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An Improvement on Disc Separation of the Schur Complement and Bounds for Determinants of Diagonally Dominant Matrices

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Abstract. In this paper, we improve the disc separation of the Schur complement of strictly diagonally dominant matrices presented in Liu [SIAM. J. Matrix Anal. Appl., 27 (2005): 665-674]. As applications, we present some new bounds for determinants of original matrices and estimations for eigenvalues of Schur complement. By theoretical analysis, we improve the bounds of determinants established in Huang [*Comput. Math. Appl.*, 50 (2005): 1677-1684].

Keywords: *H*-matrix; strictly (doubly) diagonally dominant matrix; Schur complement; Geršgorin's theorem.

1. Introduction

For localization of eigenvalues and estimations of determinants, many researches have been proposed, e.g., [1-5]. Recently Liu [6] discussed the diagonally dominant degree of the Schur complement of strictly diagonally dominant matrices and presented the localization for eigenvalues of the Schur complement and some bounds for determinants of the strictly diagonally dominant matrices. Huang [7] estimated the bounds for determinants of diagonally dominant matrices, general H-matrices and certain not diagonally dominant matrices. In this paper, we improve the diagonally dominant degree of the Schur complement of diagonally dominant matrices in [6]. Further, we obtain new bounds for determinants of diagonally dominant matrices and the estimations of eigenvalues of the Schur complement, these results improve the estimations of [6,7].

Let $A \in C^{n\times n}$ be a strictly diagonally (row) dominant matrix (SD_n), if and only if

$$|a_{ii}| > P_i(A),\ \ P_i(A) = \sum_{j=1,\, j\neq i}^{n} |a_{ij}| \,(\text{abbreviated } P_i),\ \ \forall i = 1,2,\cdots,n. \tag{1}$$

Let $A \in C^{n\times n}$ be a strictly doubly diagonally (row) dominant matrix (SDD_n), if and only if

$$|a_{ii}||a_{jj}| > P_i(A)P_j(A), \forall i, j = 1,2,\cdots,n. \tag{2}$$

If $A \in \mathrm{SDD}_n$, but $A \notin \mathrm{SD}_n$, then, by (2), there exists a unique i_0 such that

$$|a_{i_0,i_0}| \le P_{i_0}(A). \tag{3}$$

For $A = (a_{ij})$ and $B = (b_{ij}) \in C^{m\times n}$, we write $A \ge B$, if $a_{ij} \ge b_{ij}$ for all i, j. A real $n\times n$ matrix A is called an M-matrix (M_n) if $A = sI_n - B$, where $s \ge 0, B \ge 0$ and $s > \rho(B)$, $\rho(B)$ is the spectral radius of B.

Suppose $A \in C^{n\times n}$, A be called an H-matrix (H_n) if $\mu(A) \in M_n$, where, the comparison matrix $\mu(A) = (\mu_{ij})$ be defined by

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$$\mu_{ij} = \begin{cases} -|a_{ij}|, i \neq j, \\ |a_{ij}|, i = j, \end{cases} \quad i, j = 1, 2, \cdots, n.$$

Let x^T denote the transpose of the vector x, and I_n denote the $n \times n$ identity matrix. Let $A \in C^{n \times n}$, and $N = \{1, 2, \cdots, n\}$. If $\alpha \subseteq N$, $|\alpha|$ equals the cardinality of α. For nonempty index sets $\alpha, \beta \subseteq N$, we denote by $A(\alpha, \beta)$ the submatrix of A lying in the rows indicated by α and the columns indicated by β. The submatrix $A(\alpha, \alpha)$ be abbreviated to $A(\alpha)$. Let $\alpha \subset N$ and $\alpha^c = N - \alpha$, both arranged in increasing order. Then

$$A/\alpha = A/A(\alpha) = A(\alpha^c) - A(\alpha^c, \alpha)[A(\alpha)]^{-1} A(\alpha, \alpha^c)$$

be called the Schur complement with respect to $A(\alpha)$.

Lemma 1.1 (See [8]). Let $A \in M_n$. Then there exists a positive diagonal matrix D such that $AD \in \mathrm{SD}_n$.

Lemma 1.2. (See [12]). Let $A \in \mathrm{SD}_n, \mathrm{SDD}_n$. Then $\mu(A) \in M_n$, $A \in H_n$.

Lemma 1.3 (See [9]). Let $A \in C^{n \times n}$, $B \in M_n$. If $\mu(A) \geq B$, then $A \in H_n$ and $B^{-1} \geq |A^{-1}| \geq 0$.

Remark 1.1. From Lemma 1.3, we obtain immediately that

$$A \in H_n \Rightarrow [\mu(A)]^{-1} \geq |A^{-1}|.$$

Lemma 1.4 (See [10]). Let $A \in SD_n$ and m be a proper subset of n. Then

$$A/m \in \mathrm{SD}_{n-|m|}.$$

Lemma 1.5 (See [11]). Let $A \in C^{n \times n}$. A is an H matrix if the following inequality be hold

$$|a_{ii}| > \sum_{\substack{t \in N_1 \\ t \neq i}} |a_{it}| + \mu \sum_{t \in N_2} |a_{it}| \frac{P_t}{|a_{tt}|}, \ \forall i \in N_1, \tag{4}$$

where

$$0 \leq \mu \triangleq \begin{cases} \max\limits_{1 \leq \omega \leq k} \dfrac{\sum\limits_{t \in N_1} |a_{jt}|}{P_j - \sum\limits_{t \in N_2, t \neq j} |a_{jt}| \frac{P_t}{|a_{tt}|}} \leq 1, & \text{if} \quad P_t \neq 0, \forall j \in N_2, \\ 1 & \text{else if } \exists j \in N_2, \hat{P}_j = 0, \end{cases}$$

$$\hat{P}_t \triangleq \sum_{\substack{t \in N_1 \\ t \neq i}} |a_{it}| + \sum_{t \in N_2} |a_{it}| \frac{P_t}{|a_{tt}|}, \ \forall i \in N_1,$$

$$N_1 \triangleq \{i \in N \mid 0 < |a_{ii}| \leq P_i(A)\}, N_2 \triangleq \{i \in N \mid |a_{ii}| > P_i(A)\}.$$

2. Disc separation of the Schur complements of SD_n and SDD_n

In this section, by discussing the criteria of H_n, we improve the diagonally dominant degree of the Schur complement of SD_n and SDD_n in [6].

Lemma 2.1. Let $A \in \mathrm{SD}_n(\text{or } \mathrm{SDD}_n)$, $\alpha = \{i_1, i_2, \cdots, i_k\}$ be a proper subset of N and $\alpha^c = N - \alpha$ $= \{j_1, j_2, \cdots, j_l\}$, $k + l = n$. For any $j_t \in \alpha^c$, denote

$$B_{j_t}=\begin{pmatrix} x-|a_{j_ti_1}| & \cdots & -|a_{j_ti_k}| \\ -\sum_{u=1}^{l}|a_{i_1j_u}| & & \\ & \mu[A(\alpha)] & \\ -\sum_{u=1}^{l}|a_{i_kj_u}| & & \end{pmatrix}$$

(i) For $A\in \mathrm{SD}_n$, then $B_{j_t}\in H_{|\alpha^c|}$ if

$$x\geq \mu\max_{1\leq\omega\leq k}\frac{P_{i_\omega}}{|a_{i_\omega,i_\omega}|}\sum_{v=1}^{k}|a_{j_ti_v}|, \tag{5}$$

where

$$0\leq\mu\triangleq\begin{cases}\max\limits_{1\leq\omega\leq k}\dfrac{\sum_{v=1}^{l}|a_{i_\omega,j_v}|}{|a_{i_\omega i_\omega}|-\max\limits_{1\leq u\leq k}\frac{P_{i_u}}{|a_{i_u,i_u}|}\sum\limits_{v=1,v\neq\omega}^{k}|a_{i_\omega,i_v}|}<1, & \text{if } P_{i_\omega}\neq 0,\\ 2 & \text{else if } \hat{P}_{i_\omega}=0.\end{cases}$$

(ii) For $A\in \mathrm{SDD}_n$, and $i_0\in\alpha$ be such as in (3), then $B_{j_t}\in H_{|\alpha^c|}$ if

$$x\geq|a_{j_ti_0}|+\tilde{\mu}\max_{i_\omega\in\alpha-\{i_0\}}\frac{P_{i_\omega}}{|a_{i_\omega,i_\omega}|}\sum_{i_v\in\alpha-\{i_0\}}|a_{j_ti_v}|, \tag{6}$$

where

$$0\leq\tilde{\mu}\triangleq\begin{cases}\max\limits_{i_\omega\in\alpha-\{i_0\}}\dfrac{\sum_{v=1}^{l}|a_{i_\omega,j_v}|+|a_{i_\omega,i_0}|}{|a_{i_\omega i_\omega}|-\max\limits_{i_u\in\alpha-\{i_0\}}\frac{P_{i_u}}{|a_{i_u,i_u}|}\sum\limits_{\substack{i_v\in\alpha-\{i_0\}\\ v\neq\omega}}|a_{i_\omega,i_v}|}<1, & \text{if } P_{i_\omega}\neq 0,\\ 1, & \text{else if } \hat{P}_{i_\omega}=0.\end{cases}$$

Proof. Consider the following two cases:

(i): $N_1=\{j_t\}, N_2=\alpha$;

(ii): $N_1=\{j_t,i_0\}, N_2=\alpha-\{i_0\}$.

According to Lemma 1.5, we obtain inequalities (5) and (6). Further, by Lemma 1.2,

$$B_{j_t}=\mu(B_{j_t})\in M_{|\alpha^c|},$$

then $\det B_{j_t}>0$. The equality case follows from a continuity argument (with $x+\varepsilon$ in B_{j_t} and letting $\varepsilon\to 0^+$).

Liu [6] defined the following ω_{j_t}:

$$\omega_{j_t}=\min_{1\leq v\leq k}\frac{|a_{i_vi_v}|-P_{i_v}(A)}{|a_{i_vi_v}|}\sum_{u=1}^{k}|a_{j_ti_u}|. \tag{7}$$

In this paper, for the simplicity, we let

(a) If $A\in \mathrm{SD}_n$, then

$$\hat{\omega}_{j_t} = \left(1 - \mu \max_{1\le v\le k} \frac{P_{i_v}}{|a_{i_v,i_v}|}\right) \sum_{u=1}^{k} |a_{j_t i_u}|; \tag{8}$$

(b) If $A \in \mathrm{SDD}_n$, but $A \notin \mathrm{SD}_n$, then

$$\tilde{\omega}_{j_t} = \left(1 - \tilde{\mu} \max_{i_v \in \alpha - \{i_0\}} \frac{P_{i_v}}{|a_{i_v,i_v}|}\right) \sum_{i_u \in \alpha - \{i_0\}} |a_{j_t i_u}|. \tag{9}$$

For the convenience of comparison, we give some results of [6]:

Theorem 2.1 [6, Theorem 1]. Let $A \in \mathrm{SD}_n$, $\alpha = \{i_1, i_2, \cdots, i_k\} \subset N$, $\alpha^c = N - \alpha = \{j_1, j_2, \cdots, j_l\}$, $k + l = s$, $A/\alpha = (\tilde{a}_{tr})$, ω_{j_t} be defined as in (7). Then

$$|\tilde{a}_{tt}| - P_t(A/\alpha) \ge |a_{j_t j_t}| - P_{j_t}(A) + \omega_{j_t} \ge |a_{j_t j_t}| - P_{j_t}(A) > 0$$

and

$$|\tilde{a}_{tt}| + P_t(A/\alpha) \le |a_{j_t j_t}| + P_{j_t}(A) - \omega_{j_t} \le |a_{j_t j_t}| + P_{j_t}(A).$$

Corollary 2.1 [6, Corollary 1]. Let $A \in \mathrm{SD}_n$ and take $\alpha = \{1, 2, \cdots, n-1\}$. Then

$$|a_{nn}| - \max_{1\le i\le n-1} \frac{P_i(A)}{|a_{ii}|} P_n(A) \le |A/\alpha| \le |a_{nn}| - \max_{1\le i\le n-1} \frac{P_i(A)}{|a_{ii}|} P_n(A).$$

Theorem 2.2 [6, Theorem 2]. Let $A \in \mathrm{SDD}_n$, and $i_0, 1 \le i_0 \le n$, be such as in (3). Then for any index set α containing i_0, writing $\alpha = \{i_1, i_2, \cdots, i_k\}, \alpha^c = N - \alpha = \{j_1, j_2, \cdots, j_l\}, k + l = n$, and $A/\alpha = (\tilde{a}_{tr})$. Then

$$|\tilde{a}_{tt}| - P_t(A/\alpha) \ge |a_{j_t j_t}| - P_{j_t}(A) + \left(1 - \frac{P_{i_0}(A)}{|a_{i_0 i_0}|}\right) \sum_{v=1}^{k} |a_{j_t i_v}|$$

$$\ge |a_{j_t j_t}| - \frac{P_{i_0}(A)}{|a_{i_0 i_0}|} P_{j_t}(A) > 0$$

and

$$|\tilde{a}_{tt}| + P_t(A/\alpha) \le |a_{j_t j_t}| + P_{j_t}(A) - \left(1 - \frac{P_{i_0}(A)}{|a_{i_0 i_0}|}\right) \sum_{v=1}^{k} |a_{j_t i_v}|$$

$$\le |a_{j_t j_t}| + \frac{P_{i_0}(A)}{|a_{i_0 i_0}|} P_{j_t}(A).$$

In this paper, we replace ω_{j_t} in [6] by $\hat{\omega}_{j_t}$ and $\tilde{\omega}_{j_t}$, by the similar way to the proof of Theorem 1 and 2 in [6], then we can obtain the similar results as Theorem 2.1, 2.2 and Corollary 2.1.

Theorem 2.3. Let $A \in \mathrm{SD}_n$, $\alpha = \{i_1, i_2, \cdots, i_k\} \subset N$, $\alpha^c = N - \alpha = \{j_1, j_2, \cdots, j_l\}, k + l = s$. $\hat{\omega}_{j_t}$ be defined as in (8), $A/\alpha = (\tilde{a}_{tr})$. Then

$$|\tilde{a}_{tt}| - P_t(A/\alpha) \ge |a_{j_t j_t}| - P_{j_t}(A) + \hat{\omega}_{j_t} \ge |a_{j_t j_t}| - P_{j_t}(A) > 0$$

and

$$|\tilde{a}_{tt}| + P_t(A/\alpha) \le |a_{j_t j_t}| + P_{j_t}(A) - \hat{\omega}_{j_t} \le |a_{j_t j_t}| + P_{j_t}(A).$$

Corollary 2.2. Let $A \in \mathrm{SD}_n$ and take $\alpha = \{1, 2, \cdots, n-1\}$. Then

$$|a_{nn}| - \mu \max_{1\le i\le n-1} \frac{P_i(A)}{|a_{ii}|} P_n(A) \le |A/\alpha| \le |a_{nn}| - \mu \max_{1\le i\le n-1} \frac{P_i(A)}{|a_{ii}|} P_n(A).$$

Theorem 2.4. Let $A \in \mathrm{SDD}_n$, and $i_0, 1 \le i_0 \le n$, be such as in (3). Then for any index set α containing i_0, writing $\alpha=\{i_1,i_2,\ldots,i_k\},\alpha^c=N-\alpha=\{j_1,j_2,\ldots,j_l\},k+l=n$, $\tilde{\omega}_{j_t}$ is defined as in (9) and $A/\alpha=(\tilde{a}_{ts})$. Then

$$|\tilde{a}_{tt}|-P_t(A/\alpha)\geq|a_{j_tj_t}|-P_{j_t}(A)+\tilde{\omega}_{j_t}\geq|a_{j_tj_t}|-P_{j_t}(A)>0$$

and

$$|\tilde{a}_{tt}|+P_t(A/\alpha)\leq|a_{j_tj_t}|+P_{j_t}(A)-\tilde{\omega}_{j_t}\leq|a_{j_tj_t}|+P_{j_t}(A).$$

Proof. Since $A\in\mathrm{SDD}_n$, by (9), we have

$$\tilde{\omega}_{j_t}=\sum_{i_u\in\alpha-\{i_0\}}|a_{j_ti_u}|+|a_{j_ti_0}|-|a_{j_ti_0}|-\mu\max_{i_v\in\alpha-\{i_0\}}\frac{P_{i_v}}{|a_{i_v,i_v}|}\sum_{i_u\in\alpha-\{i_0\}}|a_{j_ti_u}|$$

$$=\sum_{i_u\in\alpha}|a_{j_ti_u}|-|a_{j_ti_0}|-\mu\max_{i_v\in\alpha-\{i_0\}}\frac{P_{i_v}}{|a_{i_v,i_v}|}\sum_{i_u\in\alpha-\{i_0\}}|a_{j_ti_u}|$$

Further, according to Lemma 2.1, by the similar way to the proof of Theorem 2 in [6], we can complete the proof of Theorem 2.4.

Remark 2.1. By comparison, we obtain that $\hat{\omega}_{j_t}<\omega_{j_t}$, and $\tilde{\omega}_{j_t}<\omega_{j_t}$. Thus, we improve Theorem 1, 2 and Corollary 2 in [6].

3. Bounds for determinants of SD_n and SDD_n

For the convenience of comparison, we use the same denotes as in [6]. Let {J1,J2, … ,Jn}be a rearrangement of the elements in $N=\{1,2,\cdots,n\}$. We Denote $\alpha_1=\{j_n\}$, $\alpha_2=\{j_n,j_{n-1}\}$, $\cdots,\alpha_s=\{j_n,j_{n-1},\cdots,j_1\}=N$. Then with $\alpha_{n-k+1}-\alpha_{n-k}=\{j_k\}$, $k=1,2,\cdots,n$, $\alpha_0=\varnothing$, and

$$P_{j_k}[A(\alpha_{n-k+1})]=\sum_{u\in\alpha_{n-k}}|a_{j_ku}|.$$

Let χ represent any rearrangement $\{j_1,j_2,\cdots,j_n\}$ of the elements in N with $\alpha_1,\alpha_2,\ldots,\alpha_n$.

In this section, we use $\mu\max_{u\in\alpha_{n-k}}\frac{P_u[A(\alpha_{n-k+1})]}{|a_{uu}|}$ and $\tilde{\mu}\max_{i_u\in\alpha-\{i_0\}}\frac{P_{i_u}}{|a_{i_u,i_u}|}$ to replace $\max_{u\in\alpha_{n-k}}\frac{P_u[A(\alpha_{n-k+1})]}{|a_{uu}|}$ and $\frac{P_{i_0}[A(\alpha_{n-k+1})]}{|a_{i_0i_0}|}$ in [6, Theorem 3 and Theorem 4], respectively. Then we can obtain the following results.

Theorem 3.1. Let $A\in\mathrm{SD}_n$. Then

$$\max_{\chi}\prod_{k=1}^{n}\left\{|a_{j_kj_k}|-\mu\max_{u\in\alpha_{n-k}}\frac{P_u[A(\alpha_{n-k+1})]}{|a_{uu}|}P_{j_k}[A(\alpha_{n-k+1})]\right\}\leq|\det A|$$

$$\leq\min_{\chi}\prod_{k=1}^{n}\left\{|a_{j_kj_k}|+\mu\max_{u\in\alpha_{n-k}}\frac{P_u[A(\alpha_{n-k+1})]}{|a_{uu}|}P_{j_k}[A(\alpha_{n-k+1})]\right\}.$$

Remark 3.1. Since $\mu\leq1$, then

$$\max_{u\in\alpha_{n-k}}\frac{P_u[A(\alpha_{n-k+1})]}{|a_{uu}|}\geq\mu\max_{u\in\alpha_{n-k}}\frac{P_u[A(\alpha_{n-k+1})]}{|a_{uu}|}.$$

Thus, bounds for determinants in Theorem 3.1 are better than that of [6, Theorem 3].

Especially, we can assume

$\alpha_1=\{n\},\alpha_2=\{n-1,n\},\alpha_2=\{n-2,n-1,n\},\cdots,\alpha_s=\{1,2,\cdots,n\}=N.$

Then $\alpha_{n-k+1}-\alpha_{n-k}=\{k\},k=1,2,\cdots,n,$ with $\alpha_0=\varnothing,$ and

$$P_u[A(\alpha_{n-u+1})] = \sum_{v\in\alpha_{n-u}} |a_{uv}| = \sum_{v=u+1}^{n} |a_{uv}|,$$
$$\mu = \max_{i+1\le u\le n} \frac{|a_{ui}|}{|a_{uu}| - \max_{i+1\le\omega\le n} \frac{P_\omega}{|a_{\omega,\omega}|} \sum_{v=u+1}^{n} |a_{u,v}|}. \tag{10}$$

Then, we obtain the following Theorem.

Theorem 3.2. Let $A\in \mathrm{SD}_n$ and P_u, μ is defined in (10). Then

$$\prod_{k=1}^{n}\left\{|a_{kk}| - \mu \max_{u\in\alpha_{n-k}} \frac{\sum_{v=u+1}^{n}|a_{uv}|}{|a_{uu}|}\sum_{v=k+1}^{n}|a_{kv}|\right\} \le |\det A| \le \prod_{k=1}^{n}\left\{|a_{kk}| + \mu \max_{u\in\alpha_{n-k}} \frac{\sum_{v=u+1}^{n}|a_{uv}|}{|a_{uu}|}\sum_{v=k+1}^{n}|a_{kv}|\right\}$$

Obviously, we improve the following Theorem 3.3 [7, Theorem 1].

Theorem 3.3 [7, Theorem 1]. Let $A\in \mathrm{SD}_n$. Then

$$\prod_{k=1}^{n}\left\{|a_{kk}| - m_k\sum_{v=k+1}^{n}|a_{kv}|\right\} \le |\det A| \le \prod_{k=1}^{n}\left\{|a_{kk}| + m_k\sum_{v=k+1}^{n}|a_{kv}|\right\},$$

where

$$m_k = \max_{i+1\le k\le n} \frac{|a_{ki}|}{|a_{kk}| - \sum_{v=k+1}^{n}|a_{k,v}|}.$$

For an analogous result of SDD_n, let χ denote all rearrangements of the elements in N with $\alpha_1 = \{i_0 \equiv j_n\}$, withe i_0 be such as in (3).

Theorem 3.4. Let $A\in \mathrm{SDD}_n$, and $A\notin \mathrm{SD}_n$ and with i_0 be such as in (3). Then

$$\max_{\chi} |a_{i_0 i_0}| \prod_{k=1}^{n-1}\left\{|a_{j_k j_k}| - \tilde{\mu} \max_{i_v\in\alpha-\{i_0\}} \frac{P_{i_v}[A(\alpha_{n-k+1})]}{|a_{i_v i_v}|} P_{j_k}[A(\alpha_{n-k+1})]\right\} \le |\det A|$$
$$\le \min_{\chi} |a_{i_0 i_0}| \prod_{k=1}^{n-1}\left\{|a_{j_k j_k}| + \tilde{\mu} \max_{i_v\in\alpha-\{i_0\}} \frac{P_{i_v}[A(\alpha_{n-k+1})]}{|a_{i_v i_v}|} P_{j_k}[A(\alpha_{n-k+1})]\right\}.$$

Remark 3.2. Since $\tilde{\mu}\le 1$, then

$$\frac{P_{i_0}[A(\alpha_{n-k+1})]}{|a_{i_0 i_0}|} > \tilde{\mu} \max_{i_v\in\alpha-\{i_0\}} \frac{P_{i_v}}{|a_{i_v,i_v}|}.$$

Thus, we can obtain the better bounds for determinants than the bounds in [6, Theorem 4].

Theorem 3.5. Let $A\in H_n$. Then

$$\min_{\chi}\prod_{k=1}^{n}\{|a_{j_k j_k}| - R_{j_k}\} \le |\det A| \le \max_{\chi}\prod_{k=1}^{n}\{|a_{j_k j_k}| + R_{j_k}\},$$

where

$$R_{j_k} = \left\{\frac{1}{x_{j_k}}\mu \max_{u\in\alpha_{n-k}} \frac{P_u[(AX)(\alpha_{n-k+1})]}{|a_{uu}|} P_{j_k}[(AX)(\alpha_{n-k+1})]\right\},$$

and $X = \mathrm{diag}(x_1, x_2, \cdots, x_n)$ be a positive diagonal matrix and satisfies $AX\in \mathrm{SD}_n$.

Proof. Since $A \in H_n$, by Lemma 1.1, then, there exists an positive diagonal matrix X satisfy $AX \in \mathrm{SD}_n$. Further, according to Theorem 3.1, we obtain the results.

Corollary 3.1. Let $A \in C^{n\times n}$ and satisfies (4). Then

$$\prod_{k=1}^{n}\left\{|a_{kk}|-\frac{1}{x_k}\sum_{v=k+1}^{n}|x_v a_{kv}|\right\} \le |\det A| \le \prod_{k=1}^{n}\left\{|a_{kk}|+\frac{1}{x_k}\sum_{v=k+1}^{n}|x_v a_{kv}|\right\},$$

where

$$x_t = \begin{cases} \dfrac{P_t(A)}{|a_{tt}|}, & t \in N_2, \\ \varepsilon, & t \in N_1, \end{cases}$$

$$\max_{i\in N_1}\frac{\sum\limits_{t\in N_2}|a_{it}|\frac{P_t}{|a_{tt}|}}{|a_{ii}|-\sum\limits_{\substack{t\in N_1\\ t\ne i}}|a_{it}|} < \varepsilon < \min_{j\in N_2}\frac{P_j-\sum\limits_{t\in N_2, t\ne j}|a_{jt}|\frac{P_t}{|a_{tt}|}}{\sum\limits_{t\in N_1}|a_{jt}|}, \quad \forall i\in N_1, j\in N_2. \tag{11}$$

Proof. According to Lemma 1.5, we select the positive diagonal matrix X and its elements such as in (11), then $AX \in \mathrm{SD}_n$. Further, by Theorem 3.5, we complete the proof of Corollary 3.1.

4. Bounds for the Schur complement of SD_n and SDD_n

In this section, according to Geshgorin's theorem, we give the localization for eigenvalues of the Schur complement of SD_n and SDD_n. Further, we improve the lower bound for eigenvalues of Schur complement of SD_n in [6,Theorem 5].

Theorem 4.1. Let $A \in \mathrm{SD}_n$, $\hat{\omega}_{j_t}$ be defined as in (7) and α, α^c be defined as in Lemma 2.1, $\lambda(A/\alpha)$ denote the set of eigenvalues of A/α, and $A/\alpha = (\tilde{a}_{tr})$. Then, for any eigenvalue λ of the Schur complement of SD_n, we have.

$$\min_{j_t\in\alpha^c}[|a_{j_t j_t}|-P_{j_t}(A)+\hat{\omega}_{j_t}] \le |\lambda| \le \max_{j_u\in\alpha^c}[|a_{j_u j_u}|+P_{j_u}(A)-\hat{\omega}_{j_u}].$$

Proof. By Geshgorin's theorem, we obtain that

$$|\lambda - \tilde{a}_{tt}| \le P_{j_t}(A/\alpha).$$

Thus

$$|\tilde{a}_{tt}| - P_{j_t}(A/\alpha) \le |\lambda| \le |\tilde{a}_{tt}| + P_{j_t}(A/\alpha).$$

Further, according to Theorem 2.1, we have

$$|a_{j_t j_t}| - P_{j_t}(A) + \hat{\omega}_{j_t} \le |\lambda| \le |a_{j_t j_t}| + P_{j_t}(A) - \hat{\omega}_{j_t}.$$

Thus, we complete the proof of Theorem 4.1.

Remark 4.1. By comparison, we know that the above bounds for eigenvalues are more accurate than the bounds in [6, Theorem 5].

Theorem 4.2. Let $A \in \mathrm{SDD}_n$ and $A \notin \mathrm{SD}_n$, with i_0 be such as in (3), $\tilde{\omega}_{j_t}$ be defined as in (3), α, α^c be defined as in Lemma 2.1, $\lambda(A/\alpha)$ denote the set of eigenvalues of A/α, and denote $A/\alpha = (\tilde{a}_{tr})$. Then, for any eigenvalue λ of the Schur complement of SD_n, we have.

$$\min_{j_t\in\alpha^c}\left[|a_{j_t j_t}|-P_{j_t}(A)+\tilde{\omega}_{j_t}\right] \le |\lambda| \le \max_{j_u\in\alpha^c}\left[|a_{j_u j_u}|+P_{j_u}(A)-\tilde{\omega}_{j_u}\right].$$

Proof. According to Theorem 3.2, by the similar way to the proof of Theorem 4.1, we obtain Theorem 4.2.

5. Examples

In this section, we present some examples to illustrate these bounds in this paper are more efficiency than bounds in [6,7].

Example 1. Let

$$A=\begin{pmatrix}3 & -1 & 1\\ 2 & 4 & 1\\ 0 & 1 & 2\end{pmatrix}, \det A=27.$$

By Theorem 3.1: $14.6968<\det A<35.0049$. By [6, Theorem 3]: $9.75<\det A<42.75$.

By Theorem 3.3 ([7, Theorem 1]): $11.1056<\det A<40.4536$.

Example 2. Let

$$A=\begin{pmatrix}3 & 1 & 1\\ 1 & 2 & 0\\ 2 & 3 & 4\end{pmatrix},\ \det A=19.$$

By Theorem 3.2: $9<\det A<39$. By Theorem 3 of [7]: $6<\det A<53$.

Example 3. Let

$$A=\begin{pmatrix}5 & -1 & 1 & 2\\ 2 & 6 & 1 & 2\\ 0 & 1 & 4 & 1\\ 2 & -2 & 1 & 8\end{pmatrix}.$$

Obviously, A be a strictly diagonally dominant matrix. Without loss of generality, we assume $\alpha=\{1,2\}, \alpha^c=\{3,4\}$. Then

$$A/\alpha=\begin{pmatrix}3.9063 & 0.8125\\ 0.7500 & 7.5000\end{pmatrix},\ \lambda(A/\alpha)=\{3.7440, 7.6622\}.$$

According to Theorem 4.1, we have

$$\min_{j_t\in\alpha^c}[|a_{j_tj_t}|-P_{j_t}(A)+\hat{\omega}_{j_t}]=2.4<|\lambda|<\max_{j_u\in\alpha^c}[|a_{j_uj_u}|+P_{j_u}(A)-\hat{\omega}_{j_u}]=11.6.$$

According to Theorem 5 in [7], we have

$$|\lambda|>\min_{j_t\in\alpha^c}[|a_{j_tj_t}|-P_{j_t}(A)+\omega_{j_t}](=2.17).$$

By numerical comparison, we know that the lower bound for eigenvalues is more accurate than the lower bound in [7].

Example 4. Let

$$A=\begin{pmatrix}5 & -2 & 1.5 & 2\\ 2 & 6 & 1 & 2\\ 0 & 1 & 4 & 1\\ 2 & -2 & 1 & 8\end{pmatrix}.$$

Obviously, $A\in \mathrm{SDD}_n$ but $A\notin \mathrm{SD}_n$, and $i_0=1$ be such as in (3). Without loss of generality, we assume $\alpha=\{1,2\}, \alpha^c=\{3,4\}$. Then

$$A/\alpha=\begin{pmatrix}3.9412 & 0.8235\\ 0.4706 & 7.4118\end{pmatrix},\ \lambda(A/\alpha)=\{3.8329, 7.5201\}.$$

$$\min_{j_t\in\alpha^c}\left[|a_{j_tj_t}|-P_{j_t}(A)+\tilde{\omega}_{j_t}\right]=3.22<|\lambda|<\max_{j_u\in\alpha^c}\left[|a_{j_uj_u}|+P_{j_u}(A)-\tilde{\omega}_{j_u}\right]=12.7.$$

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7. References

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