Exact and Numerical Solution of Lienard's Equation by the Variational Homotopy Perturbation Method

M. Matinfar[+] , M. Mahdavi and Z. Raeisy

Science of Mathematics Faculty, Department of Mathematics, University of Mazandaran, P. O. Box: 47416-95447, Babolsar, Iran

Abstract. In this paper, exact and numerical solutions are obtained for the Lienard's equation by variational homotopy perturbation method (VHPM). Comparisons are made among the variational iteration method (VIM), the exact solutions and the proposed method. The results reveal that the proposed method is very effective and simple and can be applied for other nonlinear problems in mathematical.

Keywords: Variational Homotopy Perturbation Method, Lagrange multiplier, Lienard's equation.

1. Introduction

In this paper, we consider Lienard equation:

$$x'' + f(x)x' + g(x) = h(t), \tag{1}$$

which is not only regarded as a generalization of the damped pendulum equation or a damped spring-mass system (where $f(x)x'$ is the damping force, g(x) is the restoring force and h(t) is the external force), but also used as nonlinear models in many physically significant fields when taking different choices for f(x), g(x) and h(t). For example, the choices $f(x) = \varepsilon(x^2 - 1)$, $g(x) = x$ and $h(t) = 0$ lead equation of (1) to the Van der Pol equation served as a nonlinear model of electronic oscillation. Therefore, studying equation of (1) is of physical significance. In the general case, it is commonly believed that it is very difficult to find its exact solution by usual ways [3]. The following special case of equation (1) was studied in [1, 2, 3]:

$$x'' + lx + mx^3 + nx^5 = 0, \tag{2}$$

where l, m and n are real coefficients. Finding explicit exact and numerical solutions of nonlinear equations efficiently is of major importance and has widespread applications in numerical methods and applied mathematics. In this study, we will implement the variational homotopy perturbation method (VHPM) to find exact solution and approximate solutions to the Lienard equation for a given nonlinearity.

The VHPM provides the solution in a rapid convergent series which may lead the solution in a closed form. It is worth mentioning that the VHPM is applied with out any discretization, restrictive assumption, or transformation and is free from round-off errors. Also the VHPM provides an analytical solution by using the initial conditions only and the boundary conditions can be used only to justify the obtained result. Numerical results reveal that the VHPM is easy to implement and reduces the computational work to a tangible level while still maintaining a very higher level of accuracy [7].

2. Variational Homotopy Perturbation Method

To convey the basic idea of the variational homotopy perturbation method, we consider the following general differential equation:

$$Lu + Nu = g(x), \tag{3}$$

[+] Corresponding author. Tel.: +98-112-534 2496; fax: +98-112-534 2460.
 E-mail address: m:m.matinfar@umz.ac.ir.

where L is a linear operator, N is a nonlinear operator and g(x) is an inhomogeneous term. According to variational iteration method [4-6, 8-12], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\{L\,u_n + N\,\tilde{u}_n - g(\tau)\}\,d\tau, \tag{4}$$

where $\lambda(\tau)$ is a Lagrange multiplier [4-6, 8-12] which can be identified optimally via the variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta\tilde{u}_n = 0$ and (4) is called a correct functional. Now, we apply the homotopy perturbation method;

$$\sum_{i=0}^{\infty} p^i u_i = u_0 + p\int_0^x \lambda(\tau)\left\{N\left(\sum_{i=0}^{\infty} p^i\,\tilde{u}_i\right)\right\}d\tau - p\int_0^x \lambda(\tau)g(\tau)d\tau, \tag{5}$$

which is the variational homotopy perturbation method and is formulated by the coupling of variational iteration method and Adomian's polynomials. The embedding parameter $p\in[0,1]$ can be considered as an expanding parameter [13-18]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [13-18] to obtain:

$$f = \sum_{i=0}^{\infty} p^i u_i = u_0 + p\,u_1 + p^2\,u_2 + \cdots \tag{6}$$

If $p \rightarrow 1$, then (6) becomes the approximate solution of the form:

$$u = \lim_{p\to 1} f = u_0 + u_1 + u_2 + \cdots. \tag{7}$$

A comparison of like powers of p gives solutions of various orders.

3. VHPM for Lienard's equation

In this section, we consider Lienard equation (2) with initial conditions:

$$x(0) = C_1, x'(0) = C_2, \tag{8}$$

by using of initial conditions (8), we choose:

$$x_0(t) = C_1 + C_2\,t, \tag{9}$$

Where (9) is an initial approximation of Eq. (2). For solving Eq. (2) via VHPM, we consider:

$$L(x) = x'', \tag{10}$$

$$N(x) = lx + mx^3 + nx^5, \tag{11}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [4-6, 8-12], we can construct a correct functional as follows:

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda(\tau)\left\{x_{n_{\tau\tau}} + l\,\tilde{x}_n + m\,\tilde{x}_n^3 + n\,\tilde{x}_n^5\right\}d\tau, \tag{12}$$

where $\tilde{x}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \tau - t$, which yields the following iteration formula:

$$x_{n+1}(t) = x_n(t) + \int_0^t (\tau - t)\left\{x_{n_{\tau\tau}} + l\,x_n + m\,x_n^3 + n\,x_n^5\right\}d\tau. \tag{13}$$

Applying the variational homotopy perturbation method, we have:

$$x_0 + p\,x_1 + p^2 x_2 + \cdots = C_1 + C_2 t$$

$$+ p\int_0^t (\tau - t)l\left(x_0 + p\,x_1 + p^2 x_2 + p^3 x_3 + \cdots\right)d\tau$$

$$+ p\int_0^t (\tau - t)m\left(x_0 + p\,x_1 + p^2 x_2 + p^3 x_3 + \cdots\right)^3 d\tau$$

$$+ p \int\limits_{0}^{t} (\tau - t) n \left(x_0 + p\, x_1 + p^2 x_2 + p^3 x_3 + \cdots \right)^5 d\tau \,. \qquad (14)$$

Comparing the coefficient of like powers of p we have:

$$p^0 : x_0(t) = C_1 + C_2\, t \,,$$

$$p^1 : x_1(t) = \int\limits_{0}^{t} (\tau - t) l\, x_0\, d\tau + \int\limits_{0}^{t} (\tau - t) m\, x_0^{\,3}\, d\tau + \int\limits_{0}^{t} (\tau - t) n\, x_0^{\,5}\, d\tau,$$

$$\vdots$$

So we obtain the components which constitute $x(t)$, thus we will have:

$$x(t) = x_0 + x_1 + x_2 + \cdots .$$

For later numerical computation, we let the expression:

$$\varphi_n = \sum_{i=0}^{n} x_i(t), \qquad (15)$$

to denote the n-term approximation to $x(t)$.

4. Implementation of the method

In this section two important cases of Lienard equation will be investigated to show the reliability of the proposed scheme.

Example 1. We consider the equation (2) with the following initial conditions:

$$x(0) = C_1 = \sqrt{\frac{-2l}{m}}, \; x'(0) = C_2 = -\frac{l\sqrt{-l}}{m\sqrt{\dfrac{-2l}{m}}}, \qquad (16)$$

where m and l are arbitrary constants. By using the equation (14) we have:

$$x_0 = \sqrt{\frac{-2l}{m}} - \frac{l\sqrt{-l}}{m\sqrt{\dfrac{-2l}{m}}}\, t \,,$$

$$x_1 = \frac{5n\, l^5 t^7 (-l)^{\frac{5}{2}}}{210\, m^5 \left(\dfrac{-2l}{m}\right)^{\frac{5}{2}}} + \frac{70\, n\, l^7 t^6}{210\, m^5 \left(\dfrac{-2l}{m}\right)^{\frac{5}{2}}} - \frac{21\, m^2\, l^4 t^5 (-l)^{\frac{3}{2}}}{210\, m^5 \left(\dfrac{-2l}{m}\right)^{\frac{5}{2}}}$$

$$+ \frac{420 n\, l^5 t^5 (-l)^{\frac{3}{2}}}{210\, m^5 \left(\dfrac{-2l}{m}\right)^{\frac{5}{2}}} + \frac{210\, m^2 l^5 t^4}{210\, m^5 \left(\dfrac{-2l}{m}\right)^{\frac{5}{2}}} - \frac{1400\, n l^6 t^4}{210\, m^5 \left(\dfrac{-2l}{m}\right)^{\frac{5}{2}}}$$

$$-\frac{700\,m^2\,l^4 t^3 \left(-l\right)^{\frac{1}{2}}}{210\,m^5\left(\dfrac{-2l}{m}\right)^{\frac{5}{2}}}+\frac{2800\,n\,l^5 t^3\left(-l\right)^{\frac{1}{2}}}{210\,m^5\left(\dfrac{-2l}{m}\right)^{\frac{5}{2}}}-\frac{840\,m^2\,l^4 t^2}{210\,m^5\left(\dfrac{-2l}{m}\right)^{\frac{5}{2}}}+\frac{3360\,n\,l^5 t^2}{210\,m^5\left(\dfrac{-2l}{m}\right)^{\frac{5}{2}}},$$

$$\vdots$$

So we obtain the components which constitute $x(t)$, thus we will have:

$$x(t) = x_0 + x_1 + x_2 + \cdots.$$

The exact value of $x(t)$ in a closed form is:

$$x(t) = \sqrt{\frac{-2l\left(1 + \tanh\left(t\sqrt{-l}\right)\right)}{m}}, \tag{17}$$

as presented in [3].

Now in table 1, we present the absolute errors between φ_1 and the exact solution and the absolute errors between φ_2 and the exact solution for the values of $t = 0.1(0.1)0.5$, $l = -1$, $m = 4$, $n = -3$.

Table 1: The numerical results for φ_1, φ_2 in comparison with the exact solution of x.

t	$\left|x - \varphi_1\right|$	$\left|x - \varphi_2\right|$
0.1	8.8312 e-007	5.3785 e-010
0.2	1.2932 e-005	6.4118 e-008
0.3	5.6140 e-005	1.1344 e-006
0.4	1.3721 e-004	8.9651 e-006
0.5	2.1031 e-004	4.5009 e-005

Also in Table 2, we present absolute errors between the 1-iterate of VIM (x_{1VIM}) and the exact solution and absolute errors between the 2-iterate of VIM (x_{2VIM}) and the exact solution for the values of $t = 0.1(0.1)0.5$, $l = -1$, $m = 4$, $n = -3$, as presented in [3].

Table 2: The numerical results for x_{1VIM}, x_{2VIM} in comparison with the exact solution of x.

t	$\left|x - x_{1VIM}\right|$	$\left|x - x_{2\,VIM}\right|$
0.1	8.8312 e-007	5.3588 e-010
0.2	1.2932 e-005	1.7493 e-008
0.3	5.6140 e-005	1.4734 e-007
0.4	1.3721 e-004	5.3415 e-007
0.5	2.1031 e-004	1.0564 e-006

Example 2. Now, we consider the equation (2) with the following initial conditions:

$$x(0) = \sqrt{\frac{K}{2+D}},\ x'(0) = 0, \tag{18}$$

where

$$K = 4\sqrt{\frac{2l^2}{(3m^2 - 16nl)}}, \quad D = -1 + \frac{m\sqrt{3}}{\sqrt{(3m^2 - 16nl)}}, \tag{19}$$

by using the equation (14) we have:

$$x_0 = 2\sqrt{\frac{\sqrt{\frac{2l^2}{(3m^2 - 16nl)}}}{1 + \frac{m\sqrt{3}}{(3m^2 - 16nl)}}},$$

$$x_1 = \frac{-4\sqrt{6}\,t^2 l\,m^2 \sqrt{(l\sqrt{2})(m\sqrt{3} + \sqrt{(3m^2 - 16nl)})}}{(m\sqrt{3} + \sqrt{(3m^2 - 16nl)})^3}$$

$$- \frac{6\,t^2 l\,m^2 \sqrt{(l\sqrt{2})(m\sqrt{3} + \sqrt{(3m^2 - 16nl)})}}{(m\sqrt{3} + \sqrt{(3m^2 - 16nl)})^3}$$

$$- \frac{2\sqrt{3}\,t^2 l\,m\sqrt{(3m^2 - 16nl)}\sqrt{(l\sqrt{2})(m\sqrt{3} + \sqrt{(3m^2 - 16nl)})}}{(m\sqrt{3} + \sqrt{(3m^2 - 16nl)})^3}$$

$$- \frac{4\sqrt{2}\,t^2 l\,m\sqrt{(3m^2 - 16nl)}\sqrt{(l\sqrt{2})(m\sqrt{3} + \sqrt{(3m^2 - 16nl)})}}{(m\sqrt{3} + \sqrt{(3m^2 - 16nl)})^3}$$

$$- \frac{16\,t^2 l^2\,n\sqrt{(l\sqrt{2})(m\sqrt{3} + \sqrt{(3m^2 - 16nl)})}}{(m\sqrt{3} + \sqrt{(3m^2 - 16nl)})^3},$$

$$\vdots$$

So we obtain the components which constitute $x(t)$, thus we will have:

$$x(t) = x_0 + x_1 + x_2 + \cdots.$$

The exact value of $x(t)$ in a closed form is:

$$x(t) = \sqrt{\frac{K \sec h(t\sqrt{-l})}{2 + D \sec h^2(t\sqrt{-l})}}, \tag{20}$$

where $\quad K = 4\sqrt{\dfrac{2l^2}{\left(3m^2 - 16nl\right)}}$, $\quad D = -1 + \dfrac{m\sqrt{3}}{\sqrt{\left(3m^2 - 16nl\right)}}$, as presented in [3].

Now in table 3, we present the absolute errors between φ_1 and the exact solution and the absolute errors between φ_2 and the exact solution for the values of $t = 0.1(0.1)0.5$, $l = -1$, $m = 4$, $n = 3$.

Table 3: The numerical results for φ_1, φ_2 in comparison with the exact solution of x.

t	$\left|x - \varphi_1\right|$	$\left|x - \varphi_2\right|$
0.1	8.0317 e-005	7.2266 e-005
0.2	4.3147 e-004	3.0265 e-004
0.3	1.3656 e-003	7.1345 e-004
0.4	3.3474 e-003	1.2863 e-003
0.5	6.9285 e-003	1.8965 e-003

Also in Table 4, we present absolute errors between the 1-iterate of VIM (x_{1VIM}) and the exact solution and absolute errors between the 2-iterate of VIM (x_{2VIM}) and the exact solution for the values of $t = 0.1(0.1)0.5$, $l = -1$, $m = 4$, $n = 3$, as presented in [3].

Table 4: The numerical results for x_{1VIM}, x_{2VIM} in comparison with the exact solution of x.

t	$\left|x - x_{1VIM}\right|$	$\left|x - x_{2\,VIM}\right|$
0.1	2.0441 e-005	4.4279 e-008
0.2	3.2151 e-004	2.7277 e-006
0.3	1.5829 e-003	2.9172 e-005
0.4	4.8181 e-003	1.5024 e-004
0.5	1.1232 e-002	5.1335 e-004

The numerical results reveal that the VHPM is easy to implement and reduces the computational work to a tangible level while still maintaining a very higher level of accuracy.

5. Conclusions

In this paper, variational homotopy perturbation method was employed successfully for solving the Lienard equation. The exact solutions are compared with the numerical solutions of VHPM and VIM. The small amount of computation compared to that required in other methods such as the variational iteration method and the rapid convergence show that the method is reliable and provides a significant improvement in solving the nonlinear equations over existing methods. The computations in this paper are done by MATLAB software.

6. References

[1]   Z. Feng. On explicit exact solutions for the Lienard equation and its applications. *Phys. Lett. A*. 2002, **239**: 50-56.

[2]   D. Kong. Explicit exact solutions for the Lienard equation and its applications. *Phys. Lett. A*. 1995, **196**: 301-306.

[3]   M. Matinfar, H. Hosseinzadeh, and M. Ghanbari. A numerical implementation of the variational iteration method for the Lienard equation. *World Journal of Modelling and simulation.* 2008, **4**: 205-210.

[4]   J.H. He. Variational iteration method-a kind of non-linear analytical technique: some examples. *International*

Journal of Non-Linear Mechanics. 1999, **34**: 699-708.

[5] J.H. He, and X.H. Wu. Variational iteration method: new development and applications. *Computers and Mathematics with Applications*. 2007, **54**: 881-894.

[6] M. Inokuti, H. Sekine, and T. Mura. General use of the Lagrange multiplier in nonlinear mathematical physics. *Variational Method in the Mechanics of Solids.* Pergamon Press, 1978.

[7] M.A. Noor, and S.T. Mahyud-Din. Variational Homotopy Perturbation Method for solving Higher Dimentional initial boundary value problems. *Mathematical Problems in Engineering*. Hindawi publishing corporation. Volume 2008, Article ID 696734, 11 pages, doi:10.1155/2008/696734.

[8] J.H. He. Some asymptotic methods for strongly nonlinear equations. *International Journal of Modern Physics B*. 2006, **20**: 1141-1199.

[9] J.H. He. Non-perturbative method for strongly nonlinear problems. dissertation, de Verlag im Internet GmbH, 2006.

[10] J.H. He. Approximate solution of nonlinear differential equations with convolution product nonlinearities. *Comput. Methods Appl. Mech. Eng.* 1998, **167**: 69-73.

[11] J.H. He, and X.H. Wu. Construction of solitary solution and compaction-like solution by Variational iteration method. *Chaos Solitons Fractals.* 2006, **29**: 108-113.

[12] J.H. He. New interpretation of homotopy perturbation method. *Int. J. Mod. Phys*. B. 2006, **20** (18): 2561-2568.

[13] J.H. He. Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering.* 1999, **178**: 257-262.

[14] J.H. He. Homotopy perturbation method for solving boundary value problems. *Physics Letters A.* 2006, **350**: 87-88.

[15] J.H. He. Comparison of homotopy perturbation method and homotopy analysis method. *Applied Mathematics and Computation.* 2004, **156**: 527-539.

[16] J.H. He. Homotopy perturbation method for bifurcation of nonlinear problems. *International Journal of Nonlinear Sciences and numerical Simulation.* 2005, **6**: 207-208.

[17] J.H. He. The homotopy perturbation method nonlinear oscillators with discontinuities. *Applied Mathematics and Computation*. 2004, **151**: 287-292.

[18] J.H. He. A coupling method of a homotopy technique and a perturbation technique for nonlinear problems. *International Journal of Non-Linear Mechanics.* 2000, **35**: 37-43.