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A New 6-point Ternary Interpolating Subdivision Scheme and its Differentiability

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Abstract. We present a new 6-point ternary interpolating scheme with a shape parameter. The scheme is C^2 continuous over the parametric interval. The differentiable properties of proposed as well as two other existing 6-point ternary interpolating schemes have been explored. Application of proposed scheme is given to show its visual smoothness.

Keywords: Interpolating subdivision scheme, continuity, smoothness, shape parameter, Laurent polynomial.

1. Introduction

Computer Aided Geometric Design (CAGD) is a branch of applied mathematics, which deals with algorithms for the shape and structure of smooth curves and surfaces and for their competent mathematical demonstration. Subdivision is a very common approach which is related to CAGD. We can survey subdivision as process of taking a coarse shape and refining it to produce another shape that is more visually nice-looking and smooth. We can divide the subdivision schemes into two major types: one is interpolating, in which original points stay undistributed while new points are included and the other is approximating, in which new points are included as well as old points are moved at each refinement level.

Now a days wide variety of 6-point interpolating binary/ternary schemes have been introduced in the literature. Deslauries and Dubuc [1] introduced 6-point ternary interpolating scheme in 1989. In [7] Weisman described a 6-point binary interpolating scheme. Khan and Mustafa [6] introduced ternary 6-point subdivision scheme. Lian [5] generalized classical 4-point and 6-point interpolating schemes to a-ary interpolating schemes for any integer $a \geq 3$. The Laurents polynomial method has been used by [2] to discuss analysis of binary/ternary schemes.

A general ternary subdivision scheme S which maps a coarse polygon $f^k = \{f_i^k\}_{i\in Z}$ to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in Z}$ is defined by

$$f_i^{k+1} = \sum_{j\in Z} a_{3j-i} f_j^k, \qquad i \in Z, \tag{1.1}$$

where the set $a = \{a_i \mid i \in Z\}$ of coefficients is called mask of the scheme. A necessary condition for uniform convergence of the subdivision scheme (1.1) is that

$$\sum_{j\in Z} a_{3j} = \sum_{j\in Z} a_{3j+1} = \sum_{j\in Z} a_{3j+2} = 1 \tag{1.2}$$

The z -transform of the mask a of subdivision scheme can be given as

$$a(z) = \sum_{i\in Z} a_i z^i, \tag{1.3}$$

which is called the symbol or Laurent polynomial of the scheme. From (1.2) and (1.3) the Laurent polynomial of a convergent subdivision scheme satisfies

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$$a\left(e^{2i\pi/3}\right)=a\left(e^{4i\pi/3}\right)=0 \text{ and } a(1)=3. \tag{1.4}$$

The existence of associated subdivision scheme for the divided differences of the original control polygon and of related Laurent polynomial $a_1(z)$ is assured by this condition

$$a_1(z)=\frac{3z^2}{z^2+z+1}a(z).$$

The subdivision scheme S_1 with symbol $a_1(z)$ is connected to scheme S with symbol $a(z)$ by the following theorem.

Theorem 1.1. [3] Let a subdivision scheme is denoted by S with symbol $a(z)$ satisfying (1.4). Then there exist a subdivision scheme S_1 with the property

$$\Delta f^k = S_1 \Delta f^{k-1},$$

where $f^k = S^k f^0$ and $\Delta f^k = \left\{(\Delta f^k)_i = 3^k\left(f_{i+1}^k - f_i^k\right): i \in Z\right\}$. Moreover, S is uniformly convergent if and only if $\frac{1}{3}S_1$ converges uniformly to the zero function for all initial data f^0, such that

$$\lim_{k\to\infty}\left(\frac{1}{3}S_1\right)^k f^0 = 0.$$

We define the norm of scheme as

$$\left\|\left(\frac{1}{3}S_n\right)^L\right\|_\infty = \max\left\{\sum_{j\in Z}\left|b_{i+3^L j}^{[n,L]}\right| : i=0,1,\dots,3^L-1\right\}, \tag{1.5}$$

where

$$b^{[n,L]}(z)=\tfrac{1}{3^L}\prod_{j=0}^{L-1}a_n\left(z^{3^j}\right). \tag{1.6}$$

and

$$a_n(z)=\left(\frac{3z^2}{z^2+z+1}\right)a_{n-1}(z)=\left(\frac{3z^2}{z^2+z+1}\right)^n a(z), \quad n\ge 1. \tag{1.7}$$

Theorem 1.2. [3] Let S be subdivision scheme with a characteristic II-polynomial $a(z)=\left(\frac{z^2+z+1}{3z^2}\right)^n q(z), q\in \mathrm{II}$. If the subdivision scheme S_n, corresponding to the II-polynomial $q(z)$, converges uniformly then $S^\infty f^0 \in C^n(R)$ for any initial control polygon f^0.

Corollary 1.3. [3] If S is a subdivision scheme of the form above and $\frac{1}{3}S_{n+1}$ converges uniformly to the zero function for all initial data f^0 then $S^\infty f^0 \in C^n(R)$ for any initial control polygon f^0

2. A 6-point ternary interpolating scheme

In this section, we construct a 6-point ternary interpolating subdivision scheme.

2.1. Construction of the scheme

Consider the following three recursive relations which refine given kth level polygon $f^k=\{f_i^k\}_{i\in Z}$ to $(k+1)$th level polygon $f^{k+1}=\{f_i^{k+1}\}_{i\in Z}$

$$\begin{aligned} f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= a_0 f_{i-2}^k + a_1 f_{i-1}^k + a_2 f_i^k + a_3 f_{i+1}^k + a_4 f_{i+2}^k + a_5 f_{i+3}^k, \\ f_{3i+2}^{k+1} &= a_5 f_{i-2}^k + a_4 f_{i-1}^k + a_3 f_i^k + a_2 f_{i+1}^k + a_1 f_{i+2}^k + a_0 f_{i+3}^k. \end{aligned} \tag{2.1}$$

We get following mask from above recursive relations

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$$a = \{..., 0, a_5, a_0, 0, a_4, a_1, 0, a_3, a_2, 1, a_2, a_3, 0, a_1, a_4, 0, a_0, a_5, 0, ...\} .$$

The Laurent polynomial of this mask is

$$a(z) = a_5 z^8 + a_0 z^7 + a_4 z^5 + a_1 z^4 + a_3 z^2 + a_2 z + 1 + a_2 z^{-1} + a_3 z^{-2} + a_1 z^{-4} + a_4 z^{-5} + a_0 z^{-7} + a_5 z^{-8}. \quad (2.2)$$

We require $a(1) = 3$, to generate C^0 functions, which gives the following condition

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 1. \quad (2.3)$$

In order to find the mask of our scheme, first we find Laurent polynomial $a_n(z)$ corresponding to the subdivision scheme S_n by (1.7), subject to the condition that mask of a_n satisfy (1.2). By taking n = 1, 2, 3, 4 in (1.7) we get

$$\begin{aligned}
a_{(1)} &= 3\{a_5, a_0 - a_5, -a_0, a_4 + a_5, a_1 + a_0 - a_4 - a_5, -a_1 - a_0, a_3 + a_4 + a_5, \\
&\quad 1 - 2a_3 - 2a_4 - 2a_5, a_3 + a_4 + a_5, -a_1 - a_0, a_1 + a_0 - a_4 - a_5, a_4 + a_5, \\
&\quad -a_0, a_0 - a_5, a_5\}, \\
a_{(2)} &= 9\{a_5, a_0 - 2a_5, a_5 - 2a_0, a_4 + a_0 + 2a_5, a_1 - 2a_4 + 2a_0 - 2a_5, -2a_1 + a_4 \\
&\quad - 4a_0 + 2a_5, 1/3 + 2a_1 + 4a_0, -2a_1 + a_4 - 4a_0 + 2a_5, a_1 - 2a_4 + 2a_0 - 2a_5, \\
&\quad a_4 + a_0 + 2a_5, a_5 - 2a_0, a_0 - 2a_5, a_5\}, \\
a_{(3)} &= 27\{a_5, a_0 - 3a_5, 3a_5 - 3a_0, a_4 + 3a_0 + 2a_5, 11a_0 + 4a_1 + 1/3, \\
&\quad 6a_0 + 18a_5 + 1/3, 11a_0 + 4a_1 + 1/3, a_4 + 3a_0 + 2a_5, \\
&\quad 3a_5 - 3a_0, a_0 - 3a_5, a_5\}, \\
a_{(4)} &= 81\{a_5, a_0 - 4a_5, 6a_5 - 4a_0, a_4 + 6a_0, -12a_0 + 4a_4 + 18a_5 + 1/3, \\
&\quad a_4 + 6a_0, 6a_5 - 4a_0, a_0 - 4a_5, a_5\},
\end{aligned} \quad (2.4)$$

where

$$\begin{aligned}
6a_0 + 3a_1 - 3a_3 - 6a_4 - 9a_5 &= -1, \\
27a_0 + 9a_1 + 9a_4 + 27a_5 &= -1, \\
21a_0 + 4a_1 - 5a_4 - 24a_5 &= 0, \\
36a_0 - 9a_4 - 9a_5 &= 1.
\end{aligned} \quad (2.5)$$

Solving (2.3) and (2.5) by taking $a_5 = w$, we get the following mask of 6-point ternary interpolating subdivision scheme (2.1).

$$a_0 = \frac{5}{243} - w, \; a_1 = -\frac{35}{243} + 5w, \; a_2 = \frac{70}{81} - 10w,$$

$$a_3 = \frac{70}{243} + 10w, \; a_4 = -\frac{7}{243} - 5w, \; a_5 = w.$$

By taking $w = \frac{7}{729}$, we get mask of DD 6-point ternary interpolating scheme [1].

2.2. Smoothness analysis

All Let S be a scheme defined by (2.1) then its Laurent polynomial can be written as

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$$a(z)=wz^{8}+\left(\frac{5}{243}-w\right)z^{7}+\left(-\frac{7}{243}-5w\right)z^{5}+\left(-\frac{35}{243}+5w\right)z^{4}$$
$$+\left(\frac{70}{243}+10w\right)z^{2}+\left(\frac{70}{81}-10w\right)z^{1}+1+\left(\frac{70}{81}-10w\right)z^{-1}$$
$$+\left(\frac{70}{243}+10w\right)z^{-2}+\left(-\frac{35}{243}+5w\right)z^{-4}+\left(-\frac{7}{243}-5w\right)z^{-5}$$
$$+\left(\frac{5}{243}-w\right)z^{-7}+wz^{-8}.$$

If $S_i;\ i=1,\ 2,\ 3,\ 4$ are divided difference subdivision schemes of S corresponding to the Laurent polynomials $a_i(z);\ i=1,\ 2,\ 3,\ 4$ then by (2.4) we get

$$a_1(z)=3\left\{wz^{8}+\left(\frac{5}{243}-2w\right)z^{7}+\left(-\frac{5}{243}+w\right)z^{6}+\left(-\frac{7}{243}-4w\right)z^{5}\right.$$
$$+\left(-\frac{23}{243}+8w\right)z^{4}+\left(\frac{30}{243}-4w\right)z^{3}+\left(\frac{63}{243}+6w\right)z^{2}+\left(\frac{117}{243}-12w\right)z$$
$$+\left(\frac{63}{243}+6w\right)+\left(\frac{30}{243}-4w\right)z^{-1}+\left(-\frac{23}{243}+8w\right)z^{-2}+\left(-\frac{7}{243}-4w\right)z^{-3}$$
$$\left.+\left(-\frac{5}{243}+w\right)z^{-4}+\left(\frac{5}{243}-2w\right)z^{-5}+wz^{-6}\right\},$$

$$a_2(z)=9\left\{wz^{8}+\left(\frac{5}{243}-3w\right)z^{7}+\left(-\frac{10}{243}+3w\right)z^{6}+\left(-\frac{2}{243}-4w\right)z^{5}\right.$$
$$+\left(-\frac{11}{243}+9w\right)z^{4}+\left(\frac{43}{243}-9w\right)z^{3}+\left(\frac{31}{243}+6w\right)z^{2}+\left(\frac{43}{243}-9w\right)z$$
$$+\left(-\frac{11}{243}+9w\right)+\left(-\frac{2}{243}-4w\right)z^{-1}+\left(-\frac{10}{243}+3w\right)z^{-2}$$
$$\left.+\left(\frac{5}{243}-3w\right)z^{-3}+wz^{-4}\right\},$$

$$a_3(z)=27\left\{wz^{8}+\left(\frac{5}{243}-4w\right)z^{7}+\left(-\frac{5}{81}+6w\right)z^{6}+\left(\frac{8}{243}-6w\right)z^{5}\right.$$
$$+\left(-\frac{4}{243}+9w\right)z^{4}+\left(\frac{13}{81}-12w\right)z^{3}+\left(-\frac{4}{243}+9w\right)z^{2}+\left(\frac{8}{243}-6w\right)z$$
$$\left.+\left(-\frac{5}{81}+6w\right)+\left(\frac{5}{243}-4w\right)z^{-1}+wz^{-2}\right\},$$

and

$$a_4(z)=81\left\{wz^{8}+\left(\frac{5}{243}-5w\right)z^{7}+\left(-\frac{20}{243}+10w\right)z^{6}+\left(\frac{23}{243}-11w\right)z^{5}\right.$$
$$+\left(-\frac{7}{243}+10w\right)z^{4}+\left(\frac{23}{243}-11w\right)z^{3}+\left(-\frac{20}{243}+10w\right)z^{2}$$
$$\left.+\left(\frac{5}{243}-5w\right)z^{1}+w\right\}.$$

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It is easy to show that $a(z)$ and $a_i(z)$; $i=1, 2, 3, 4$ satisfies (1.4).

Using (1.6) for $L = 1$, we get

$$b^{[1,1]}(z)=\frac{1}{3}a_1(z),$$

so by (1.5)

$$\left\|\frac{1}{3}S_1\right\|_\infty=\max\left\{\sum_{j\in Z}\left|b^{[1,1]}_{i+3j}\right| : i=0, 1, 2\right\}.$$

This implies

$$\left\|\frac{1}{3}S_1\right\|_\infty = \max\left\{\left|\frac{63}{243}+6w\right|+\left|\frac{30}{243}-4w\right|+\left|-\frac{5}{243}+w\right|+\left|-\frac{7}{243}-4w\right|+|w|,\right.$$
$$\left.\left|\frac{117}{243}-12w\right|+\left|-\frac{23}{243}+8w\right|+\left|\frac{5}{243}-2w\right|+\left|-\frac{23}{243}+8w\right|+\left|\frac{5}{243}-2w\right|\right\}.$$

Similarly

$$\left\|\frac{1}{3}S_2\right\|_\infty = 3\max\left\{\left|-\frac{11}{243}+9w\right|+\left|\frac{43}{243}-9w\right|+\left|-\frac{10}{243}+3w\right|+\left|\frac{5}{243}-3w\right|,\right.$$
$$\left|\frac{43}{243}-9w\right|+\left|-\frac{11}{243}+9w\right|+\left|\frac{5}{243}-3w\right|+\left|-\frac{10}{243}+3w\right|,$$
$$\left.\left|\frac{31}{243}+6w\right|+2\left|-\frac{2}{243}-4w\right|+2|w|\right\},$$

$$\left\|\frac{1}{3}S_3\right\|_\infty = 9\max\left\{\left|-\frac{15}{243}+6w\right|+\left|\frac{13}{81}-12w\right|+\left|-\frac{5}{81}+6w\right|,\ \left|\frac{8}{243}-6w\right|\right.$$
$$\left.+\left|-\frac{4}{243}+9w\right|+\left|\frac{5}{243}-4w\right|+|w|\right\},$$

$$\left\|\frac{1}{3}S_4\right\|_\infty = 27\max\left\{|w|+\left|-\frac{20}{243}+10w\right|+\left|-\frac{23}{243}+11w\right|,\ 2\left|-\frac{5}{243}+5w\right|\right.$$
$$\left.+\left|-\frac{7}{243}+10w\right|,\ |w|+\left|-\frac{20}{243}+10w\right|+\left|-\frac{23}{243}+11w\right|\right\}.$$

As we see for $-\frac{35}{3888}<w<\frac{13}{243}$, $\left\|\frac{1}{3}S_1\right\|_\infty<1$ then by Theorem 1.2 for this range of shape parameter w the scheme is C^0. Similarly $\left\|\frac{1}{3}S_2\right\|_\infty<1$ for $-\frac{1}{486}<w<\frac{23}{1944}$, then by Corollary 1.3, the scheme is C^1 and for $\frac{7}{972}<w<\frac{11}{1215}$, $\left\|\frac{1}{3}S_3\right\|_\infty<1$, then by Corollary 1.3, the scheme is C^2.

For C^3-continuity, the condition $\left\|\frac{1}{3}S_4\right\|_\infty<1$ must be satisfied. This condition is true for $\frac{17}{2430}<w<\frac{26}{2673}$ and $\frac{2}{1215}<w<\frac{131}{2430}$, but there is no common range of w for which $\left\|\frac{1}{3}S_4\right\|_\infty<1$. In the same way, we can show that for $L = 2, 3, \ldots,$ we have $\left\|\left(\frac{1}{3}S_4\right)^L\right\|_\infty\geq 1$ therefore scheme S is not C^3-continuous.

By summarizing above discussion, we get following result.

Theorem 2.1. Given initial control points $\{f_i^0\}_{i\in Z}$, let $\{f_i^k\}_{i\in Z}$ defined by (2.1) be

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the values corresponding to $\frac{i}{3^k}$ and let f(t) be the limit function of scheme (2.1), then f(t) is C^0, C^1 and C^2-continuous for the ranges $-\frac{35}{3888} < w < \frac{13}{243}$, $-\frac{1}{486} < w < \frac{23}{1944}$ and $\frac{7}{972} < w < \frac{11}{1215}$, respectively.

3. Differentiability of 6-point Ternary Interpolating Scheme

In this section, we will find exact expressions for the first and the second derivatives of the limit functions of the proposed scheme, DD [1] and Khan & Mustafa [6] 6-point ternary schemes by following the procedure of [4].

Theorem 3.1. Given initial real numbers $\left\{f_i^0\right\}$, let $\left\{f_i^k\right\}$ be the values defined by 6-point ternary interpolating scheme (2.1) corresponding to $\frac{i}{3^k}$ $\left(i,\, k \in Z,\, k \geq 0\right)$ and $f \in C^2$ be the corresponding limit function with $\frac{7}{972} < w < \frac{11}{1215}$, then for arbitrary fixed $m, n_0 \in Z$, $m \geq 0$, the derivatives of the limit function f are

$$f^{/}\left(\frac{n_0}{3^m}\right) = \frac{3^{m+k}}{6(202-4374w)}\Big\{(-5+2187w)\left(f^{m+k}_{3^k n_0+3} - f^{m+k}_{3^k n_0-3}\right) + 81(1+81w)\left(f^{m+k}_{3^k n_0-2} - f^{m+k}_{3^k n_0+2}\right) + 27(-29+243w)\left(f^{m+k}_{3^k n_0-1} - f^{m+k}_{3^k n_0+1}\right)\Big\},$$

$$f^{//}\left(\frac{n_0}{3^m}\right) = \frac{3^{2(m+k)}}{18(-5+27459w)}\Big\{(5-1431w)\left(f^{m+k}_{3^k n_0-3} + f^{m+k}_{3^k n_0+3}\right) + 27(1+729w)\left(f^{m+k}_{3^k n_0-2} + f^{m+k}_{3^k n_0+2}\right) + 27(-31+729w)\left(f^{m+k}_{3^k n_0-1} + f^{m+k}_{3^k n_0+1}\right) - 10(-167+7047w)\, f^{m+k}_{3^k n_0}\Big\}.$$

Proof. Suppose

$$F^k = \left(f^{m+k}_{3^k n_0-3},\, f^{m+k}_{3^k n_0-2},\, f^{m+k}_{3^k n_0-1},\, f^{m+k}_{3^k n_0},\, f^{m+k}_{3^k n_0+1},\, f^{m+k}_{3^k n_0+2},\, f^{m+k}_{3^k n_0+3}\right)^T. \tag{3.1}$$

Since scheme under consideration is interpolating, so we have $f^{m+k}_{3^k n_0} = f^m_{n_0}$. During each subdivision level control points can be evaluated by using following rule

$$F^{k+1} = AF^k,$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

is the vertex subdivision matrix. The configuration around the vertex is shown in Figure 1. For $\frac{7}{972} < w < \frac{11}{1215}$, matrix A has seven different eigenvalues as $\lambda_1 = 1$, $\lambda_2 = \frac{1}{3}$, $\lambda_3 = \frac{1}{9}$, $\lambda_4 = \frac{1}{27}$, $\lambda_5 = \frac{1}{81}$, $\lambda_6 = 21w - \frac{10}{81}$, $\lambda_7 = 9w - \frac{20}{243}$ and has seven orthogonal eigenvectors. Let $\hat{r}_i$, $\hat{l}_i$ be the right and left eigenvectors of matrix A corresponding to the eigenvalues λ_i, $i = 1, \ldots, 7$, then direct computations leads to

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$$\hat{r}_1=\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\end{pmatrix},\ \hat{r}_2=\begin{pmatrix}-\frac{3}{2}\\-1\\-\frac{1}{2}\\0\\\frac{1}{2}\\1\\\frac{3}{2}\end{pmatrix},\ \hat{r}_3=\begin{pmatrix}9\\4\\1\\0\\1\\4\\9\end{pmatrix},\ \hat{r}_4=\begin{pmatrix}-27\\-8\\-1\\0\\1\\8\\27\end{pmatrix},\ \hat{r}_5=\begin{pmatrix}81\\16\\1\\0\\1\\16\\81\end{pmatrix},$$

$$\hat{r}_6=\begin{pmatrix}1\\-\frac{(531441w^2-87966w+625)}{729(81w+1)}\\9w-\frac{20}{243}\\0\\-9w+\frac{20}{243}\\\frac{(531441w^2-87966w+625)}{729(81w+1)}\\-1\end{pmatrix},\ \hat{r}_7=\begin{pmatrix}1\\-\frac{(1240029w^2-72414w+325)}{81(729w+1)}\\21w-\frac{10}{81}\\0\\21w-\frac{10}{81}\\-\frac{(1240029w^2-72414w+325)}{81(729w+1)}\\1\end{pmatrix}.$$

Similarly, the left eigenvectors are

$$\hat{l}_2=\left(-\frac{-5+2187w}{12(101-2187w)},\frac{27(1+81w)}{4(101-2187w)},\frac{9(-29+243w)}{4(101-2187w)},0,\right.$$
$$\left.-\frac{9(-29+243w)}{4(101-2187w)},-\frac{27(1+81w)}{4(101-2187w)},\frac{-5+2187w}{12(101-2187w)}\right),$$

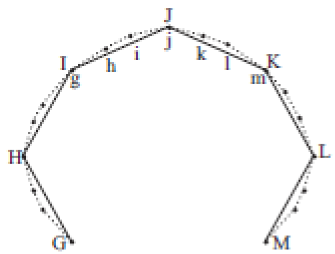

Fig. 1: Configuration around vertex *J*. Here $\{G, H, I, J, K, L, M\}$ and $\{g, h, i, j, k, l, m\}$ are old and new vertices respectively.

$$\hat{l}_3=\left(\frac{5-1431w}{36(-5+27459w)},\frac{3(1+729w)}{4(-5+27459w)},\frac{3(-31+729w)}{4(-5+27459w)},-\frac{10(-167+7047w)}{36(-5+27459w)},\right.$$
$$\left.\frac{3(-31+729w)}{4(-5+27459w)},\frac{3(1+729w)}{4(-5+27459w)},\frac{5-1431w}{36(-5+27459w)}\right).$$

Since proposed scheme is C^2 for $\frac{7}{972}<w<\frac{11}{1215}$, then we have

$$\lim_{k\to\infty}F^k=f_{n_0}^m\hat{r}_1=f\left(\frac{n_0}{3^m}\right)\hat{r}_1,$$

$$\lim_{k\to\infty}\frac{f^{m+k}_{3^k n_0+j}-f^m_{n_0}}{\frac{j}{3^{m+k}}}=\lim_{k\to\infty}\frac{f\left(\frac{3^k n_0+j}{3^{m+k}}\right)-f\left(\frac{n_0}{3^m}\right)}{\frac{j}{3^{m+k}}}=f'\left(\frac{n_0}{3^m}\right),\quad j=\pm1,\pm2,\pm3, \tag{3.2}$$

and

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$$\lim_{k\to\infty}\frac{f^{m+k}_{3^k n_0+j}+f^{m+k}_{3^k n_0-j}-2f^m_{n_0}}{\left(\frac{j}{3^{m+k}}\right)^2}=\lim_{k\to\infty}\frac{f\left(\frac{3^k n_0+j}{3^{m+k}}\right)+f\left(\frac{3^k n_0-j}{3^{m+k}}\right)-2f\left(\frac{n_0}{3^m}\right)}{\left(\frac{j}{3^{m+k}}\right)^2}$$

$$= f^{//}\left(\frac{n_0}{3^m}\right),\ \ j=\pm 1,\pm 2,\pm 3\,. \tag{3.3}$$

Let $\hat{e}=(1,\ 1,\ 1,\ 0,\ 1,\ 1,\ 1)^T$, $D=diag\left(-\frac{1}{3},\ -\frac{1}{2},\ -1,\ 1,\ 1,\ \frac{1}{2},\ \frac{1}{3}\right)$ and

$$J=\begin{pmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0 \end{pmatrix}$$

Then by (3.2) and (3.3), we derive

$$\lim_{k\to\infty}3^{m+k}\,D\left(F^k-f^m_{n_0}\,\hat{r}_1\right)=f^{/}\left(\frac{n_0}{3^m}\right)\hat{e}\,, \tag{3.4}$$

$$\lim_{k\to\infty}3^{2(m+k)}\,D^2\left(F^k+JF^k-2f^m_{n_0}\,\hat{r}_1\right)=f^{//}\left(\frac{n_0}{3^m}\right)\hat{e}\,. \tag{3.5}$$

Since matrix A has seven linearly independent eigenvectors, then there exist α_1 , α_2 , … , α_7 such that F^0 can be written as

$$F^0=\sum_{i=1}^{7}\alpha_i\,\hat{r}_i\,.$$

From the expressions of $\hat{r}_i$, $i=1,\ 2,\ 3,\ \ldots,\ 7$ and

$$F^0=\left(f^m_{n_0-3},\ f^m_{n_0-2},\ f^m_{n_0-1},\ f^m_{n_0},\ f^m_{n_0+1},\ f^m_{n_0+2},\ f^m_{n_0+3}\right)^T,$$

we have $\alpha_1=f^m_{n_0}$. Therefore F^k can be written

$$\begin{aligned} F^k &= A^k F^0=\sum_{i=1}^{7}\alpha_i\,(\lambda_i)^k\,\hat{r}_i\\ &= f^m_{n_0}\hat{r}_1+\alpha_2\left(\frac{1}{3}\right)^k\hat{r}_2+\alpha_3\left(\frac{1}{9}\right)^k\hat{r}_3\\ &\quad+\alpha_4\left(\frac{1}{27}\right)^k\hat{r}_4+\alpha_5\left(\frac{1}{81}\right)^k\hat{r}_5+\sum_{i=6}^{7}\alpha_i(\lambda_i)^k\,\hat{r}_i. \end{aligned} \tag{3.6}$$

By utilizing (3.6) in (3.4), we get

$$\begin{aligned} f^{/}\left(\frac{n_0}{3^m}\right)\hat{e}=\lim_{k\to\infty}3^{m+k}\,D\Bigg\{ & f^m_{n_0}\,\hat{r}_1+\alpha_2\left(\frac{1}{3}\right)^k\hat{r}_2+\alpha_3\left(\frac{1}{9}\right)^k\hat{r}_3+\alpha_4\left(\frac{1}{27}\right)^k\hat{r}_4\\ &+\sum_{i=5}^{7}\alpha_i\,(\lambda_i)^k\,\hat{r}_i-f^m_{n_0}\,\hat{r}_1\Bigg\} \end{aligned}$$

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$$= \lim_{k\to\infty} 3^m \left\{ \alpha_2\, D\hat{r}_2 + \alpha_3 \left(\frac{1}{3}\right)^k D\hat{r}_3 + \alpha_4 \left(\frac{1}{9}\right)^k D\hat{r}_4 \right.$$

$$\left. + 3^k \sum_{i=5}^{7} \alpha_i \left(\lambda_i\right)^k D\hat{r}_i \right\}.$$

This implies that

$$f^{/}\left(\frac{n_0}{3^m}\right)\hat{e} = 3^m\, \alpha_2\, D\hat{r}_2 . \tag{3.7}$$

In view of

$$J\hat{r}_i = \begin{cases} \hat{r}_i, & i = 1, 3, 5, 7, \\ -\hat{r}_i, & i = 2, 4, 6, \end{cases}$$

and by utilizing (3.6) in (3.5), we have

$$f''\left(\frac{n_0}{3^m}\right)\hat{e} = \lim_{k\to\infty} 3^{2(m+k)} D^2 \left\{ f_{n_0}^m \hat{r}_1 + \alpha_2 \left(\frac{1}{3}\right)^k \hat{r}_2 + \alpha_3 \left(\frac{1}{9}\right)^k \hat{r}_3 + \alpha_4 \left(\frac{1}{27}\right)^k \hat{r}_4 \right.$$

$$+ \sum_{i=5}^{7} \alpha_i \left(\lambda_i\right)^k \hat{r}_i + f_{n_0}^m \hat{r}_1 - \alpha_2 \left(\frac{1}{3}\right)^k \hat{r}_2 + \alpha_3 \left(\frac{1}{9}\right)^k \hat{r}_3 - \alpha_4 \left(\frac{1}{27}\right)^k \hat{r}_4$$

$$\left. + \alpha_5 \left(\lambda_5\right)^k \hat{r}_5 - \alpha_6 \left(\lambda_6\right)^k \hat{r}_6 + \alpha_7 \left(\lambda_7\right)^k \hat{r}_7 - 2 f_{n_0}^m \hat{r}_1 \right\}.$$

This implies that

$$f''\left(\frac{n_0}{3^m}\right)\hat{e} = \lim_{k\to\infty} 3^{2m} \left\{ 2\alpha_3 D^2 \hat{r}_3 + 2\alpha_5 \left(9\lambda_5\right)^k D^2 \hat{r}_5 + 2\alpha_7 \left(9\lambda_7\right)^k D^2 \hat{r}_7 \right\}.$$

So, we have

$$f''\left(\frac{n_0}{3^m}\right)\hat{e} = 3^{2m} 2\alpha_3 D^2 \hat{r}_3 . \tag{3.8}$$

Since we have $D\hat{r}_2 = \hat{e}$ and $D^2\hat{r}_3 = \hat{e}$. By using them in (3.7) and (3.8), we get

$$f'\left(\frac{n_0}{3^m}\right) = 3^m \alpha_2 , \tag{3.9}$$

and

$$f''\left(\frac{n_0}{3^m}\right) = 3^{2m} 2\alpha_3 . \tag{3.10}$$

Now by multiplying (3.6) with left eigenvector $\hat{l}_2$ corresponding to eigenvalue $\lambda_2 = \dfrac{1}{3}$ and in view of

$$\hat{l}_j\, \hat{r}_i = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

we have

$$\hat{l}_2 F^k = \hat{l}_2 \left(f_{n_0}^m \hat{r}_1 + \alpha_2 \left(\frac{1}{3}\right)^k \hat{r}_2 + \alpha_3 \left(\frac{1}{9}\right)^k \hat{r}_3 + \alpha_4 \left(\frac{1}{27}\right)^k \hat{r}_4 + \sum_{i=5}^{7} \alpha_i \left(\lambda_i\right)^k \hat{r}_i \right).$$

This implies that

$$\hat{l}_2 F^k = \alpha_2 \left(\frac{1}{3}\right)^k .$$

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Again we have

$$\alpha_2 = 3^k \hat{l}_2 F^k .$$

Using above result in (3.9), we have

$$f'\left(\frac{n_0}{3^m}\right) = 3^m \alpha_2 = 3^{m+k} \hat{l}_2 F^k . \tag{3.11}$$

So by using (3.1) and $\hat{l}_2$, we have

$$f'\left(\frac{n_0}{3^m}\right) = 3^{m+k} \begin{pmatrix} -\frac{-5+2187w}{12(101-2187w)} \\ \frac{27(1+81w)}{4(101-2187w)} \\ \frac{9(-29+243w)}{4(101-2187w)} \\ 0 \\ -\frac{9(-29+243w)}{4(101-2187w)} \\ -\frac{27(1+81w)}{4(101-2187w)} \\ \frac{-5+2187w}{12(101-2187w)} \end{pmatrix}^T \begin{pmatrix} f^{m+k}_{3^k n_0 -3} \\ f^{m+k}_{3^k n_0 -2} \\ f^{m+k}_{3^k n_0 -1} \\ f^{m+k}_{3^k n_0} \\ f^{m+k}_{3^k n_0 +1} \\ f^{m+k}_{3^k n_0 +2} \\ f^{m+k}_{3^k n_0 +3} \end{pmatrix}.$$

This implies that

$$\begin{aligned} f'\left(\frac{n_0}{3^m}\right) = \frac{3^{m+k}}{6(202-4374w)} \Big\{ & (-5+2187w)\left(f^{m+k}_{3^k n_0+3} - f^{m+k}_{3^k n_0-3}\right) \\ & + 81(1+81w)\left(f^{m+k}_{3^k n_0-2} - f^{m+k}_{3^k n_0+2}\right) + 27(-29+243w)\left(f^{m+k}_{3^k n_0-1} - f^{m+k}_{3^k n_0+1}\right) \Big\}. \end{aligned}$$

Similarly by multiplying (3.6) with left eigenvector $\hat{l}_3$ corresponding to $\lambda_3 = \frac{1}{9}$, we get

$$\hat{l}_3 F^k = \alpha_3 \left(\frac{1}{9}\right)^k .$$

This implies that

$$\alpha_3 = 9^k \hat{l}_3 F^k = 3^{2k} \hat{l}_3 F^k .$$

Using above equation in (3.10), we have

$$f''\left(\frac{n_0}{3^m}\right) = 3^{2(m+k)} 2 \hat{l}_3 F^k . \tag{3.12}$$

So by using (3.1) and value of $\hat{l}_3$, we have

$$f''\left(\frac{n_0}{3^m}\right) = 2.3^{2(m+k)} \begin{pmatrix} \frac{5-1431w}{36(-5+27459w)} \\ \frac{3(1+729w)}{4(-5+27459w)} \\ \frac{3(-31+729w)}{4(-5+27459w)} \\ -\frac{10(-167+7047w)}{36(-5+27459w)} \\ \frac{3(-31+729w)}{4(-5+27459w)} \\ \frac{3(1+729w)}{4(-5+27459w)} \\ \frac{5-1431w}{36(-5+27459w)} \end{pmatrix}^T \begin{pmatrix} f^{m+k}_{3^k n_0 -3} \\ f^{m+k}_{3^k n_0 -2} \\ f^{m+k}_{3^k n_0 -1} \\ f^{m+k}_{3^k n_0} \\ f^{m+k}_{3^k n_0 +1} \\ f^{m+k}_{3^k n_0 +2} \\ f^{m+k}_{3^k n_0 +3} \end{pmatrix}$$

This completes the proof. □

Theorem 3.2. Given initial real numbers $\{f_i^0\}$, let $\{f_i^k\}$ be the values defined by DD 6-point ternary interpolating scheme [1] corresponding to $\frac{i}{3^k}$ $(i,k \in Z, k \geq 0)$ and $f \in C^2$ be the corresponding limit function, then for arbitrary fixed m , $n_0 \in Z, m \geq 0$, the derivatives of the limit function f are

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$$f'\left(\frac{n_0}{3^m}\right)=\frac{3^{m+k}}{60}\left\{\left(f^{m+k}_{3^k n_0-3}-f^{m+k}_{3^k n_0+3}\right)+9\left(f^{m+k}_{3^k n_0+2}-f^{m+k}_{3^k n_0-2}\right)+45\left(f^{m+k}_{3^k n_0-1}-f^{m+k}_{3^k n_0+1}\right)\right\},$$

$$f''\left(\frac{n_0}{3^m}\right)=\frac{3^{2(m+k)}}{36}\left\{13\left(f^{m+k}_{3^k n_0+3}+f^{m+k}_{3^k n_0-3}\right)-81\left(f^{m+k}_{3^k n_0+2}+f^{m+k}_{3^k n_0-2}\right)+243\left(f^{m+k}_{3^k n_0+1}+f^{m+k}_{3^k n_0-1}\right)-350\,f^{m+k}_{3^k n_0}\right\}.$$

Theorem 3.3. Given initial real numbers $\{f_i^0\}$, let $\{f_i^k\}$ be the values defined by Khan and Mustafa scheme [6] be the values corresponding to $\frac{i}{3^k}$ $(i,k\in Z, k\geq 0)$ and $f\in C^2$ be the corresponding limit function with $0.01152<w<0.01183$, then for arbitrary fixed m, $n_0\in Z, m\geq 0$, the derivatives of the limit function f are

$$f'\left(\frac{n_0}{3^m}\right)=\frac{3^{m+k}}{6(34-7614w)}\left\{(-11+405w)\left(f^{m+k}_{3^k n_0+3}-f^{m+k}_{3^k n_0-3}\right)\right.$$
$$\left.+27(1+81w)\left(f^{m+k}_{3^k n_0+2}-f^{m+k}_{3^k n_0-2}\right)+81(-1+351w)\left(f^{m+k}_{3^k n_0-1}-f^{m+k}_{3^k n_0+1}\right)\right\}.$$

$$f''\left(\frac{n_0}{3^m}\right)=\frac{3^{2(m+k)}}{32(19-4779w+275562w^2)}\left\{\left(143-24705w+1102248w^2\right)\left(f^{m+k}_{3^k n_0+3}+f^{m+k}_{3^k n_0-3}\right)\right.$$
$$+9\left(-13+891w+1102248w^2\right)\left(f^{m+k}_{3^k n_0+2}+f^{m+k}_{3^k n_0-2}\right)+135\left(-1+135w\right.$$
$$\left.+1102248w^2\right)\left(f^{m+k}_{3^k n_0+1}+f^{m+k}_{3^k n_0-1}\right)+2\left(109-1539w-1102248w^2\right)f^{m+k}_{3^k n_0}\right\}.$$

4. Application of proposed scheme

Here we demonstrate the proposed scheme by applying it to the different polygons in Figure 2. Doted lines show initial polygon, whereas continuous curve are generated at parametric value $w=0.008$.

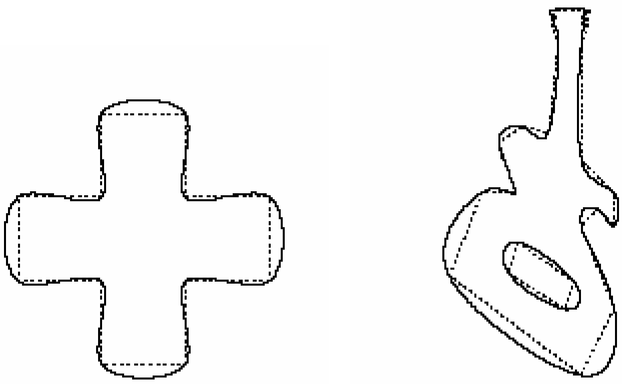

Fig. 2: Doted lines indicate initial polygon, whereas continuous curve are generated by proposed scheme.

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