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A New Skew Linear Interpolation Characteristic Difference Method for Sobolev Equation

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Abstract. A new kind of characteristic-difference scheme for Sobolev equations is constructed by combining characteristic method with the finite-difference method and with the skew linear interpolation method. The convergence of the characteristic-difference scheme is studied. The advantage of this scheme is very effectual to eliminate the numerical oscillations and have potential advantages in other equations.

Keywords: Sobolev equation; characteristic-difference scheme; skew linear interpolation; convergence

1. Introduction

Many mathematical physics problems can be described by Sobolev equations, such as in fluid flow, heat diffusion and other areas of application. The primal numerical solution of using finite difference method and finite element method for one dimensional Sobolev equations is in [1],[2]. In the year of 1982, Douglas and Russel[3] presented the method of characteristics with finite element or finite difference procedures to solve convection diffusion equations, and then You[4] applied this characteristics difference element method to solve Sobolev equation. During the computation, this method used the algebraic interpolation of the last time step, thus for some problems the stability of the computational scheme is not good enough, even can cause some numerical oscillation. To avoid happen the phenomena of numerical oscillation, Qin[5] introduced a new linear interpolation method (see figure 1) in solving convection diffusion problem: use the values of four points $(x_{j-1},t_{n-1}),(x_j,t_{n-1}),(x_{j-1},t_n),(x_j,t_n)$ to make bilinear interpolation for the value of point P_*.

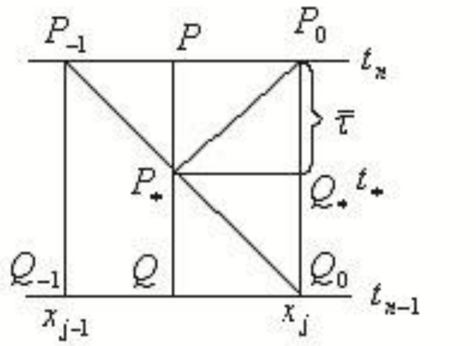

Figure 1. Bilinear interpolation

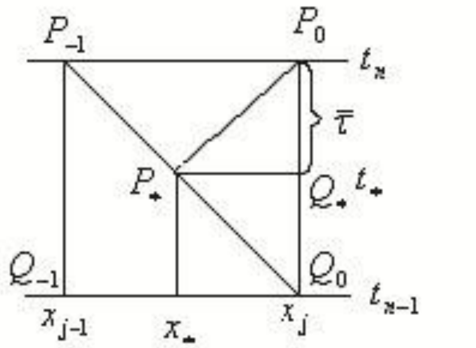

Figure 2. Skew linear inpolation

In this paper, we present a improved characteristic-difference method of [5](see figure 2) in solving the Sobolev equations: only use the values of two points $(x_j,t_{n-1}),(x_{j-1},t_n)$ (see figure 2) to make Skew linear interpolation of point P_*. Compare with the method in [3],[4], our new method show itself more stability. Compare with [5], our method is easier to realize in algorithm. This method can also be applied to solve

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convection dominated diffusion equations.

2. Construction of the finite difference scheme

Consider one dimensional initial-boundary Sobolev equations:

$$\begin{cases} c(x)\dfrac{\partial u}{\partial t}+b(x)\dfrac{\partial u}{\partial x}-\dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial u}{\partial x}+d(x)\dfrac{\partial^2 u}{\partial x\partial t}\right)=f(x,t),(x,t)\in(0,L)\times(0,T] \\ u(x,0)=u_0(x),x\in(0,L) \\ u(0,t)=g_0(t),u(L,t)=g_1,t\in[0,T] \end{cases} \tag{1}$$

where $a(x),b(x),c(x),d(x)\in C[0,L]$, $f(x,t)\in L^2([0,L]\times[0,T])$, and there exists positive constants a_0,b^*,c_0,d_0, $a(x)\geq a_0>0$, $|b(x)|\leq b^*$, $c(x)\geq c_0>0$, $d(x)\geq d_0\geq 0$, for $\forall x\in[0,L]$.

The solvability and uniqueness of (1) can be found in [2]. We assume that (1) has a unique solution and have some necessary smoothness. Denote the characteristic direction of operator $c(x)\dfrac{\partial u}{\partial t}+b(x)\dfrac{\partial u}{\partial x}$ to be $\lambda=\lambda(x)$, and then the characteristic derivative is defined by

$$\frac{\partial}{\partial\lambda}=\frac{1}{\varphi(x)}\left[c(x)\frac{\partial}{\partial t}+b(x)\frac{\partial}{\partial x}\right],$$

where $\varphi(x)=\left[b(x)^2+c(x)^2\right]^{1/2}$. Therefore, the first equation of (1) can then be write as the following form:

$$\varphi(x)\frac{\partial u}{\partial\lambda}-\frac{\partial}{\partial x}\left[a(x)\frac{\partial u}{\partial x}+d(x)\frac{\partial^2 u}{\partial x\partial t}\right]=f(x,t),\quad 0<x<L,0<t<T. \tag{2}$$

In figure 2, suppose the values on $n-1$ time step is either initial value or already be computed by initial value. When $b(x)>0$, the characteristic direction at point P_0 is the direction along P_0P_*, where P_* is the intersection point of Q_0P_{-1} and characteristic direction. Thus by using linear interpolation, we can use the values at points Q_0 and P_{-1} to get the value at point P_*, and then applying finite difference method and characteristic method to construct an implicit difference scheme. The advantage of this method is: we only need to make skew linear interpolation in the segment Q_0P_{-1}, need not do other extra work to determine the interpolation point. This technique is better than the normal finite difference method by decreasing the truncation error along time.

Take space step $h>0$, mesh grid $x_j=jh$, $j=0,1,2,\cdots M=[L/h]$, time step $\tau>0$, mesh grid $t_n=n\tau$, $n=0,1,\cdots,\mathrm{N}=[\mathrm{T}/\tau]$, along the characteristic line P_0P_*, we make following finite difference discrete:

$$\left(\varphi\frac{\partial u}{\partial\lambda}\right)_j^n\approx\varphi_j\frac{u(x_j,t_n)-u(x_*,t_*)}{|P_0P_*|}, \tag{3}$$

where (x_*,t_*) denotes the coordinate at point P_*, $x_*=x_j-b_j\bar{\tau}/c_j$, $\varphi_j=\varphi(x_j)$. Denote $c_j=c(x_j)$, $b_j=b(x_j)$, $a_j=a(x_j)$, $d=d(x_j)$, $j=1,2,\cdots,M$. Thus from $\dfrac{|Q_0Q_*|}{|Q_0P_0|}=\dfrac{|P_*Q_*|}{|P_0P_{-1}|}$, we have $\bar{\tau}=\dfrac{c_jh\tau}{c_jh+b_j\tau}$, $t_*=n\tau-\bar{\tau}$. Therefore from

$$|P_0P_*|=\sqrt{(x_j-x_*)^2+\bar{\tau}^2}=\varphi_j\bar{\tau}/c_j,$$

we can get the exact expression of (3)

$$\left(\varphi\frac{\partial u}{\partial\lambda}\right)_j^n=\varphi_j\frac{u\left(x_j,t_n\right)-u\left(x_*,t_*\right)}{\left|P_0P_*\right|}+r_j^n=\frac{c_jh+b_j\tau}{h}\frac{u\left(x_j,t_n\right)-u\left(x_*,t_*\right)}{\tau}+r_j^n, \tag{4}$$

where r_j^n is the local truncation error:

$$r_j^n=\left(\varphi\frac{\partial u}{\partial\lambda}\right)_j^n-\frac{c_jh+b_j\tau}{h}\frac{u\left(x_j,t_n\right)-u\left(x_*,t_*\right)}{\tau}. \tag{5}$$

Let $\delta_{\bar{x}},\delta_x$ be the backwards and forward finite difference quotients along x direction respectively, and $\delta_{\bar{t}}$ denote the backwards finite difference quotient along t direction, denote by

$$\delta_{\bar{x}}\left[a\delta_x u\right]_j^n=\frac{1}{h}\left[a_{j+1/2}\frac{u\left(x_{j+1},t_n\right)-u\left(x_j,t_n\right)}{h}-a_{j-1/2}\frac{u\left(x_j,t_n\right)-u\left(x_{j-1},t_n\right)}{h}\right],$$

$$\delta_{\bar{x}}\left[d\delta_x\delta_{\bar{t}}u\right]_j^n$$

$$=\frac{1}{h}\left[d_{j+1/2}\frac{\delta_{\bar{t}}u\left(x_{j+1},t_n\right)-\delta_{\bar{t}}u\left(x_j,t_n\right)}{h}-d_{j-1/2}\frac{\delta_{\bar{t}}u\left(x_j,t_n\right)-\delta_{\bar{t}}u\left(x_{j-1},t_n\right)}{h}\right],$$

$$\tilde{r}_j^n=-\left[\frac{\partial}{\partial x}\left(a\left(x\right)\frac{\partial u}{\partial x}+d(x)\frac{\partial^2 u}{\partial x\partial t}\right)\right]_j^n+\delta_{\bar{x}}\left[a\delta_x u+d\delta_x\delta_{\bar{t}}u\right]_j^n. \tag{6}$$

By (4)-(6), we have

$$\frac{c_jh+b_j\tau}{h}\frac{u\left(x_j,t_n\right)-u\left(x_*,t_*\right)}{\tau}-\delta_{\bar{x}}\left[a\delta_x u+d\delta_x\delta_{\bar{t}}u\right]_j^n=f_j^n-R_j^n, \tag{7}$$

where

$$f_j^n=f(x_j,t_n), \tag{8}$$

$$R_j^n=r_j^n+\tilde{r}_j^n. \tag{9}$$

Omit R_j^n in (7), and then the characteristic difference scheme of solving problem (1) can be established as follows

$$\begin{cases}\dfrac{c_jh+b_j\tau}{h}\dfrac{U_j^n-U_j^*}{\tau}-\delta_{\bar{x}}\left[a\delta_x U+d\delta_x\delta_{\bar{t}}U\right]_j^n=f_j^n,\\ U_j^0=u_0\left(x_j\right),U_0^n=g_1\left(t_n\right),U_M^n=g_2\left(t_n\right),\\ j=0,1,2,\cdots,M,n=1,2,\cdots,N,\end{cases} \tag{10}$$

where U_j^n is the numerical solution of $u\left(x_j,t_n\right)$.

3. skew linear interpolation

We first point out that U_j^* in (10) are determined by the skew linear interpolation with two points Q_0, P_{-1}, let $U_j^*=w\left(x_*,t_*\right)$, where $w\left(x,t\right)=I_1\left\{U_{j-1}^n,U_j^{n-1}\right\}\left(x,t\right)$ denotes the linear function obtained by interpolation data $\left\{U_{j-1}^n,U_j^{n-1}\right\}$, I_1 is the linear interpolation operator. The result is as follows:

$$I_1\left\{U_{j-1}^n,U_j^{n-1}\right\}\left(x_*,t_*\right)=\frac{\left|Q_0P_*\right|}{\left|P_{-1}Q_0\right|}U_{P_{-1}}+\frac{\left|P_{-1}P_*\right|}{\left|P_{-1}P_0\right|}U_{Q_0},\tag{11}$$

where

$$\frac{\left|Q_0P_*\right|}{\left|P_{-1}Q_0\right|}=\frac{\left|Q_0Q_*\right|}{\left|Q_0P_0\right|}=\frac{b_j\tau}{c_jh+b_j\tau},\tag{12}$$

$$\frac{\left|P_{-1}P_*\right|}{\left|P_{-1}P_0\right|}=1-\frac{\left|Q_0P_*\right|}{\left|P_{-1}Q_0\right|}=\frac{c_jh}{c_jh+b_j\tau}.\tag{13}$$

Apply (12), (13) to (11), then

$$I_1\left\{U_{j-1}^n,U_j^{n-1}\right\}\left(x_*,t_*\right)=\frac{b_j\tau}{c_jh+b_j\tau}U_{P_{-1}}+\frac{c_jh}{c_jh+b_j\tau}U_{Q_0},\tag{14}$$

i.e.

$$U_j^*=\frac{b_j\tau}{c_jh+b_j\tau}U_{j-1}^n+\frac{c_jh}{c_jh+b_j\tau}U_j^{n-1}.\tag{15}$$

Apply (15) to (10), we obtain a computational scheme. When $h=O(\tau)$, the coefficient matrix formed by difference scheme (10) is strictly diagonal dominated, therefore (10) has a unique solution.

4. convergence analysis

Let $W^{m,p}$ denotes Sobolev space with norm $\|\cdot\|_{m,p}$. For grid function v,w, define inner product and norm:

$$(v,w)=\sum_{j=1}^{M-1}v_jw_jh,\ \|v\|^2=(v,v),$$

$$[v,w)=\sum_{j=0}^{M-1}v_jw_jh,\ \left\|\left[v\right\|\right.^2=[v,v),$$

$$[a\delta_xv,\delta_xw)=\sum_{j=0}^{M-1}a_{j+1/2}\delta_xv_j\delta_xw_j,\ \left\|\left[a^{1/2}\delta_xv\right\|\right.^2=[a\delta_xv,\delta_xv).$$

By Taylor formula [4], when $u\in W^{2,\infty}(0,T;W^{4,\infty}(0,L))$,

$$\left|\tilde r_j^n\right|\le M_1(\|u\|_{W^{2,\infty}(W^{4,\infty})})h^2.\tag{16}$$

Apply the Taylor formula with integral remaining term, we get

$$r_j^n=\left(\varphi\frac{\partial u}{\partial\lambda}\right)_j^n-\frac{c_jh+b_j\tau}{h}\frac{u\left(x_j,t_n\right)-u\left(x_*,t_*\right)}{\tau}=\frac{\varphi_j^2h\tau}{2\left(c_jh+b_j\tau\right)}\left(\frac{\partial^2u}{\partial\lambda^2}\right)_j^*,\tag{17}$$

where $\left(\frac{\partial^2u}{\partial\lambda^2}\right)_j^*$ is the second tangent derivative of u along segment P_0P_* at some point of line P_0P_*. Combine (9), (16) and (17), then when $u\in W^{2,\infty}(W^{4,\infty})$,

$$\left|R_j^n\right|\le K_1\left[\left\|\frac{\partial^2u}{\partial\lambda^2}\right\|_{L^\infty(L^\infty)}+\|u\|_{W^{2,\infty}(W^{4,\infty})}\right](\tau+h^2),\tag{18}$$

where K_1 is positive constant, $\left\|\frac{\partial^2u}{\partial\lambda^2}\right\|_{L^\infty(L^\infty)}=\max_{[0,L]\times[0,T]}\left|\frac{\partial^2u}{\partial\lambda^2}\right|$.

Denote $e_j^n=u\left(x_j,t_n\right)-U_j^n$, then from (7) and (10) we get the error equation

$$\begin{cases} \dfrac{c_j h + b_j \tau}{h} \dfrac{e_j^n - e_j^*}{\tau} - \delta_{\bar{x}} \left[a\delta_x e + d\delta_x \delta_{\bar{t}} e \right]_j^n = -R_j^n \\ e_j^0 = 0, e_0^n = 0, e_M^0 = 0, j = 0,1,2,\cdots,M, n = 1,2,\cdots,N, \end{cases} \tag{19}$$

where $e_j^* = u(x_*, t_*) - w(x_*, t_*)$, then from (19)

$$\frac{c_j h + b_j \tau}{h} \frac{e_j^n - e_j^*}{\tau} - \delta_{\bar{x}} \left[a\delta_x e + d\delta_x \delta_{\bar{t}} e \right]_j^n = -R_j^n.$$

Denote $g_j = \dfrac{c_j h + b_j \tau}{h}$, then $g_j \geq c_0$, ($j = 0,1,2,\cdots M$). Multiply the last equation by $\delta_{\bar{t}} e_j^n h$, and sum for j from 1 to $M-1$, then

$$(g \frac{e^n - e^*}{\tau}, \delta_{\bar{t}} e^n) + [a\delta_x e^n, \delta_x \delta_{\bar{t}} e^n) + [d\delta_x \delta_{\bar{t}} e^n, \delta_x \delta_{\bar{t}} e^n) = -(R^n, \delta_{\bar{t}} e^n), \tag{20}$$

where

$$(g \frac{e^n - e^*}{\tau}, \delta_{\bar{t}} e^n) = \sum_{j=1}^{M-1} g_j \frac{e_j^n - e_j^*}{\tau} (\delta_{\bar{t}} e_j^n) h, [d\delta_x \delta_{\bar{t}} e^n, \delta_x \delta_{\bar{t}} e^n) \geq d_0 \left\| \delta_x \delta_{\bar{t}} e^n \right\|^2,$$

$$[a\delta_x e^n, \delta_x \delta_{\bar{t}} e^n) = [a\delta_x e^n, \delta_x \frac{e^n - e^{n-1}}{\tau}) \geq \frac{1}{2\tau} \{ [a\delta_x e^n, \delta_x e^n) - [a\delta_x e^{n-1}, \delta_x e^{n-1}) \},$$

$$\left| -(R^n, \delta_{\bar{t}} e^n) \right| \leq \varepsilon \left\| \delta_{\bar{t}} e^n \right\|^2 + M_1 \left\| R^n \right\|^2.$$

The difficulty is the estimate of first term in (20). Since

$$\begin{aligned} e_j^* &= u(x_*, t_*) - w(x_*, t_*) = u(x_*, t_*) - I_1 \left\{ U_{j-1}^n, U_j^{n-1} \right\} (x_*, t_*) \\ &= u(x_*, t_*) - I_1 \left\{ u(x_{j-1}, t_n), u(x_j, t_{n-1}) \right\} (x_*, t_*) \\ &\quad + I_1 \left\{ u(x_{j-1}, t_n), u(x_j, t_{n-1}) \right\} (x_*, t_*) - I_1 \left\{ U_{j-1}^n, U_j^{n-1} \right\} (x_*, t_*) \\ &= \bar{r}_j^n + I_1 \left\{ e_{j-1}^n, e_j^{n-1} \right\} (x_*, t_*), \end{aligned} \tag{21}$$

where $\bar{r}_j^n = u(x_*, t_*) - I_1 \left\{ u(x_{j-1}, t_n), u(x_j, t_{n-1}) \right\} (x_*, t_*)$, from (15) and (21), we have:

$$\begin{aligned} g_j \frac{e_j^n - e_j^*}{\tau} &= g_j [e_j^n - \bar{r}_j^n - I_1 \{ e_{j-1}^n, e_j^n \} (x_*, t_*)] / \tau \\ &= g_j [e_j^n - \bar{r}_j^n - \frac{b_j \tau}{c_j h + b_j \tau} e_{j-1}^n - \frac{c_j h}{c_j h + b_j \tau} e_j^{n-1}] / \tau \\ &= \frac{c_j h + b_j \tau}{\tau h} [\frac{b_j \tau}{c_j h + b_j \tau} (e_j^n - e_{j-1}^n) + \frac{c_j h}{c_j h + b_j \tau} (e_j^n - e_j^{n-1}) - \bar{r}_j^n] \\ &= \frac{b_j}{h} (e_j^n - e_{j-1}^n) + c_j \delta_{\bar{t}} e_j^n - \frac{c_j h + b_j \tau}{\tau h} \bar{r}_j^n, \end{aligned}$$

$$(g \frac{e^n - e^*}{\tau}, \delta_{\bar{t}} e^n) = (c\delta_{\bar{t}} e^n, \delta_{\bar{t}} e^n) + \sum_{j=1}^{M-1} b_j (e_j^n - e_{j-1}^n) \delta_{\bar{t}} e_j^n - \sum_{j=1}^{M-1} \frac{c_j h + b_j \tau}{\tau} \bar{r}_j^n \delta_{\bar{t}} e_j^n. \tag{22}$$

Therefore

$$\begin{aligned} c_0 \left\| \delta_{\bar{t}} e^n \right\|^2 + d_0 \left\| \delta_x \delta_{\bar{t}} e^n \right\|^2 + \frac{1}{2\tau} \{ [a\delta_x e^n, \delta_x e^n) - [a\delta_x e^{n-1}, \delta_x e^{n-1}) \} \\ \leq \varepsilon \left\| \delta_{\bar{t}} e^n \right\|^2 + M_1 \left\| R^n \right\|^2 \end{aligned}$$

$$+\left|\sum_{j=1}^{M-1} b_j(e_j^n - e_{j-1}^n)\delta_{\bar t} e_j^n\right| + \left|\sum_{j=1}^{M-1}\frac{c_j h + b_j\tau}{\tau}\bar r_j^n \delta_{\bar t} e_j^n\right|. \tag{23}$$

We also need to estimate error term $\bar r_j^n$ of the skew linear interpolation.

$$\left|\bar r_j^n\right| = \left|u(x_*,t_*) - I_1\left\{u(x_{j-1},t_n), u(x_j,t_{n-1})\right\}(x_*,t_*)\right|$$
$$= \frac{1}{2}\left|\left(\frac{\partial^2 u}{\partial\rho^2}\right)_{\bar j}^{\bar n}\left|P_{-1}P_*\right|\left|P_*Q_0\right|\right| = \frac{1}{2}\frac{b_j\tau h\left(h^2+\tau^2\right)}{\left(c_j h + b_j\tau\right)^2}\left|\frac{\partial^2 u}{\partial\rho^2}\right|_{\bar j}^{\bar n}, \tag{24}$$

where ρ denotes the direction of $P_{-1}Q_0$, $\left(\frac{\partial^2 u}{\partial\rho^2}\right)_{\bar j}^{\bar n}$ denote the value at some point on segment $\overline{P_{-1}Q_0}$ of second direction derivative along the direction of $\overline{P_{-1}Q_0}$. When $h = O(\tau)$, by (24), we have

$$\max_j\left|\bar r_j^n\right| \le M_2 \max_{[x_{j-1},x_j]\times[t_{n-1},t_n]}\left|\frac{\partial^2 u}{\partial\rho^2}\right| h\tau, \tag{25}$$

where M_2 is a positive constant. Notice that the boundedness of $b(x)$, apply (25) to (23),

$$c_0\left\|\delta_{\bar t} e^n\right\|^2 + d_0\left\|\delta_x\delta_{\bar t} e^n\right\|^2 + \frac{1}{2\tau}\{[a\delta_x e^n, \delta_x e^n) - [a\delta_x e^{n-1}, \delta_x e^{n-1})\}$$
$$\le 3\varepsilon\left\|\delta_{\bar t} e^n\right\|^2 + M_1\left\|R^n\right\|^2 + C(\left\|\delta_{\bar x} e^n\right\|^2 + \left\|g\bar r^n/\tau\right\|^2),$$

where constant C depends on ε.

Multiply by 2τ, take $\varepsilon = \frac{c_0}{8}$, and sum for n, applying (18), (25), we have

$$c_0\sum_{m=1}^{n}\left\|\delta_{\bar t} e^n\right\|^2\tau + d_0\sum_{m=1}^{n}\left\|\delta_x\delta_{\bar t} e^n\right\|^2\tau + a_0\left\|\left[\delta_x e^n\right\|\right.^2$$
$$\le M_1 L\{K_1[\left\|\frac{\partial^2 u}{\partial\lambda^2}\right\|_{L^\infty(L^\infty)} + \left\|u\right\|_{W^{2,\infty}(W^{4,\infty})}](\tau + h^2)\}^2$$
$$+C\{\sum_{m=1}^{n}\left\|\delta_{\bar x} e^n\right\|^2\tau + L\max_j (c_j h + b_j\tau)^2\cdot\max_{[x_{j-1},x_j]\times[t_{n-1},t_n]}\left|\frac{\partial^2 u}{\partial\rho^2}\right|^2\}.$$

Denote $\left\|\frac{\partial^2 u}{\partial\rho^2}\right\|_{L^\infty(L^\infty)} = \max_{[0,L]\times[0,T]}\left|\frac{\partial^2 u}{\partial\rho^2}\right|$, notice that $\left\|\delta_{\bar x} e^n\right\| \le \left\|\left[\delta_x e^n\right\|\right.$, and then applying discrete Gronwall inequality, we have

$$(\sum_{m=1}^{n}\left\|\delta_{\bar t} e^n\right\|^2\tau)^{\frac{1}{2}} + (\sum_{m=1}^{n}\left\|\delta_x\delta_{\bar t} e^n\right\|^2\tau)^{\frac{1}{2}} + \left\|\left[\delta_x e^n\right\|\right.$$
$$\le M\{[\left\|\frac{\partial^2 u}{\partial\lambda^2}\right\|_{L^\infty(L^\infty)} + \left\|u\right\|_{W^{2,\infty}(W^{4,\infty})}](\tau + h^2) + \left\|\frac{\partial^2 u}{\partial\rho^2}\right\|_{L^\infty(L^\infty)}(\tau + h)\}, \tag{26}$$

where M is positive constant.

The upward discussion based on $b_j > 0$. When $b_j < 0$, similar discussions can also be made and establish the corresponding computational schemes.

Theorem 1 Suppose that the solution of problem (1) satisfies

$$u \in W^{2,\infty}(W^{4,\infty}), \frac{\partial^2 u}{\partial \lambda^2}, \frac{\partial^2 u}{\partial \rho^2} \in L^\infty(L^\infty),$$

$\{U_j^n\}$ is the solution of skew linear interpolation characteristic difference scheme (10), then while $h = O(\tau)$, the error estimates satisfies (26).

Remark If $d(x) \equiv 0$, the problem (1) degenerate to a convection diffusion problem, the computation scheme (10) and the conclusion of theorem 1 also hold for convection diffusion equations.

5. Numerical examples

Consider following Sobolev equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial t}\right) = f(x,t), (x,t) \in [0,1] \times (0, 0.99).$$

Take the solution of this equation to be:

$$u(x,t) = 10^{-16} e^{20(x+t)}.$$

The initial-boundary value conditions and $f(x,t)$ be determined by solution $u(x,t)$. Solve this problem by using skew linear interpolation characteristic difference method (10) and the characteristic difference method given in[4], when $h = \frac{1}{30}, \tau = \frac{1}{10}$, we can compare the numerical result with the accurate solution. For example, for $t = 0.99$, denote err= $(\sum_{j=1}^{M} (u_j - U_j)^2)^{\frac{1}{2}} h$ the average absolute errors, then the average absolute errors are 0.0212 and 0.0352 with respect to the two methods. Table 1 listed some results at certain grid points for $t = 0.99$:

Table 1. The numerical compare of finite difference schemes

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
u	2.935e-7	2.169e-6	1.603e-5	1.185e-4	8.749e-4	6.465e-3	4.777e-2	3.530e-1	2.609
LCD	2.932e-7	2.168e-6	1.614e-5	1.180e-4	8.753e-4	6.471e-3	4.781e-2	3.576e-1	2.621
SLCD	2.933e-7	2.170e-6	1.612e-5	1.182e-4	8.750e-4	6.467e-3	4.779e-2	3.538e-1	2.613

Computations show the skew linear interpolation characteristic difference method is an efficient algorithm, compared with [4], this method is more stable, and it is simpler than [5]. This method can also be used to solve convection diffusion equations with variable coefficients.

6. References

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