quad m \geq n, \quad \text{Rank}(W) = n$$

is said to be a common infinity-norm Lyapunov function for the set of systems(4) if there exist matrices $Q_i \in R^{m \times m}$, $i = 1, \dots, N$ such that we have the matrix relations

$$WA_i = Q_i W \quad (5)$$

and

$$\|Q_i\|_{\infty} < 1 \quad (6)$$

for all $1 \leq i \leq N$.

3. Computation of common infinity-norm Lyapunov functions

Let us consider A_i , $i = 1, \dots, N$ in (1) or (4) have the form as follows

$$\begin{pmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1n}^i \\ 0 & a_{22}^i & \cdots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^i \end{pmatrix}$$

or

$$\begin{pmatrix} a_{11}^i & 0 & \cdots & 0 \\ a_{21}^i & a_{22}^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^i & a_{n2}^i & \cdots & a_{nn}^i \end{pmatrix}.$$

The following we give the theorem below.

Theorem 1 Let $A_i \in R^{n \times n}$, $i = 1, \dots, N$ be Hurwitz such that A_i are upper triangular matrices, then systems (1) share a common infinity-norm Lyapunov function

$$V(x) = \|Wx\|_\infty,$$

where

$$W = \begin{pmatrix} p_1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & p_{n-1} & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

and

$$p_1 > 0, \dots, p_{n-1} > 0.$$

Proof. From (2), we have

$$\begin{aligned} Q_i &= WA_i W^{-1} \\ &= \begin{pmatrix} p_1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & p_{n-1} & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11}^i & a_{12}^i & \dots & a_{1n}^i \\ 0 & a_{22}^i & \dots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^i \end{pmatrix} \begin{pmatrix} 1/p_1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1/p_{n-1} & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_{11}^i & \frac{p_1}{p_2} a_{12}^i & \dots & p_1 a_{1n}^i \\ 0 & a_{22}^i & \dots & p_2 a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^i \end{pmatrix}, i = 1, \dots, N. \end{aligned} \tag{7}$$

Noticing (3), we have

$$\begin{cases} a_{11}^i + \left| \frac{p_1}{p_2} a_{12}^i \right| + \dots + |p_1 a_{1n}^i| < 0 \\ a_{22}^i + \left| \frac{p_2}{p_3} a_{23}^i \right| + \dots + |p_2 a_{2n}^i| < 0 \\ \dots \dots \dots \\ a_{ll}^i + \left| \frac{p_l}{p_{l+1}} a_{l(l+1)}^i \right| + \dots + |p_l a_{ln}^i| < 0 \\ \dots \dots \dots \\ a_{nn}^i < 0 \end{cases}, i = 1, \dots, N. \tag{8}$$

Since A_i are Hurwitz, in (8), $a_{nn}^i < 0$ are always true. By $a_{(n-1)(n-1)}^i + |p_{n-1} a_{(n-1)n}^i| < 0$, we get $0 < p_{n-1} < -\frac{a_{(n-1)(n-1)}^i}{|a_{(n-1)n}^i|}$. So p_{n-1} can be equal to any number in $(0, \min_{1 \leq i \leq N} (-\frac{a_{(n-1)(n-1)}^i}{|a_{(n-1)n}^i|}))$, and we can fix

p_{n-1} . Similarly, in general we $0 < p_l < -\frac{a_{ll}^i}{\left| \frac{a_{l(l+1)}^i}{p_{l+1}} + \dots + |a_{ln}^i| \right|}$ and $p_l \in (0, \min_{1 \leq i \leq N} -\frac{a_{ll}^i}{\left| \frac{a_{l(l+1)}^i}{p_{l+1}} + \dots + |a_{ln}^i| \right|})$.

Finally, we obtain $0 < p_1 < -\frac{a_{11}^i}{\left|\frac{a_{12}^i}{p_2}\right| + \dots + |a_{1n}^i|}$. This completes the proof.

Based on the proof above, we can get the algorithm of computing common infinity-norm Lyapunov functions.

Algorithm 1

Denote $A_l = \min_{1 \leq i \leq N} -\frac{a_{ll}^i}{\left|\frac{a_{l(l+1)}^i}{p_{l+1}}\right| + \dots + |a_{ln}^i|}$, $l = 1, \dots, n - 2$.

Step 1 Compute $\min_{1 \leq i \leq N} \left(-\frac{a_{(n-1)(n-1)}^i}{|a_{(n-1)n}^i|}\right)$. For any $p_{n-1} \in (0, \min_{1 \leq i \leq N} \left(-\frac{a_{(n-1)(n-1)}^i}{|a_{(n-1)n}^i|}\right))$, output p_{n-1} ;

Step 2 Set $k = n - 2$;

Step 3 Compute A_k , for any $p_k \in (0, A_k)$, output p_k ;

Step 4 $k = k - 1$;

Step 5 If $k \geq 1$, goto step 2, otherwise stop;

Step 6 Output $V(x) = \|Wx\|_\infty$.

In what follows, we give an example.

Suppose $A_1 = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ in (1). By computing, we can get

$$W = \begin{pmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

When A_i are lower triangular matrices, we have the following theorem.

Theorem 2 Let $A_i \in R^{n \times n}$, $i = 1, \dots, N$ be Hurwitz such that A_i are lower triangular matrices, then systems (2) share a common infinity-norm Lyapunov function

$$V(x) = \|Wx\|_\infty$$

where

$$W = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & p_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_{n-1} \end{pmatrix}$$

and

$$p_1 > 0, \dots, p_{n-1} > 0.$$

The proof of this theorem is similar to Theorem 1, so the proof is omitted here.

In what follows, we give the algorithm.

Algorithm 2

Denote $B_l = \min_{1 \leq i \leq N} -\frac{a_{(l+1)(l+1)}^i}{|a_{(l+1)1}^i| + \left| \frac{a_{(l+1)2}^i}{p_1} \right| + \dots + \left| \frac{a_{(l+1)l}^i}{p_{l-1}} \right|}$, $l = 2, \dots, n-1$.

Step 1 Compute $\min_{1 \leq i \leq N} \left(-\frac{a_{22}^i}{|a_{21}^i|} \right)$, for any $p_1 \in (0, \min_{1 \leq i \leq N} \left(-\frac{a_{22}^i}{|a_{21}^i|} \right))$, output p_1 ;

Step 2 Set $k = 2$;

Step 3 Compute B_k . For any $p_k \in (0, B_k)$, output p_k ;

Step 4 $k = k + 1$;

Step 5 If $k \leq n - 1$, goto step 2, otherwise stop;

Step 6 Output $V(x) = \|Wx\|_\infty$.

In what follows, we give an example.

Suppose $A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$ in (1). By computing, we can get

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix}.$$

Next we consider discrete-time LTI systems.

Theorem 3 Let $A_i \in R^{n \times n}$, $i = 1, \dots, N$ be Schur such that A_i are upper triangular matrices, then systems (4) share a common infinity-norm Lyapunov function

$$V(x) = \|Wx\|_\infty,$$

Where

$$W = \begin{pmatrix} p_1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & p_{n-1} & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

and

$$p_1 > 0, \dots, p_{n-1} > 0.$$

Proof: Similar to Theorem 1, $Q_i = \begin{pmatrix} a_{11}^i & \frac{p_1}{p_2} a_{12}^i & \dots & p_1 a_{1n}^i \\ 0 & a_{22}^i & \dots & p_2 a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^i \end{pmatrix}$, $i = 1, \dots, N$.

Noticing (3), we have

$$\left\{ \begin{array}{l} \left| a_{11}^i \right| + \left| \frac{p_1}{p_2} a_{12}^i \right| + \dots + \left| p_1 a_{1n}^i \right| < 0 \\ \left| a_{22}^i \right| + \left| \frac{p_2}{p_3} a_{23}^i \right| + \dots + \left| p_2 a_{2n}^i \right| < 0 \\ \dots\dots\dots \\ \left| a_{ll}^i \right| + \left| \frac{p_l}{p_{l+1}} a_{l(l+1)}^i \right| + \dots + \left| p_l a_{ln}^i \right| < 0 \\ \dots\dots\dots \\ \left| a_{nn}^i \right| < 0 \end{array} \right. \tag{9}$$

Since A_i are Schur, in (9), $a_{nn}^i < 0$ are always true. By $\left| a_{(n-1)(n-1)}^i \right| + \left| p_{n-1} a_{(n-1)n}^i \right| < 1$, we get $0 < p_{n-1} < \frac{1 - \left| a_{(n-1)(n-1)}^i \right|}{\left| a_{(n-1)n}^i \right|}$. So p_{n-1} can be equal to any number in $(0, \min_{1 \leq i \leq N} (\frac{1 - \left| a_{(n-1)(n-1)}^i \right|}{\left| a_{(n-1)n}^i \right|}))$, and we can

fix p_{n-1} . Similarly, in general we $0 < p_l < \frac{1 - \left| a_{ll}^i \right|}{\left| \frac{a_{l(l+1)}^i}{p_{l+1}} \right| + \dots + \left| a_{ln}^i \right|}$ and $p_l \in (0, \min_{1 \leq i \leq N} (\frac{1 - \left| a_{ll}^i \right|}{\left| \frac{a_{l(l+1)}^i}{p_{l+1}} \right| + \dots + \left| a_{ln}^i \right|})$.

Finally, we obtain $0 < p_1 < \frac{1 - \left| a_{11}^i \right|}{\left| \frac{a_{12}^i}{p_2} \right| + \dots + \left| a_{1n}^i \right|}$. Noticing A_i are Schur, so $1 - \left| a_{ll}^i \right| > 0, l = 1, \dots, n - 1$. This

completes the proof.

Based on the proof above, we can get the algorithm 3.

Algorithm 3

Denote $C_l = \min_{1 \leq i \leq N} \frac{1 - \left| a_{ll}^i \right|}{\left| \frac{a_{l(l+1)}^i}{p_{l+1}} \right| + \dots + \left| a_{ln}^i \right|}, l = 1, \dots, n - 2$.

Step 1 Compute $\min_{1 \leq i \leq N} (\frac{1 - \left| a_{(n-1)(n-1)}^i \right|}{\left| a_{(n-1)n}^i \right|})$, for any $p_{n-1} \in (0, \min_{1 \leq i \leq N} (\frac{1 - \left| a_{(n-1)(n-1)}^i \right|}{\left| a_{(n-1)n}^i \right|}))$, output p_{n-1} ;

Step 2 Set $k = n - 2$;

Step 3 Compute C_k . For any $p_k \in (0, C_k)$, output p_k ;

Step 4 $k = k - 1$;

Step 5 If $k \geq 1$, goto step 2, otherwise stop;

Step 6 Output $V(x) = \|Wx\|_\infty$

In what follows, we give an example.

Suppose $A_1 = \begin{pmatrix} \frac{1}{2} & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$, $A_2 = \begin{pmatrix} \frac{1}{2} & 1 & 1 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ in (4). By computing, we can get

$$W = \begin{pmatrix} \frac{1}{20} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, we can obtain Theorem 4.

Theorem 4 Let $A_i \in R^{n \times n}$, $i = 1, \dots, N$ be Schur such that A_i are lower triangular matrices, then systems (4) share a common infinity-norm Lyapunov function

$$V(x) = \|Wx\|_\infty,$$

where

$$W = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & p_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_{n-1} \end{pmatrix}$$

and

$$p_1 > 0, \dots, p_{n-1} > 0.$$

Based on the proof above, we can get the algorithm 4.

Algorithm 4

Denote $D_l = \min_{1 \leq i \leq N} \frac{1 - |a_{ll}^i|}{\frac{|a_{l(l+1)}^i|}{p_{l+1}} + \dots + |a_{ln}^i|}$, $l = 2, \dots, n - 1$.

Step 1 Compute $\min_{1 \leq i \leq N} \left(\frac{1 - |a_{22}^i|}{|a_{21}^i|} \right)$. For any $p_1 \in (0, \min_{1 \leq i \leq N} \left(\frac{1 - |a_{22}^i|}{|a_{21}^i|} \right))$, output p_1 ;

Step 2 Set $k = 2$;

Step 3 Compute D_k , for any $p_k \in (0, D_k)$, output p_k ;

Step 4 $k = k + 1$;

Step 5 If $k \leq n - 1$, goto step 2, otherwise stop;

Step 6 Output $V(x) = \|Wx\|_\infty$.

In what follows, we give an example.

Suppose $A_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 2 & \frac{1}{3} & 0 \\ 1 & 2 & \frac{1}{4} \end{pmatrix}$, $A_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{4} & 0 \\ 1 & 1 & \frac{1}{4} \end{pmatrix}$ in (4). By computing, we can get

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{3}{100} \end{pmatrix}.$$

4. Conclusions

In this paper, we deal with the problem of the computation of common infinity-norm Lyapunov functions for continuous-time LTI systems or discrete-time LTI systems. For systems matrices are upper triangular form or lower triangular form, we present analytical methods and feasible algorithms are addressed.

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