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Anti-Synchronization of two Different Chaotic Systems via Optimal Control with Fully Unknown Parameters*

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Abstract. This paper presents anti-synchronization of two different chaotic systems using optimal control method. The proposed technique is applied to achieve chaos anti-synchronization for the new four-dimensional and hyperchaotic Lü systems with fully unknown parameters. Numerical simulations results are presented to demonstrate the effectiveness of the method.

Key Words: Anti-synchronization;Optimal; Adaptive.

1. Introduction

Since the idea of synchronizing chaotic systems was introduced by Pecora and Carroll [1] in 1990, chaos synchronization has received increasing attention due to its theoretical challenge and its great potential applications in secure communication, chemical reaction and biological systems [2].

Recently several different types of synchronizations have been proposed in the literature, for example, generalized synchronization [3,4,5,6], phase synchronization[7,8], lag synchronization[9,10,11], anti-synchronization[12,13,14] and so on . The anti-synchronization (AS) method is that the state vectors of synchronized systems have the same absolute values but opposite signs, i.e. the sum of the output signals of two systems can converge to zero.

Recently many methods have been used to achieve anti-synchronization between chaotic systems see [15, 16, 17]. There are a few papers that studied the problem of optimal controlling chaos and optimal synchronization of the chaotic systems with unknown system parameters see [18, 19, 20]. Most of the methods mentioned above synchronize two different chaotic systems using adaptive methods. However, these methods of the synchronization are far from optimal synchronization. In general, the adaptive synchronization is not necessary satisfy the optimality conditions. Our attention in the present paper is to study optimal synchronization of two different systems with complete uncertain system parameters.

This paper is organized as follows: Section 2 explain the principle of optimal control. In Section 3 describes the new four-dimensional system and hyperchaotic Lü system. In Section 4,the anti-synchronization of the new four-dimensional system and hyperchaotic Lü system with completely uncertain system parameters is achieved with the optimal and adaptive control technique. In Section 5 numerical results demonstrate the effectiveness of the proposed control technique are presented. Finally, Section 6 contains a summarized conclusion of the results.

2. Principle of Optimal Control

Nonlinear chaotic system

$$\begin{aligned} \dot{x} &= f(t,x) \\ \dot{y} &= g(t,y)+u(t,x,y) \end{aligned} \tag{1}$$

where $x, y \in R^n$, f and g are $R \times R^n \to R^n$ and differentiable. The first equation of the system (1) is master system and the second is slave system. where $u(t,x,y)$ is control function . Let $e(t) = x(t) + y(t)$, our goal is to design the controller and satisfy:

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$$E = min\{\int_{t_0}^{+\infty}[q(x)+u^T Ru]dt\} \tag{2}$$

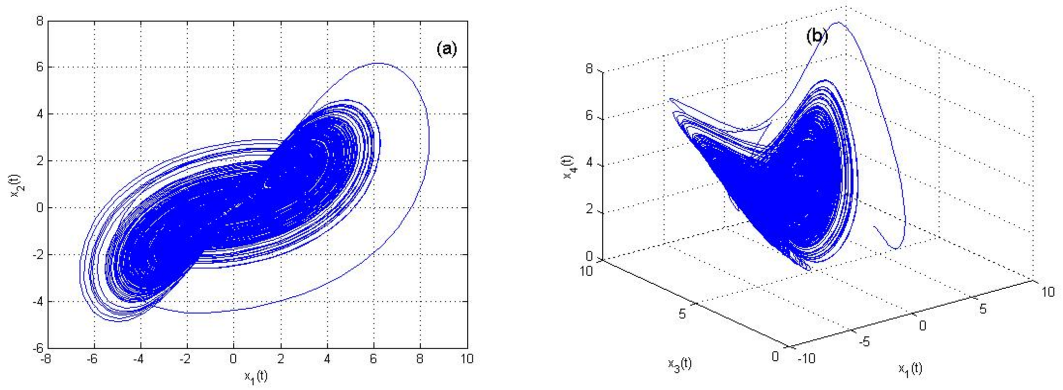

Figure 1. chaotic attractors for the new four-dimensional system when $a=35, b=10, c=1, d=10$. (a) chaotic attractor in (x_1, x_2) space; (b) chaotic attractor in (x_1, x_3, x_4) space.

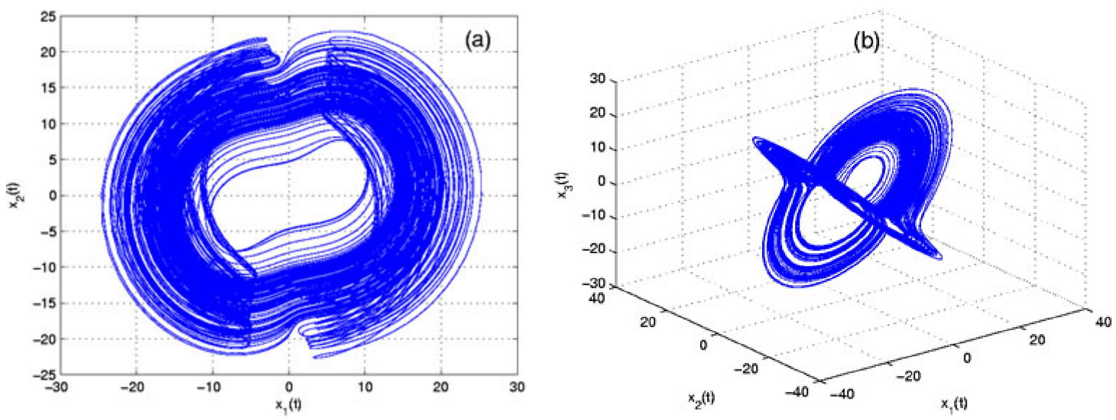

Figure 2. chaotic attractors for the new four-dimensional system when $a=30, b=10, c=37, d=10$. (a) chaotic attractor in (x_1, x_2) space; (b) chaotic attractor in (x_1, x_2, x_3) space.

and then

$$\lim_{t\to\infty}\|e(t)\| = 0,$$

where $q(x)$ is a function which is continuous ,differentiable and positive .According to Dynamic Programming the optimal control is based on Hamilton-Jacobi-Bellman equation:

$$\min_{u^*}[\dot{E}+w] = \frac{\partial}{\partial u^*}[\dot{E}+w] = 0 \tag{3}$$

where $w = q(x)+u^T Ru$ and u^* is optimal controller.

3. Systems Description

Recently a new Four-Dimensional Chaotic System is studied [21].The system can be described by:

$$\begin{aligned}
\dot{x}_1 &= -a(x_1 - x_2) + x_2x_3x_4 \\
\dot{x}_2 &= b(x_1 + x_2) - x_1x_3x_4 \\
\dot{x}_3 &= -cx_3 + x_1x_2x_4 \\
\dot{x}_4 &= -dx_4 + x_1x_2x_3
\end{aligned} \tag{4}$$

where $x_i (i=1,2,3,4)$ are the state variables of the system and a,b,c,d are all positive real constant parameters. When $a=35, b=10, c=1$ and $d \in (0, 21.88]$ the system (4) is chaotic with a positive Lyapunov exponent (for example, with $d=10$, the chaotic attractor is shown in Fig.1). When $a=30, b=10, c=37$ and $d=10$ the system (4) is also chaotic .The chaotic attractor as shown in Fig.2.

Lü system as a typical transaction system, found by Lü and Chen, which connects the Lorenz and Chen attractors and represents the transition from one to the other .Recently, Chen Ai-min others proposed hyperchaotic Lü system[22].The hyperchaotic Lü system is described by:

$$\begin{aligned} \dot{y}_1 &= -a(y_1 - y_2) + y_4 \\ \dot{y}_2 &= cy_2 - y_1 y_3 \\ \dot{y}_3 &= y_1 y_2 - by_3 \\ \dot{y}_4 &= y_1 y_3 + dy_4 \end{aligned} \tag{5}$$

The system is based on Lü system, the first equation with a controller , the second and three equation non-controller. Then fourth equation contains an additional controller. Which a, b, c is admission to the system parameters. Gain control parameter d is to be determined. By analysis of the dynamics of the system, including the bifurcation diagram, Lyapunov exponent spectrum Poincare mapping, the system and circuit simulation experiments confirm hyperchaotic .

When $a=36, b=3, c=20$ and d takes other different values, system performance of the different dynamics: when $-1.03 \le d \le -0.46$, system is a periodic orbit; when $-0.46 < d \le -0.35$,system is chaotic attractor; and when $-0.35 < d \le 1.30$,there are two index greater than zero system is super chaotic attractor.

In this paper, we consider the system is hyperchaotic. Selecting parameters $a=36, b=3, c=20$ and $-0.35 < d \le 1.30$ .When $a=36, b=3, c=20$ and $d=1$, hyperchaotic attractor as shown in Fig.3.

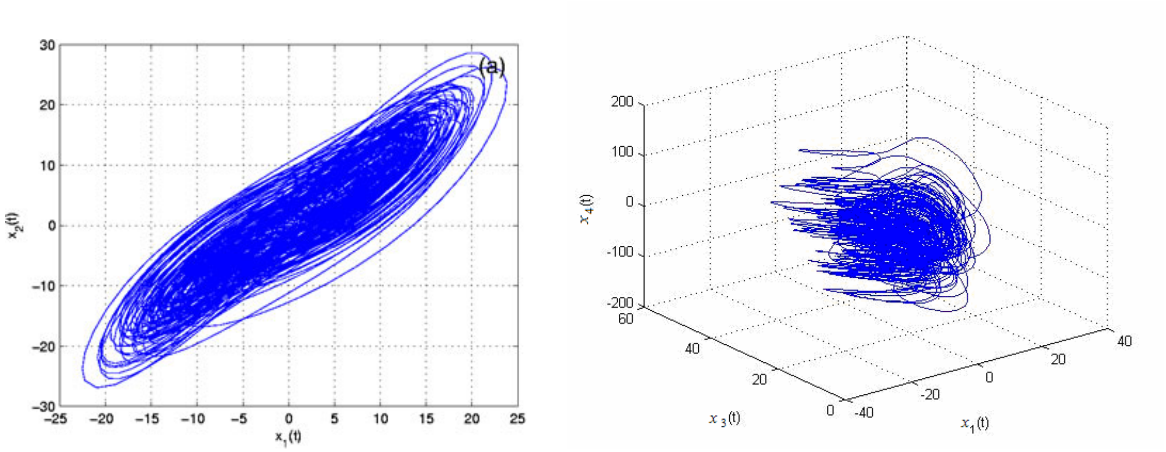

Figure 3. Hyperchaotic attractors for the Lü system when $a=36, b=3, c=20$ and $d=1$. (a) hyperchaotic attractor in (x_1, x_2) space; (b) hyperchaotic attractor in (x_1, x_3, x_4) space.

4. Anti-Synchronization Via Optimal Control

The purpose of this section is to introduce a development optimal control law for resolving the optimal anti-synchronization between the new Four-Dimensional system and hyperchaotic Lü system with completely unknown system parameters. All derivations of the optimal controllers are based on the Lyapunov Bellman technique[23].

In order to observe the anti-synchronization behavior in the new Four-Dimensional system and hyperchaotic Lü system we assume that Four-Dimensional system drives the hyperchaotic Lü system. Therefore, we define the master system and slave systems as follows

$$\begin{aligned}\dot{x}_1 &= -a(x_1 - x_2) + x_2x_3x_4\\ \dot{x}_2 &= b(x_1 + x_2) - x_1x_3x_4\\ \dot{x}_3 &= -cx_3 + x_1x_2x_4\\ \dot{x}_4 &= -dx_4 + x_1x_2x_3\end{aligned} \tag{6}$$

and

$$\begin{aligned}\dot{y}_1 &= -a_1(y_1 - y_2) + y_4 + v_1 + u_1\\ \dot{y}_2 &= c_1y_2 - y_1y_3 + v_2 + u_2\\ \dot{y}_3 &= y_1y_2 - b_1y_3 + v_3 + u_3\\ \dot{y}_4 &= y_1y_3 + d_1y_4 + v_4 + u_4\end{aligned} \tag{7}$$

where the controllers are consist of two parts ,one is nonlinear feedback control $v_i(i=1,2,3,4)$ and the other is optimal control $u_i(i=1,2,3,4)$ .In order to determine the control functions to realize anti-synchronization between systems (6) and (7).Let us define the state errors between the response and derive system which is to be controlled and the controlling derive system as

$$e_1 = x_1 + y_1, e_2 = x_2 + y_2, e_3 = e_3 + y_3, e_4 = x_4 + y_4 \tag{8}$$

Add (6) to (7) and using the notation (8) yields

$$\begin{aligned}\dot{e}_1 &= -ae_1 + (a - a_1)(y_1 - y_2) + ae_2 + x_2x_3x_4 + v_1 + u_1\\ \dot{e}_2 &= b(e_1 + e_2 - y_1 - y_2) - x_1x_3x_4 + c_1y_2 + v_2 + u_2\\ \dot{e}_3 &= -ce_3 + cy_3 + y_1y_2 + x_1x_2x_4 - b_1y_3 + v_3 + u_3\\ \dot{e}_4 &= -de_4 + (d + d_1)y_4 + x_1x_2x_3 + y_1y_3 + v_4 + u_4\end{aligned} \tag{9}$$

and then we let

$$\begin{aligned}v_1 &= \hat{a}e_1 + (\hat{a}_1 - \hat{a})(y_1 - y_2) - x_2x_3x_4\\ v_2 &= (y_1 + y_2 - e_1 - e_2)\hat{b} + x_1x_3x_4 - \hat{c}_1y_2 + y_1y_3\\ v_3 &= \hat{c}(e_3 - y_3) + \hat{b}_1y_3 - x_1x_2x_4 - y_1y_2\\ v_4 &= \hat{d}(e_4 - y_4) - \hat{d}_1y_4 - x_1x_2x_3 - y_1y_3\end{aligned} \tag{10}$$

where $\hat{a},\hat{b},\hat{c},\hat{d},\hat{a}_1,\hat{b}_1,\hat{c}_1,\hat{d}_1$ are estimates of a,b,c,d, a_1,b_1,c_1,d_1 respectively .Then with controllers (10) we get the error dynamical system (9) can be described by

$$\begin{aligned}\dot{e}_1 &= \tilde{a}e_1 - (\tilde{a} - \tilde{a}_1)(y_1 - y_2) + u_1\\ \dot{e}_2 &= -\tilde{b}(e_1 + e_2 - y_1 - y_2) + u_2\\ \dot{e}_3 &= \tilde{c}(e_3 - y_3) + \tilde{b}_1y_3 + u_3\\ \dot{e}_4 &= \tilde{d}e_4 - (\tilde{d} - \tilde{d}_1)y_4 + u_4\end{aligned} \tag{11}$$

where $\tilde{a} = \hat{a} - a, \tilde{b} = \hat{b} - b, \tilde{c} = \hat{c} - c, \tilde{d} = \hat{d} - d, \tilde{a}_1 = \hat{a}_1 - a_1, \tilde{b}_1 = \hat{b}_1 - b_1, \tilde{c}_1 = \hat{c}_1 - c_1,$ and $\tilde{d}_1 = \hat{d}_1 - d_1$.

Clearly, the optimal synchronization problem is now replaced by the equivalent problem of optimal stabilizing the error system (11) using a suitable choice of the control law u_1,u_2,u_3,u_4 with minimum cost. Let us now formulate the following theorem.

Theorem 1. With the nonlinear feedback controllers

$$u_i = (-c_i / n_i)e_i, (i = 1,2,3,4) \tag{12}$$

and the system parameters nonlinear updating rule

$$\begin{aligned}
&\dot{\hat{a}} = (1/k_1)[-c_1e_1^2 + c_1(y_1 - y_2)e_1 - \beta_1\tilde{a}] \\
&\dot{\hat{b}} = (c_2e_2/k_2)(e_1 + e_2 - y_1 - y_2 - \beta_2\tilde{b}) \\
&\dot{\hat{c}} = (c_3e_3/k_3)(-e_3 + y_3 - \beta_3\tilde{c}) \\
&\dot{\hat{d}} = (c_4e_4/k_4)(-e_4 + y_4 - \beta_4\tilde{d}) \\
&\dot{\hat{a}}_1 = -(c_1e_1/k_5)(y_1 - y_2 - \beta_5\tilde{a}_1) \\
&\dot{\hat{b}}_1 = -(c_3y_3/k_6)(e_3 - \beta_6\tilde{b}_1) \\
&\dot{\hat{c}}_1 = (c_2y_2/k_7)(e_2 - \beta_7\tilde{c}_1) \\
&\dot{\hat{d}}_1 = -(c_4y_4/k_8)(e_4 - \beta_8\tilde{d}_1)
\end{aligned} \tag{13}$$

where α_i, n_i and c_i satisfy the conditions

$$c_i = \sqrt{n_i\alpha_i}, (i = 1,2,3,4) \tag{14}$$

Then the new four-dimensional system (6) and hyperchaotic Lü system (7) are optimally asymptotically anti-synchronized along the optimal trajectories specified by the integral performance index

$$I = \int_{t_0}^{+\infty} \{\sum_{i=1}^{4}[\alpha_i e_i^2 + n_i u_i^2] + \beta_1(\tilde{a})^2 + \beta_2(\tilde{b})^2 + \beta_3(\tilde{c})^2 + \beta_4(\tilde{d})^2 + \beta_5(\tilde{a}_1)^2 + \beta_6(\tilde{b}_1)^2 + \beta_7(\tilde{c}_1)^2 + \beta_8(\tilde{d}_1)^2\}dt \tag{15}$$

where t_0 is a fixed time moment and $\beta_i (i = 1,2,3,\cdots,8)$ are non-negative constants.

Proof. The proof of this theorem depends upon the choice of the performance measure as given by (15) and verifying that both of the optimal controllers (12) and updating rules (13) minimize this performance measure and asymptotically stabilizes the equilibrium state (11).Assume that the function E represents the minimum value of the performance measure and the fist we let

$$w = \{\sum_{i=1}^{4}[\alpha_i e_i^2 + n_i u_i^2] + \beta_1\tilde{a}^2 + \beta_2\tilde{b}^2 + \beta_3\tilde{c}^2 + \beta_4\tilde{d}^2 + \beta_5\tilde{a}_1^{\ 2} + \beta_6\tilde{b}_1^{\ 2} + \beta_7\tilde{c}_1^{\ 2} + \beta_8\tilde{d}_1^{\ 2}\}$$

consequently

$$E = \min_{\vec{u}} \int_{t_0}^{+\infty} w dt \tag{16}$$

where $\vec{u} = (u_1, u_2, u_3, u_4)$, the function E is referred to as the value function. and it satisfies the Bellman-Hamilton-Jacobi equation(3).According to the error system(11) we consider a Lyapunov function as follows:

$$E = \{\sum_{i=1}^{4} c_i e_i^2 + k_1\tilde{a}^2 + k_2\tilde{b}^2 + k_3\tilde{c}^2 + k_4\tilde{d}^2 + k_5\tilde{a}_1^{\ 2} + k_6\tilde{b}_1^{\ 2} + k_7\tilde{c}_1^{\ 2} + k_8\tilde{d}_1^{\ 2}\} \tag{17}$$

Assume $u_i^* (i = 1,2,3,4)$ are needed optimal controller and according to the principle of the optimal control and this leads to

$$\min_{u^*}[\dot{E} + w] = \sum_{i=1}^{4} 2c_i e_i + 2k_1\tilde{a}\dot{\tilde{a}} + 2k_2\tilde{b}\dot{\tilde{b}} + 2k_3\tilde{c}\dot{\tilde{c}} + 2k_4\tilde{d}\dot{\tilde{d}} + 2k_5\tilde{a}_1\dot{\tilde{a}}_1 + 2k_6\tilde{b}_1\dot{\tilde{b}}_1 + 2k_7\tilde{c}_1\dot{\tilde{c}}_1 + 2k_8\tilde{d}_1\dot{\tilde{d}}_1 + w \tag{18}$$

substituting(12),(13)and(14)into(18),we can get

$$\min_{u^*}[\dot{E} + w] = 0 \tag{19}$$

It implies that the controller u_i satisfy the Bellman-Hamilton-Jacobi equation(3) and the controller (12) is optimal.

Next we will prove that with the optimal controller (12) and the system parameters updating rule (13) two systems achieve anti-synchronization.

Lyapunov function in (17) along the optimal trajectories of the closed-loop system, we get

$$\dot{E} = \{\sum_{i=1}^{4} c_i e_i^2 + k_1 \tilde{a}^2 + k_2 \tilde{b}^2 + k_3 \tilde{c}^2 + k_4 \tilde{d}^2 + k_5 \tilde{a}_1^{\ 2} + k_6 \tilde{b}_1^{\ 2} + k_7 \tilde{c}_1^{\ 2} + k_8 \tilde{d}_1^{\ 2}\} \tag{20}$$

substituting (12),(13),(14)into(20),we can get

$$\dot{E} = -\sum_{i=1}^{4} 2\alpha_i e_i^2 - 2(\beta_1 \tilde{a}^2 + \beta_2 \tilde{b}^2 + \beta_3 \tilde{c}^2 + \beta_4 \tilde{d}^2 + \beta_5 \tilde{a}_1^{\ 2} + \beta_6 \tilde{b}_1^{\ 2} + \beta_7 \tilde{c}_1^{\ 2} + \beta_8 \tilde{d}_1^{\ 2}) \le 0 \tag{21}$$

since E is a positive decreasing function and $\dot{E}$ is negative semidefinite ,it follows that the equilibrium $(e_1 = 0, e_2 = 0, e_3 = 0, e_4 = 0, \tilde{a} = 0, \tilde{b} = 0, \tilde{c} = 0, \tilde{d} = 0, \tilde{a}_1 = 0, \tilde{b}_1 = 0, \tilde{c}_1 = 0, \tilde{d}_1 = 0)$ of the system (8) is uniformly stable, i.e. $e_1, e_2, e_3, e_4 \in L_\infty$ and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{a}_1, \tilde{b}_1, \tilde{c}_1, \tilde{d}_1 \in L_\infty$ .From Eq.(12), we can easily show that the squares of e_1, e_2, e_3 and e_4 are integrable with respect to time t ,i.e. $e_1, e_2, e_3, e_4 \in L_2$. Next by Barbalat's Lemma, for any initial condition, the system (8) implies that $e_1, e_2, e_3, e_4 \in L_\infty$,which in turn implies $(e_1, e_2, e_3, e_4) \to (0,0,0,0)$ as $t \to \infty$. Thus in the closed-loop system $y_1 \to x_1, y_2 \to x_2, y_3 \to x_3, y_4 \to x_4$ as $t \to \infty$.

Consequently, with the optimal feedback control law (12) and updating rule (13) the optimal synchronization of both derive and response systems is achieved with complete unknown system parameters, which completes the proof.

5. Simulation

In this section, we will show a series of numerical simulations to demonstrate the effectiveness of the proposed control scheme. All simulation procedures are coded and executed using the Matlab software. Fourth order Runge-Kutta integration method is used to solve two systems of differential equations (6) and (7).In addition, a time step size 0.001 is employed.

We assume that $\beta_i = 2(i = 1, 2, \cdots, 8)$, $n_i = 1(i = 1, 2, 3, 4)$, $k_i = 1(i = 1, 2, \cdots, 8)$. We will select the parameters of the new four-dimensional system $a = 30, b = 10, c = 37, d = 10$ and the parameters of hyperchaotic Lü system $a = 36, b = 3, c = 20, d = 1$ .Therefore, both the new four-dimensional system and hyperchaotic lü system exhibit chaotic behavior. The initial values of the master and slave systems are $(x_1(0), x_2(0), x_3(0), x_4(0)) = (-10, 10, 10, 20)$ and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (0.3, 2.5, 3.2, 0.2)$. So the initial values of the error system is $(e_1(0), e_2(0), e_3(0), e_4(0)) = (-9.7, 12.5, 13.2, 20.2)$ .The anti-synchronization of systems (6) and (7) via optimal control law (12) and parameters update rule (13) are shown in Fig.4 and Fig.5.

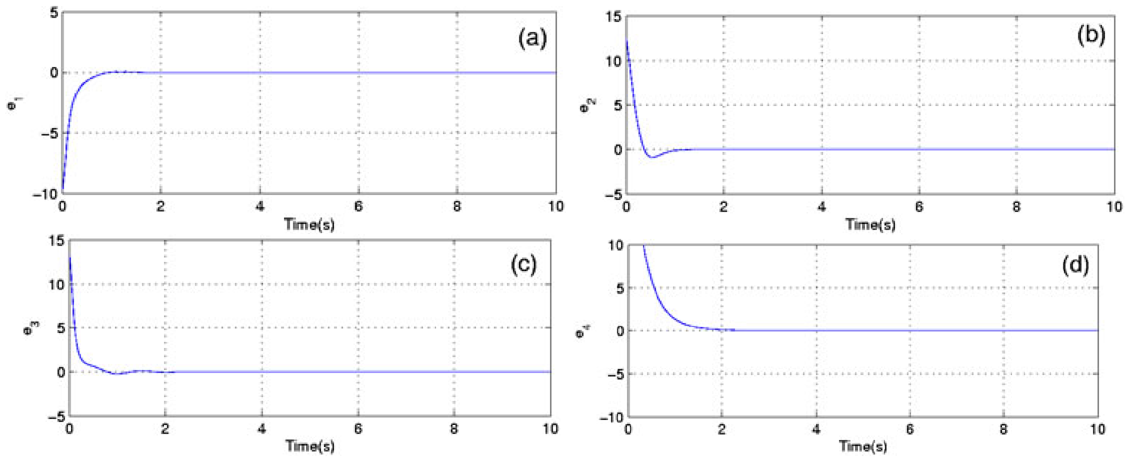

Figure 4. Dynamics of anti-synchronization errors (e_1, e_2, e_3, e_4) between the new four-dimensional system and hyperchaotic Lü system with time t .

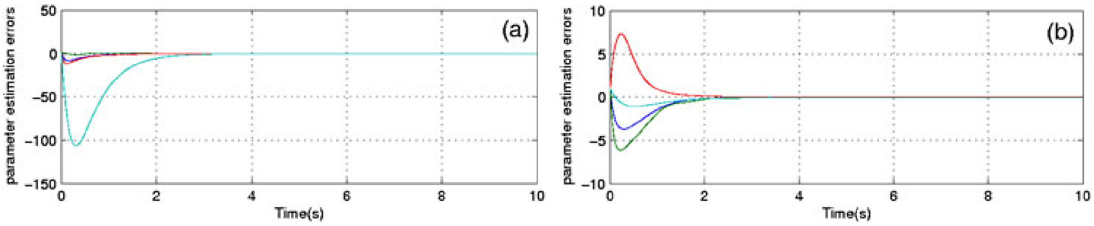

Figure 5. Adaptive parameters estimation errors:(a) $\hat{a},\hat{b},\hat{c},\hat{d}$;(b) $\hat{a}_1,\hat{b}_1,\hat{c}_1,\hat{d}_1$

6. Conclusion

In this paper, chaos anti-synchronization between two different chaotic systems with different structures and parameters via optimal and adaptive control is presented. The new four-dimensional system and the hyperchaotic Lü system are taken as an illustrative example to verify the effectiveness of the proposed method.

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