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Homotopy Analysis Method for Nonlinear Jaulent-Miodek Equation

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Abstract. The homotopy analysis method (HAM) has been developed by Liao [1-2] to obtain series solutions of controllable convergence to various nonlinear problems. In this work, we propose this method (HAM), for solving Jaulent-Miodek (JM) equation [9-10]. Numerical solutions obtained by the homotopy analysis method are compared with the exact solutions. The results for some values for the variables are shown in the tables and the solutions are presented as plots as well, showing the ability of the method

Keywords: Homotopy analysis method, Jaulent-Miodek equation

1. Introduction

Large varieties of physical, chemical, and biological phenomena are governed by nonlinear evolution equations. Except a limited number of these problems, most of them do not have precise analytical solutions so that they have to be solved using other methods. The homotopy analysis method (HAM) is a powerful analytical tool for nonlinear problems. This technique provides us with a simple way to ensure the convergence of the solution series, so that we can always get accurate enough approximations. In recent years the application of analysis theory has appeared in many researches [3-8].

In this paper, we propose homotopy analysis method (HAM), for solving Jaulent-Miodek (JM) equation [9-10]

$$\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+\frac{3}{2}v\frac{\partial^3 v}{\partial x^3}+\frac{9}{2}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}-6u\frac{\partial u}{\partial x}-6uv\frac{\partial v}{\partial x}-\frac{3}{2}\frac{\partial u}{\partial x}v^2=0,$$
$$\frac{\partial v}{\partial t}+\frac{\partial^3 v}{\partial x^3}-6\frac{\partial u}{\partial x}v-6u\frac{\partial v}{\partial x}-\frac{15}{2}\frac{\partial v}{\partial x}v^2=0.$$

With the initial condition of $u(x,0)=g_1(x), v(x,0)=g_2(x)$.

2. Basic ideas homotopy analysis method

To describe the basic ideas the homotopy analysis method, we consider the following differential equation,

$$N(u(r,t)]=0, \tag{1}$$

where N is a nonlinear operator, r and t are independent variables, $u(r,t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [2] constructs the so-called $zero$ - $order$ deformation equation

$$(1-p)L\left(\varphi(r,t;p)-u_0(r,t)\right)=p\hbar H(r,t)N(r,t;p), \tag{2}$$

where $p\in[0,1]$ is the embedding parameter, $\hbar$ is a nonzero auxiliary parameter, L is an auxiliary linear operator, $u_0(r,t)$ is an initial guess of $u(r,t)$, $\varphi(r,t;p)$ is a unknown function on independent variables r,t,p.

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It is important that one has great freedom to choose auxiliary parameter $\hbar$ in HAM. If $p=0$ and $p=1$, it holds

$$\varphi(r,t;0)=u_0(r,t), \qquad \varphi(r,t;1)=u(r,t), \tag{3}$$

Thus, as p increases from 0 to 1, the solution $\varphi(r,t;p)$ varies from the initial guesses $u_0(r,t)$ to the solution $u(r,t)$. Expanding $\varphi(r,t;p)$, in Taylor series with respect to p, we have

$$\varphi(r,t;p)=u_0(r,t)+\sum_{m=1}^{\infty}u_m(r,t)p^m, \tag{4}$$

where

$$u_m(r,t)=\frac{1}{m!}\frac{\partial^m\varphi(r,t;p)}{\partial p^m}\Big|_{p=0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (4) converges at *p* = 1, then we have

$$u(r,t)=u_0(r,t)+\sum_{m=1}^{\infty}u_m(r,t). \tag{6}$$

Define the vector $\overrightarrow{u_n}=\{u_0,u_1,\ldots u_n\}$. Differentiating equation (2) *m* times with respect to the embedding parameter p and then setting $p=0$ and finally dividing them by $m!$, we obtain the $mth-order$ deformation equation

$$L[u_m-\chi_m u_{m-1}]=\hbar H(r,t)R_m(\vec{u}_{m-1}), \tag{7}$$

where

$$R_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N(r,t;p)}{\partial p^{m-1}}\Big|_{p=0}, \tag{8}$$

and

$$\chi_m=\begin{cases}0 & m\le 1,\\ 1 & m>1.\end{cases} \tag{9}$$

Applying L^{-1} on both side of equation (7), we get

$$u_m(r,t)=\chi_m u_{m-1}(r,t)+\hbar L^{-1}[H(r,t)R_m(\vec{u}_{m-1})]. \tag{10}$$

In this way, it is easily to obtain u_m for*m* $m\ge 1,$, at $Mth-order,$ we have

$$u(r,t)=\sum_{m=0}^{M}u_m(r,t). \tag{11}$$

When $M\to\infty.$ we get an accurate approximation of the original equation (1). For the convergence of the above method we refer the reader to Liao [2]. If equation (1) admits unique solution, then this method will produce the unique solution. If equation (1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

3. Applications

Consider the following Jaulent-Miodek equation [9-10]

$$\begin{aligned}&\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+\frac{3}{2}v\frac{\partial^3 v}{\partial x^3}+\frac{9}{2}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}-6u\frac{\partial u}{\partial x}-6uv\frac{\partial v}{\partial x}-\frac{3}{2}\frac{\partial u}{\partial x}v^2=0,\\&\frac{\partial v}{\partial t}+\frac{\partial^3 v}{\partial x^3}-6\frac{\partial u}{\partial x}v-6u\frac{\partial v}{\partial x}-\frac{15}{2}\frac{\partial v}{\partial x}v^2=0.\end{aligned} \tag{12}$$

With the following initial condition

$$\begin{aligned} &u(x,0)=\frac{1}{4}c_2-\frac{1}{4}b_0^2-\frac{1}{2}b_0\sqrt{c_2}\,\sec h(\sqrt{c_2}x)-\frac{3}{4}c_2\sec h^2(\sqrt{c_2}x), \\ &v(x,0)=b_0+\sqrt{c_2}\,\sec h(\sqrt{c_2}x). \end{aligned} \tag{13}$$

With the exact solution

$$\begin{aligned} &u=\frac{1}{4}(c_2-b_0^2)-\frac{1}{2}b_0\sqrt{c_2}\sec h\left(\sqrt{c_2}(x+\frac{1}{2}(6b_0^2+c_2)t)\right)-\frac{3}{4}c_2\sec h^2\left(\sqrt{c_2}(x+\frac{1}{2}(6b_0^2+c_2)t)\right), \\ &v=b_0+\sqrt{c_2}\sec h\left(\sqrt{c_2}(x+\frac{1}{2}(6b_0^2+c_2)t)\right), \end{aligned}$$

where b_0, c_2 are arbitrary constants.

To solve the equation (12) by means of homotopy analysis method, according to the initial conditions denoted in equation (13), it is natural to choose:

$$\begin{aligned} &u_0=\frac{1}{4}(c_2-b_0^2)-\frac{1}{2}b_0\sqrt{c_2}\sec h\left(\sqrt{c_2}(x+\frac{1}{2}(6b_0^2+c_2)t)\right)-\frac{3}{4}c_2\sec h^2\left(\sqrt{c_2}(x+\frac{1}{2}(6b_0^2+c_2)t)\right), \\ &v_0=b_0+\sqrt{c_2}\sec h\left(\sqrt{c_2}(x+\frac{1}{2}(6b_0^2+c_2)t)\right). \end{aligned} \tag{14}$$

We choose the linear operator

$$\begin{cases} L(\varphi_1(x,t;p)]=\dfrac{\partial\varphi_1(x,t;p)}{\partial t}, \\ L(\varphi_2(x,t;p)]=\dfrac{\partial\varphi_2(x,t;p)}{\partial t}. \end{cases} \tag{15}$$

with the property $L[c]=0.$ Where *c* is constant. We now define a nonlinear operator as

$$\begin{cases} N[\varphi_1(x,t;p)]=\dfrac{\partial\varphi_1(x,t;p)}{\partial t}+\dfrac{\partial^3\varphi_1(x,t;p)}{\partial x^3}+\dfrac{3}{2}\varphi_2(x,t;p)\dfrac{\partial^3\varphi_2(x,t;p)}{\partial x^3}+\dfrac{9}{2}\dfrac{\partial\varphi_2(x,t;p)}{\partial x}\dfrac{\partial^2\varphi_2(x,t;p)}{\partial x^2} \\ \qquad -6\varphi_1(x,t;p)\dfrac{\partial\varphi_1(x,t;p)}{\partial x}-6\varphi_1(x,t;p)\varphi_2(x,t;p)\dfrac{\partial\varphi_2(x,t;p)}{\partial x}-\dfrac{3}{2}\dfrac{\partial\varphi_1(x,t;p)}{\partial x}\varphi_2(x,t;p)^2, \\ N[\varphi_2(x,t;p)]=\dfrac{\partial\varphi_2(x,t;p)}{\partial t}+\dfrac{\partial^3\varphi_2(x,t;p)}{\partial x^3}-6\dfrac{\partial\varphi_1(x,t;p)}{\partial x}\varphi_2(x,t;p)-6\varphi_1(x,t;p)\dfrac{\partial\varphi_2(x,t;p)}{\partial x} \\ \qquad -\dfrac{15}{2}\dfrac{\partial\varphi_2(x,t;p)}{\partial x}\varphi_2(x,t;p)^2. \end{cases} \tag{16}$$

Using above definition, with assumption $H(x,t)=1.$ We construct the $zeroth-order$ deformation equations

$$\begin{cases} (1-p)L\left(\varphi_1(x,t;p)-u_0(x,t)\right)=p\hbar N(\varphi_1(x,t;p)), \\ (1-p)L\left(\varphi_2(x,t;p)-v_0(x,t)\right)=p\hbar N(\varphi_2(x,t;p)). \end{cases} \tag{17}$$

Obviously, when $p=0$ and $p=1,$

$$\begin{aligned} &\varphi_1(x,t;0)=u_0(x,t), \quad \varphi_2(x,t;1)=v_0(x,t), \\ &\varphi_1(x,t;0)=u(x,t), \quad \varphi_2(x,t;1)=v(x,t). \end{aligned} \tag{18}$$

Thus, we obtain the $mth-order$ deformation equations

$$\begin{aligned} &L[u_m-\chi_m u_{m-1}]=\hbar R_m(\vec{u}_{m-1}), \\ &L[v_m-\chi_m v_{m-1}]=\hbar R_m(\vec{v}_{m-1}), \end{aligned} \tag{19}$$

where

$$\begin{cases} R_m(\vec{u}_{m-1}) = \dfrac{\partial U_{m-1}}{\partial t} + \dfrac{\partial^3 U_{m-1}}{\partial x^3} + \dfrac{3}{2}\sum_0^{m-1} V_k \dfrac{\partial^3 V_{m-1-k}}{\partial x^3} + \dfrac{9}{2}\sum_{k=0}^{m-1}\dfrac{\partial V_k}{\partial x}\dfrac{\partial^2 V_{m-1-k}}{\partial x^2} - 6\sum_{k=0}^{m-1} U_k \dfrac{\partial U_{m-1-k}}{\partial x} \\ \qquad - 6\sum_{i=0}^{m-1}\sum_{k=0}^{m-i-1} U_i V_k \dfrac{\partial V_{m-k-i-1}}{\partial x} - \dfrac{3}{2}\sum_{i=0}^{m-1}\sum_{k=0}^{m-i-1}\dfrac{\partial U_i}{\partial x} V_k V_{m-k-i-1}, \\ R_m(\vec{v}_{m-1}) = \dfrac{\partial V_{m-1}}{\partial t} + \dfrac{\partial^3 V_{m-1}}{\partial x^3} - 6\sum_{k=0}^{m-1}\dfrac{\partial U_k}{\partial x} V_{m-1-k} - 6\sum_{k=0}^{m-1} U_k \dfrac{\partial V_{m-1-k}}{\partial x} - \dfrac{15}{2}\sum_{i=0}^{m-1}\sum_{k=0}^{m-i-1}\dfrac{\partial V_i}{\partial x} V_k V_{m-k-i-1}. \end{cases} \quad (20)$$

Now, the solution of the $mth-order$ order deformation equation (19)

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})],$$
$$v_m(x,t) = \chi_m v_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{v}_{m-1})].$$

Finally, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t),$$
$$v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t).$$

From equations (14) and (20) and subject to initial condition

$$u_m(x,0) = v_m(x,0) = 0, \quad m \geq 1.$$

We obtain

$$u_0 = \frac{1}{4}c_2 - \frac{1}{4}b_0^2 - \frac{1}{2}b_0\sqrt{c_2}\sec h(\sqrt{c_2}x) - \frac{3}{4}c_2\sec h^2(\sqrt{c_2}x),$$
$$v_0 = b_0 + \sqrt{c_2}\sec h(\sqrt{c_2}x).$$

$$u_1 = -\hbar\frac{3}{4}\sqrt{c_2^5}\sec h^2(\sqrt{c_2}x)\tanh(\sqrt{c_2}x)t - \hbar 6b_0c_2^2\sec h(\sqrt{c_2}x)\tanh^3(\sqrt{c_2}x)t + \hbar\frac{23}{4}b_0c_2^2\sec h(\sqrt{c_2}x)\tanh(\sqrt{c_2})x)t$$
$$-\hbar\frac{3}{2}tb_0^3c_2\sec h(\sqrt{c_2}x)\tanh(\sqrt{c_2}x) - \hbar\frac{9}{2}tb_0^2\sqrt{c_2^3}\sec h^2(\sqrt{c_2}x)\tanh(\sqrt{c_2}x) - \hbar 6tb_0c_2^2\sec h^3(\sqrt{c_2}x)\tanh(\sqrt{c_2}x),$$

$$v_1 = -\hbar 6c_2^2\sec h(\sqrt{c_2}x)\tanh^3(\sqrt{c_2}x)t + \hbar\frac{13}{2}c_2^2\sec h(\sqrt{c_2}x)\tanh(\sqrt{c_2}x)t + \hbar 3tb_0^2c\sec h(\sqrt{c_2}x)\tanh(\sqrt{c_2}x)$$
$$-\hbar 6tc_2^2\sec h^3(\sqrt{c_2}x)\tanh(\sqrt{c_2}x),$$
$$\vdots$$

Hence,

$$u = u_0 + u_1 + u_2 + \ldots,$$

$$v = v_0 + v_1 + v_2 + \ldots.$$

When $\hbar = -1$, suppose $u^* = \sum_{j=0}^{3} u_j$, and $v^* = \sum_{j=0}^{3} v_j$, the results are presented in Table 1 and Fig. 1.

Table 1 The numerical results, when $b_0 = c_2 = 0.01$ and $\hbar = -1$ for solutions of Eqs. (12) for initial conditions (13).

x	t	$u^*(x,t)$	$\lvert u^* - u\rvert$	$v^*(x,t)$	$\lvert v^* - v\rvert$
0.1	0.1	-0.00552421681535966474	4×10^{-20}	0.109994947072326492	1.6×10^{-19}
0.1	0.15	-0.00552421268119929705	2.1×10^{-19}	0.109994920399016371	8.3×10^{-19}
0.15	0.1	-0.00552324416776666749	4×10^{-20}	0.109988671429124380	1.6×10^{-19}
0.2	0.3	-0.00552185135682729130	3.26×10^{-18}	0.109979684175985571	1.329×10^{-17}
0.35	0.25	-0.00551544200337122719	1.55×10^{-18}	0.109938317095391529	6.37×10^{-18}
0.45	0.45	-0.00550916080832543826	1.622×10^{-17}	0.109897761088575401	6.657×10^{-17}
0.5	0.3	-0.00550553384209772092	3.19×10^{-18}	0.109874335473663468	1.311×10^{-17}
0.45	0.7	-0.00550906801481426520	9.504×10^{-17}	0.109897161826108341	3.8981×10^{-16}
0.4	0.8	-0.00551234934376313588	1.6273×10^{-16}	0.109918350605131830	6.6674×10^{-16}
0.8	0.8	-0.00547508560544371429	1.5616×10^{-16}	0.109677468293014595	6.4730×10^{-16}
0.85	0.95	-0.00546861261467419606	3.0831×10^{-16}	0.109635567784157087	1.28058×10^{-15}
0.9	0.9	-0.00546189829138478970	2.4645×10^{-16}	0.109592087075285587	1.02594×10^{-15}
0.95	0.9	-0.00545477408645114569	2.4448×10^{-16}	0.109545931955545074	1.02005×10^{-15}
1	1	-0.00544719469399408060	3.6944×10^{-16}	0.109496805097819525	1.54524×10^{-15}

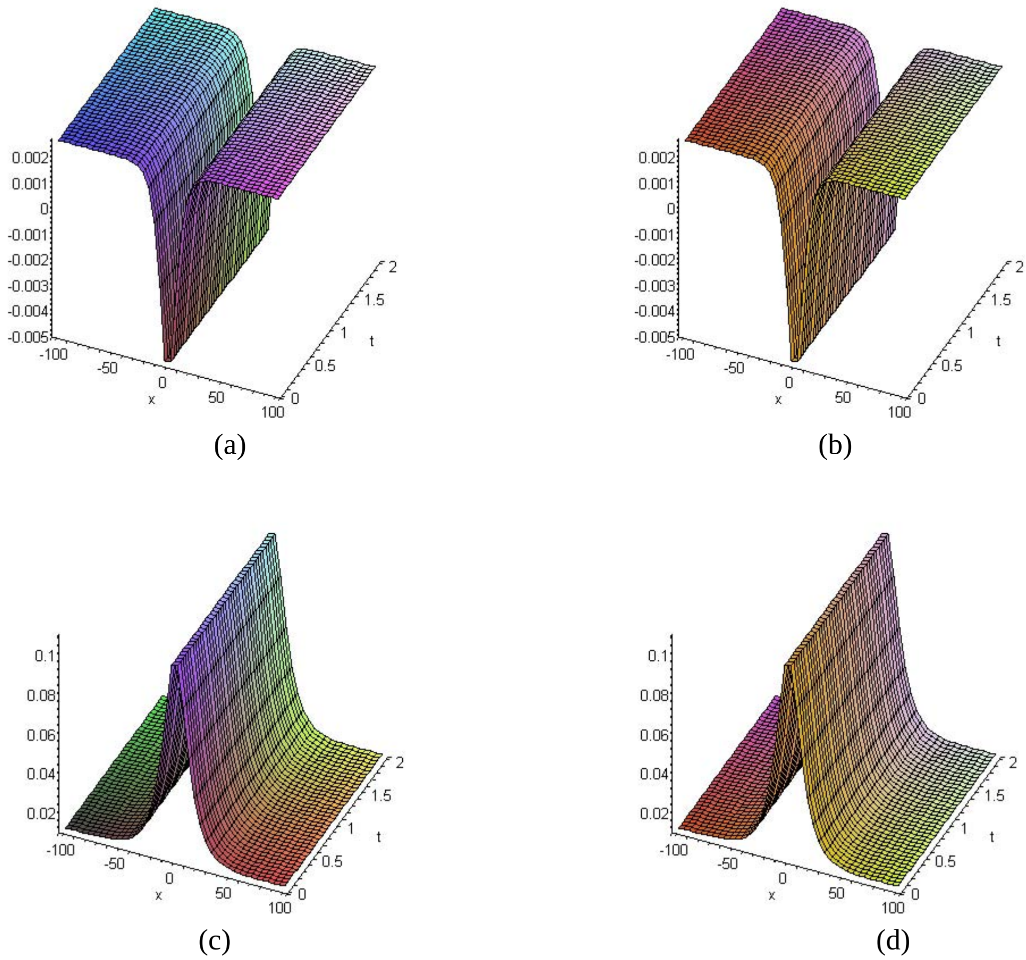

Fig. 1. The numerical results for $u^*(x,t)$, $v^*(x,t)$ are, respectively (a) and (c) in comparison with the analytical solutions $u(x,t)$ and $v(x,t)$ are, respectively (b) and (d) with the initial conditions (13) of Eq (12). when , $b_0 = 0.01$, $c_2 = 0.01$, and $\hbar = -1$.

4. Conclusions

In this article, we have applied homotopy analysis method for the solving the nonlinear Jaulent-Miodek (JM) equation. The approximate solutions obtained by the homotopy analysis method are compared with exact solutions. The results show that the homotopy analysis method is a powerful mathematical tool for solving systems of nonlinear partial differential equations having wide applications in sciences and engineering. In our work, we use the maple package to carry the computations.

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