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Non-Polynomial Spline Approach to the Solution of Fifth-Order Boundary-Value Problems

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Abstract. Non-polynomial spline in off step points is used to solve fifth-order linear boundary value problems. Associated boundary formulas are developed. We compare our results with the results produced by B-spline method [10] and Non-polynomial spline and quartic spline method [11, 12] and [14]. However, it is observed that our approach produce better numerical solutions in the sense that $\max|e_i|$ is a minimum.

Keywords: Fifth-order boundary-value problem; Non-polynomial spline functions; boundary value formulae; Numerical results.

1. Introduction

The solution of fifth order boundary value problems are not very much found in the numerical analysis literature. These problems are generally arise in the mathematical modeling of viscoelastic flows [1-2].The conditions for existence and uniqueness of solution of such boundary value problems are explained by theorems presented in Agarwal [3]. Caglar et al. [4] solved third order linear and nonlinear boundary value problems using fourth degree B-spline. Siddiqi and Twizell[5-8] presented the solutions of 6th,8th,10th and 12th order boundary value problems using the 6th,8th,10th and 12th degree spline, respectively. Siddiqi and Ghazla[9] Applied the non-polynomial spline for the solutions of the system fourth-order boundary value problems. Caglar et al.[10] presented the solutions of fifth-order boundary value problems using sixth degree B-spline. Siddiqi and Ghazla[11] presented the solutions of fifth-order linear boundary value problems using non-polynomial spline, This method is second-order convergent, in [12] siraj-ul islam et al. given a method based on sextic spline solution for the solution of fifth-order boundary value problem in grid points and also in [13] Khan et al. derived a numerical method based on non-polynomial spline. Scott and Watts [15] described the numerical solution of linear BVP using a combination of superposition and orthonormalization and in [16] described several computer codes that were developed using the superposition and orthonormalization technique and invariant imbedding. In this paper we used non-polynomial sextic spline approximation to develop a family of new numerical methods to obtain smooth approximations to the solution of fifth-order differential equation. The new methods are of order two for arbitrary α, β and γ provided that $\alpha+\beta+\gamma=\frac{1}{2}$, and are of order four If $\alpha=\frac{1}{72}, \beta=0, \gamma=0.5-(\alpha+\gamma)$, and are sixth order if $\alpha=0, \beta=\frac{1}{24}$, and $\gamma=\frac{11}{24}$. The new methods perform better than the other collocation and spline methods of same order and thus represent an improvement over existing methods. The spline functions proposed in this paper have the form $T_6=span\{1,x,x^2,x^3,x^4,Cos(kx),Sin(kx)\}$ where k is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary and which will be used to raise the accuracy of the method. Thus in each subinterval $x_i \le x \le x_{i+1}$, we have $span\{1,x,x^2,x^3,x^4,Cos(|k|x),Sin(|k|x)\}$,

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$$span\{1, x, x^2, x^3, x^4, x^5, x^6\}, \qquad (\text{when } k \to 0).$$

In this manuscript the following fifth-order boundary value problem is consider:

$$y^{(5)}(x) + q(x)y(x) = g(x), x \in [a,b], \qquad (1)$$

with boundary conditions

$$y(a) = \alpha_0, y^{(1)}(a) = \alpha_1, y^{(2)}(a) = \alpha_2, \text{ And } \quad y(b) = \beta_0, y^{(1)}(b) = \beta_1 \qquad (2),$$

Where α_i, β_i for $i = 0,1,2$ are finite real constants and the functions $g(x)$ and $q(x)$ are continuous on [a,b].

In this paper, in Section 2, the new non-polynomial spline methods are developed for solving equation (1) along with boundary condition (2). The boundary formulas are develop in Section 3 and in Section 4 numerical experiment, discussion and comparison with other known methods, are given.

2. Numerical methods

To develop the spline approximation of the fifth-order boundary-value problem (1)-(2), the interval [a,b] is divided into n equal subintervals using the grid $x_{i-\frac{1}{2}} = a + (i - \frac{1}{2})h, i = 1,\ldots,n,$ where $h = \frac{b-a}{n}$.

Consider the following non- polynomial fifth spline $S_i(x)$ is each subinterval $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}],$ $i = 0,1,\ldots,n-1$, $x_0 = a, x_n = b$,

$$S_i(x) = a_i Cos(x - x_i) + b_i Sin(x - x_i) + c_i(x - x_i)^4 + d_i(x - x_i)^3 + e_i(x - x_i)^2 + f_i(x - x_i) + g_i, \qquad (3)$$

Where $a_i, b_i, c_i, d_i, e_i,$ and f_i are real finite constants and k is free parameter.

Let y_i be an approximation of $y(x_i)$, obtained by the segment $S_i(x)$ of the mixed splines function passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}). To obtain the necessary conditions for the coefficients introduced in (3), we do not only require that $S_i(x)$ satisfies equation. (1) at x_i and x_{i+1} and that the boundary conditions are fulfilled, but also the continuity of first, second, third, fourth and fifth derivatives at the common nodes (x_i, y_i). To determine the coefficients of equation (3) we the first define

$$S_i(x_{i-\frac{1}{2}}) = y_{i-\frac{1}{2}}, \quad S_i^{(1)}(x_{i-\frac{1}{2}}) = m_{i-\frac{1}{2}}, \quad S_i^{(2)}(x_{i-\frac{1}{2}}) = M_{i-\frac{1}{2}}, \quad S_i^{(5)}(x_{i-\frac{1}{2}}) = L_{i-\frac{1}{2}},$$

$$S_i(x_{i+\frac{1}{2}}) = y_{i+\frac{1}{2}}, \quad S_i^{(1)}(x_{i+\frac{1}{2}}) = m_{i+\frac{1}{2}}, \quad S_i^{(2)}(x_{i+\frac{1}{2}}) = M_{i+\frac{1}{2}}, \quad S_i^{(5)}(x_{i+\frac{1}{2}}) = L_{i+\frac{1}{2}},$$

Algebraic manipulation yields the following expressions, whereby $\theta = kh$ and $i = 1,2,\ldots,n.$

$$a_i = h^5 \frac{Csc\,\frac{\theta}{2}(L_{i-\frac{1}{2}} - L_{i+\frac{1}{2}})}{2\theta^5},$$

$$b_i = h^5 \frac{Sec\,\frac{\theta}{2}(L_{i-\frac{1}{2}} - L_{i+\frac{1}{2}})}{2\theta^5},$$

$$c_i = \frac{h}{6\theta^5}[Csc\,\theta(6 + \theta^2 + 2(-3 + \theta^2)Cos\,\theta - 6\theta Sin\,\theta)L_{i-\frac{1}{2}} - Csc\,\theta(-6 + 2\theta^2 + (6 + \theta^2)Cos\,\theta)L_{i+\frac{1}{2}} +$$
$$\frac{\theta^5}{h^5}(6hm_{i-\frac{1}{2}} + 2h^2 M_{i-\frac{1}{2}} + h^2 M_{i+\frac{1}{2}} + 6y_{i-\frac{1}{2}} - 6y_{i+\frac{1}{2}})],$$

$$d_i = \frac{-M_{i-\frac{1}{2}} + M}{6h}$$

$$e_i = \frac{h^3}{4\theta^5}[(6\theta + (6-\theta^2)Cot\theta - 6Csc\theta)L_{i-\frac{1}{2}} + Csc\theta(-6+\theta^2+6Cos\theta)L_{i+\frac{1}{2}} -$$

$$\frac{4\theta^5}{h^5}(6hm_{i-\frac{1}{2}} + h^2M_{i-\frac{1}{2}} + 6y_{i-\frac{1}{2}} - 6y_{i+\frac{1}{2}})],$$

$$f_i = \frac{1}{h}(-y_{i-\frac{1}{2}} + y_{i+\frac{1}{2}}) - \frac{h^4}{\theta^5}(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}})Tan\frac{\theta}{2} + \frac{h}{24}(M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}}) - \frac{h^4}{24\theta^3}(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}})Tan\frac{\theta}{2},$$

$$g_i = \frac{h^5}{96\theta^5}[Csc\theta(-18-\theta^2+(-78+4\theta^2)Cos\theta - 30\theta Sin\theta)L_{i-\frac{1}{2}} + Csc\theta(78-4\theta^2+(18+\theta^2)Cos\theta)L_{i+\frac{1}{2}} +$$

$$\frac{1}{96}(30hm_{i-\frac{1}{2}} + 4h^2M_{i-\frac{1}{2}} - h^2M_{i+\frac{1}{2}} + 78y_{i-\frac{1}{2}} + 18y_{i+\frac{1}{2}})].$$

The continuity condition of the first, third and fourth derivatives at knots, $S_{i-1}^{(\lambda)}(x_{i-\frac{1}{2}}) = S_i^{(\lambda)}(x_{i-\frac{1}{2}})$, where $\lambda = 1,3, and\ 4$, yields the following equations:

$$[-(6+\frac{1}{\theta}(-12+\theta^2)Tan\frac{\theta}{2})(L_{i-\frac{3}{2}} - L_{i-\frac{1}{2}}) + \frac{6\theta^4(m_{i-\frac{3}{2}} + m_{i-\frac{1}{2}})}{h^4} + \frac{\theta^4(M_{i-\frac{3}{2}} - M_{i-\frac{1}{2}})}{h^3} + \frac{12\theta^4(y_{i-\frac{3}{2}} - y_{i-\frac{1}{2}})}{h^5}] = 0 \qquad (4)$$

$$[3(-4+\theta Cot\frac{\theta}{2} + \frac{4}{\theta}Tan\frac{\theta}{2})L_{i-\frac{3}{2}} + 2(-6-\frac{1}{\theta}(12-\theta^2)(-Cot\theta + Csc\theta))L_{i-\frac{1}{2}} + \frac{1}{\theta}((-12-\theta^2)Cot\theta + (12-5\theta^2)Csc\theta)L_{i+\frac{1}{2}} + \frac{12\theta^4(m_{i-\frac{3}{2}} + m_{i-\frac{1}{2}})}{h^4} + \frac{\theta^4(3M_{i-\frac{3}{2}} + 8M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}})}{h^3} + \frac{12\theta^4(y_{i-\frac{3}{2}} - y_{i+\frac{1}{2}})}{h^5}] = 0, \qquad (5)$$

$$[\frac{1}{\theta}Csc\theta(24+4\theta^2+\theta^4+8(-3+\theta^2)Cos\theta - 24\theta Sin\theta)L_{i-\frac{3}{2}} - 2Csc\theta((1+Cos\theta)6\theta + \theta^3Cos\theta - 12Sin\theta)L_{i-\frac{1}{2}} + \frac{1}{\theta}((1+4\theta^2)Cot\theta + (-24+8\theta^2+\theta^4)Csc\theta)L_{i+\frac{1}{2}} + \frac{24\theta^4m_{i-\frac{3}{2}} - 24m_{i-\frac{1}{2}}}{h^4} + \frac{4\theta^4(2M_{i-\frac{3}{2}} - M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}})}{h^3} + \frac{24\theta^4(y_{i-\frac{3}{2}} - 2y_{i-\frac{1}{2}} - y_{i+\frac{1}{2}})}{h^5}] = 0 \qquad (6)$$

In order to get nine additional equations is replaced by $i-1, i-2, i+1$, in each of the equations (4),(5) and (6), and help of Mathematical 5.1 we have the following consistency relation in terms of fifth derivative of spline L_i and y_i, $i = 0,1,...,n.$

$$h^5(\alpha L_{i-\frac{5}{2}} + \beta L_{i-\frac{3}{2}} + \gamma L_{i-\frac{1}{2}} + \gamma L_{i+\frac{1}{2}} + \beta L_{i+\frac{3}{2}} + \alpha L_{i+\frac{5}{2}}) = -(y_{i-\frac{5}{2}} - 5y_{i-\frac{3}{2}} + 10y_{i-\frac{1}{2}} - 10y_{i+\frac{1}{2}} + 5y_{i+\frac{3}{2}} - y_{i+\frac{5}{2}}), \quad i = 3,4,...,n-2 \qquad (7)$$

where

$$\alpha = \left(\frac{24 - 12\theta^2 + \theta^4 - 24Cos\theta}{24\theta^4 Sin\theta}\right),$$

$$\beta = -\left(\frac{-72 + 12\theta^2 + 11\theta^4 + (72 + 24\theta^2 - 2\theta^4)Cos\theta}{24\theta^5 Sin\theta}\right),$$

$$\gamma = -\left(\frac{2(-6(4+\theta^4) + (24 + 12\theta^2 + \theta^4)Cos\theta)}{24\theta^5 Sin\theta}\right).$$

Using the relation (1) and (7) we have

$$h^5[\alpha(g_{i-\frac{5}{2}} - q_{i-\frac{5}{2}} y_{i-\frac{5}{2}}) + \beta(g_{i-\frac{3}{2}} - q_{i-\frac{3}{2}} y_{i-\frac{3}{2}}) + \gamma(g_{i-\frac{1}{2}} - q_{i-\frac{1}{2}} y_{i-\frac{1}{2}}) + \gamma(g_{i+\frac{1}{2}} - q_{i+\frac{1}{2}} y_{i+\frac{1}{2}}) +$$
$$\beta(g_{i+\frac{3}{2}} - q_{i+\frac{3}{2}} y_{i+\frac{3}{2}}) + \alpha(g_{i+\frac{5}{2}} - q_{i+\frac{5}{2}} y_{i+\frac{5}{2}})] = -y_{i-\frac{5}{2}} + 5y_{i-\frac{3}{2}} - 10y_{i-\frac{1}{2}} + 10y_{i+\frac{1}{2}} - 5y_{i+\frac{3}{2}} + y_{i+\frac{5}{2}}$$
$$i = 3, 4, \dots, n-3 \qquad (8)$$

If $\theta \to 0$ than $(\alpha, \beta, \gamma) \to (\frac{1}{720}, \frac{57}{720}, \frac{302}{720})$, and we get polynomial sextic spline functions.

3. Development of the boundary formulas

To obtain unique solution we need fifth more equations to be associated with (8) so that we use the following boundary conditions. In order to obtain the sixth-order boundary formula we define the following identity

$$d'_0 y_0 + \sum_{k=0}^{4} a'_k y_{k+\frac{1}{2}} + c' h y_0^{(1)} = h^5 \sum_{k=0}^{7} b' y_{k+\frac{1}{2}}^{(5)} + t_1, \qquad (9)$$

$$d''_0 y_0 + \sum_{k=0}^{5} a''_k y_{k+\frac{1}{2}} + c'' h y_0^{(1)} = h^5 \sum_{k=0}^{8} b'' y_{k+\frac{1}{2}}^{(5)} + t_2, \qquad (10)$$

$$d'''_0 y_0 + \sum_{k=0}^{6} a'''_k y_{k+\frac{1}{2}} + c''' h y_0^{(1)} + d' h^2 y_0^{(2)} = h^5 \sum_{k=0}^{9} b''' y_{k+\frac{1}{2}}^{(5)} + t_3, \qquad (11)$$

$$\dot{d}_0 y_0 + \sum_{k=0}^{5} \dot{a}_k y_{k+n-\frac{9}{2}} + \dot{c} h y_0^{(1)} = h^5 \sum_{k=0}^{8} \dot{b} y_{k+n-\frac{9}{2}}^{(5)} + t_{n-2}, \qquad (12)$$

$$\ddot{d}_0 y_0 + \sum_{k=0}^{4} \ddot{a}_k y_{k+n-\frac{7}{2}} + \ddot{c} h y_0^{(1)} = h^5 \sum_{k=0}^{7} \ddot{b} y_{k+n-\frac{7}{2}}^{(5)} + t_{n-1}, \qquad (13)$$

where all of the coefficients are arbitrary parameters to be determined. In order to obtain the fifth-order method we find that:

$$y_0 - \frac{3675}{2816} y_{\frac{1}{2}} + \frac{1225}{2816} y_{\frac{3}{2}} - \frac{441}{2816} y_{\frac{5}{2}} + \frac{75}{2816} y_{\frac{7}{2}} + \frac{105}{352} h y_0^{(1)} = h^{(5)}\Big(\frac{9092797}{3045064704} y_{\frac{1}{2}}^{(5)} + \frac{7304661}{507510784} y_{\frac{3}{2}}^{(5)} -$$
$$\frac{3611195}{1015021568} y_{\frac{5}{2}}^{(5)} + \frac{3183845}{761266176} y_{\frac{7}{2}}^{(5)} - \frac{2359175}{1015021568} y_{\frac{9}{2}}^{(5)} + \frac{372813}{507510784} y_{\frac{11}{2}}^{(5)} - \frac{102557}{1015021568} y_{\frac{13}{2}}^{(5)}\Big), \qquad (14)$$

$$y_0+\frac{896234412465}{2668568192}y_{\frac{1}{2}}-\frac{553567834155}{667142048}y_{\frac{3}{2}}+\frac{1088158392279}{1334284096}y_{\frac{5}{2}}-\frac{268446177495}{667142048}y_{\frac{7}{2}}+\frac{212836281385}{2668568192}y_{\frac{9}{2}}+\frac{1820164500}{20848189}hy_0^{(1)}=h^{(5)}\Big(y_{\frac{1}{2}}^{(5)}+\frac{9944909969453}{237618593792}y_{\frac{3}{2}}^{(5)}+\frac{83223658283409}{5465227657216}y_{\frac{5}{2}}^{(5)}+\frac{105608680427659}{2732613828608}y_{\frac{7}{2}}^{(5)}-\frac{96351080023051}{2732613828608}y_{\frac{9}{2}}^{(5)}+\frac{115342790190447}{5465227657216}y_{\frac{11}{2}}^{(5)}-\frac{572407787673}{81570562048}y_{\frac{13}{2}}^{(5)}+y_{\frac{15}{2}}^{(5)}\Big), \qquad (15)$$

$$y_0-\frac{490481858916017144211}{1061982379022464}y_{\frac{1}{2}}-\frac{824943179583070401185}{796486784266848}y_{\frac{3}{2}}+\frac{491260466370330252073}{530991189511232}y_{\frac{5}{2}}-\frac{115522510005328799115}{265495594755616}y_{\frac{7}{2}}+\frac{267028091531920114265}{3185947137067392}y_{\frac{9}{2}}+y_{\frac{11}{2}}+\frac{1289923904141557520}{8296737336113}hy_0^{(1)}+\frac{32455453285803975}{990655204312}h^2y_0^{(2)}=h^{(5)}\Big(y_{\frac{1}{2}}^{(5)}+\frac{42362509948706550956369}{1427304317406191616}y_{\frac{3}{2}}^{(5)}-\frac{679493818240557125898023}{17127651808874299392}y_{\frac{5}{2}}^{(5)}+\frac{11963819587573598528737}{5709217269624766464}y_{\frac{7}{2}}^{(5)}-\frac{339072769981186263 8471}{5709217269624766464}y_{\frac{9}{2}}^{(5)}-\frac{197909104245087658 2827}{17127651808874299392}y_{\frac{11}{2}}^{(5)}+y_{\frac{17}{2}}^{(5)}\Big), \qquad (16)$$

$$y_n-\frac{905890244175}{2668568192}y_{n-\frac{1}{2}}+\frac{555804602325}{667142048}y_{n-\frac{3}{2}}-\frac{1091831111721}{1334284096}y_{n-\frac{5}{2}}+\frac{269284566825}{667142048}y_{n-\frac{7}{2}}-\frac{213472777175}{2668568192}y_{n-\frac{9}{2}}+\frac{1813151340}{20848189}hy_n^{(1)}=h^{(5)}\Big(y_{n-\frac{1}{2}}^{(5)}+\frac{9962826008531}{237618593792}y_{n-\frac{3}{2}}^{(5)}+\frac{83854711930479}{5465227657216}y_{n-\frac{5}{2}}^{(5)}+\frac{105615596334965}{2732613828608}y_{n-\frac{7}{2}}^{(5)}-\frac{96340480490485}{2732613828608}y_{n-\frac{9}{2}}^{(5)}+\frac{115336462375569}{5465227657216}y_{n-\frac{11}{2}}^{(5)}-\frac{572395507559}{81570562048}y_{n-\frac{13}{2}}^{(5)}+y_{n-\frac{15}{2}}^{(5)}\Big), \qquad (17)$$

$$y_n-\frac{3675}{2816}y_{n-\frac{1}{2}}+\frac{1225}{2816}y_{n-\frac{3}{2}}-\frac{441}{2816}y_{n-\frac{5}{2}}+\frac{75}{2816}y_{n-\frac{7}{2}}+\frac{105}{352}hy_n^{(1)}=h^{(5)}\Big(-\frac{9092797}{3045064704}y_{n-\frac{1}{2}}^{(5)}-\frac{7304661}{507510784}y_{n-\frac{3}{2}}^{(5)}+\frac{3611195}{1015021568}y_{n-\frac{5}{2}}^{(5)}-\frac{3183845}{761266176}y_{n-\frac{7}{2}}^{(5)}+\frac{2359175}{1015021568}y_{n-\frac{9}{2}}^{(5)}-\frac{372813}{507510784}y_{n-\frac{11}{2}}^{(5)}+\frac{102557}{1015021568}y_{n-\frac{13}{2}}^{(5)}\Big), \qquad (18)$$

The local truncation error corresponding to the scheme (8) is given by

$$\mathrm{t_i}=(1-2\alpha-2\beta-2\gamma)h^5y_i^{(5)}+\Big(\frac{5}{24}-\frac{25\alpha+9\beta+\gamma}{4}\Big)h^7y_i^{(7)}+\Big(\frac{23}{1152}-\frac{625\alpha+81\beta+\gamma}{192}\Big)h^9y_i^{(9)}+\Big(\frac{227}{193536}-\frac{15625\alpha+729\beta+\gamma}{23040}\Big)h^{11}y_i^{(11)}+\Big(\frac{631}{13271040}+\frac{390625\alpha+6561\beta+\gamma}{5160960}\Big)h^{13}y_i^{(13)}+\ldots \qquad (19)$$

We obtain the following methods for any choice of α,β and γ whose sum is $\frac{1}{2}$.

Remark (i): Second-order method

For $\alpha=\frac{1}{720},\beta=\frac{57}{720}$ and $\gamma=\frac{302}{720}$ we obtain the second order method with truncation error

$$t_i = \frac{1}{12} h^7 y^{(7)}(x_i) + o(h^8).$$

Remark (ii): Fourth-order method

For $\alpha = \frac{1}{72}, \beta = 0$ and $\gamma = \frac{35}{72}$ we obtain the second order method with truncation error

$$t_i = \frac{1}{36} h^9 y^{(9)}(x_i) + o(h^{10}).$$

Remark (iii): Sixth-order method

For $\alpha = 0, \beta = \frac{1}{24}$ and $\gamma = \frac{11}{24}$ we obtain the second order method with truncation error

$$t_i = \frac{1}{6048} h^{11} y^{(11)}(x_i) + o(h^{12}).$$

4. Non-Polynomial spline solution

The spline solution of boundary value problem (1) is based on, using (14) − (18) and (8) we obtain a system of linear equations. Considering $Y = [y_{\frac{1}{2}}, y_{\frac{3}{2}}, ..., y_{n-\frac{1}{2}}]^T$ and $C = [c_{\frac{1}{2}}, c_{\frac{3}{2}}, ..., c_{n-\frac{1}{2}}]^T$ this system can be written the following matrix equation: $(A + h^5 BF)Y = C$

Where $A =$

$$\begin{bmatrix}
a_1' & a_2' & a_3' & a_4' & 0 & 0 & 0 & . & . & . & 0 \\
a_1'' & a_2'' & a_3'' & a_4'' & a_5'' & 0 & 0 & . & . & . & 0 \\
a_1''' & a_2''' & a_3''' & a_4''' & a_5''' & 0 & 0 & . & . & . & 0 \\
-1 & 5 & -10 & 10 & -5 & 1 & & & & & \\
0 & -1 & 5 & -10 & 10 & -5 & 1 & & & & \\
0 & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 \\
0 & . & . & . & . & . & . & . & . & . & 0 \\
0 & . & . & . & . & . & . & . & . & . & 0 \\
0 & .. & .. & .. & 0 & -1 & 5 & -10 & 10 & -5 & 1 \\
0 & .. & .. & .. & 0 & 0 & \dot{a}_5 & \dot{a}_4 & \dot{a}_3 & \dot{a}_2 & \dot{a}_1 \\
0 & .. & .. & .. & 0 & 0 & 0 & \ddot{a}_4 & \ddot{a}_3 & \ddot{a}_2 & \ddot{a}_1
\end{bmatrix}$$

And

$$B=\begin{bmatrix} b_0' & b_1' & b_2' & b_3' & 0 & 0 & 0 & .. & .. & 0 \\ b_0'' & b_1'' & b_2'' & b_3'' & b_4'' & 0 & 0 & .. & .. & 0 \\ b_0''' & b_1''' & b_2''' & b_3''' & b_4''' & b_5''' & 0 & .. & .. & 0 \\ \alpha & \beta & \gamma & \gamma & \beta & \alpha & & & & \\ 0 & \alpha & \beta & \gamma & \gamma & \beta & \alpha & & & \\ 0 & .. & .. & .. & .. & .. & .. & .. & .. & 0 \\ 0 & . & . & . & . & . & . & . & . & 0 \\ 0 & .. & .. & 0 & \alpha & \beta & \gamma & \gamma & \beta & \alpha \\ 0 & .. & .. & 0 & 0 & \dot{b}_4 & \dot{b}_3 & \dot{b}_2 & \dot{b}_1 & \dot{b}_0 \\ 0 & .. & .. & 0 & 0 & 0 & \ddot{b}_3 & \ddot{b}_2 & \ddot{b}_1 & \ddot{b}_0 \end{bmatrix}.$$

The vector C is defined by

$$c_{\frac{1}{2}}=h^5\Big(\frac{9092797}{3045064704}g_{\frac{1}{2}}+\frac{7304661}{507510784}g_{\frac{3}{2}}-\frac{3611195}{1015021568}g_{\frac{5}{2}}+\frac{3183845}{761266176}g_{\frac{7}{2}}-$$
$$\frac{2359175}{1015021568}g_{\frac{9}{2}}+\frac{372813}{507510784}g_{\frac{11}{2}}-\frac{102557}{1015021568}g_{\frac{13}{2}}\Big)\Big)-\frac{105}{352}hy_0^{(1)}-y_0,$$

$$c_{\frac{3}{2}}=h^{(5)}\Big(g_{\frac{1}{2}}+\frac{9944909969453}{237618593792}g_{\frac{3}{2}}+\frac{83223658283409}{5465227657216}g_{\frac{5}{2}}+\frac{105608680427659}{2732613828608}g_{\frac{7}{2}}-$$
$$\frac{96351080023051}{2732613828608}g_{\frac{9}{2}}+\frac{115342790190447}{5465227657216}g_{\frac{11}{2}}-\frac{572407787673}{81570562048}g_{\frac{13}{2}}+g_{\frac{15}{2}}\Big)-\frac{1820164500}{20848189}hy_0^{(1)}-y_0,.$$

$$c_{i+\frac{1}{2}}=h^5\big(\alpha g_{i-\frac{5}{2}}+\beta g_{i-\frac{3}{2}}+\gamma g_{i-\frac{1}{2}}+\gamma g_{i+\frac{1}{2}}+\beta g_{i+\frac{3}{2}}+\alpha g_{i+\frac{5}{2}}\big),$$
$$i=3,4,\ldots,(n-3)$$

$$c_{n-\frac{3}{2}}=h^5\Big(g_{n-\frac{1}{2}}+\frac{9962826008531}{237618593792}g_{n-\frac{3}{2}}+\frac{83854711930479}{5465227657216}g_{n-\frac{5}{2}}+\frac{105615596334965}{2732613828608}g_{n-\frac{7}{2}}-$$
$$\frac{96340480490485}{2732613828608}g_{n-\frac{9}{2}}+\frac{115336462375569}{5465227657216}g_{n-\frac{11}{2}}-\frac{572395507559}{81570562048}y_{n-\frac{13}{2}}+g_{n-\frac{15}{2}}\Big)-$$
$$\frac{1813151340}{20848189}hy_n^{(1)}-y_n,$$

$$c_{n-\frac{1}{2}}=h^5\Big(-\frac{9092797}{3045064704}g_{n-\frac{1}{2}}-\frac{7304661}{507510784}g_{n-\frac{3}{2}}+\frac{3611195}{1015021568}g_{n-\frac{5}{2}}-\frac{3183845}{761266176}g_{n-\frac{7}{2}}+\frac{2359175}{1015021568}g_{n-\frac{9}{2}}-$$
$$\frac{372813}{507510784}g_{n-\frac{11}{2}}+\frac{102557}{1015021568}g_{n-\frac{13}{2}}\Big)-\frac{105}{352}hy_n^{(1)}-y_n.$$

5. Numerical results

Example 1. We Consider the following boundary-value problem

$$y^{(5)}(x) - y(x) = -(15 + 10x)e^{x}, \qquad 0 \le x \le 1$$
$$y(0) = y(1) = 0, y'(0) = 1, y'(1) = -e, y''(0) = 0 \tag{20}$$

The exact solution for this problem is $y(x) = x(1-x)e^{x}$. This example have be solved by our presented method with value of , $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, the computed solution are compared with exact solution, The maximum absolute errors associated with y_i for the system (20) are summarized in Table (1) and compared with [10,11,12].

Example 2. We consider the following boundary-value problem

$$y^{(5)}(x) + xy(x) = (1-x)Cos(x) - 5Sin(x) + xSin(x) - x^2 Sin(x), \qquad 0 \le x \le 1$$
$$y(0) = y(1) = 0, y'(0) = 1, y'(1) = -Sin(1), y''(0) = -2, \tag{21}$$

The exact solution for this problem is $y(x) = (1-x)Sin(x)$. We solved this example by different values of $h = 1/12, \ldots, 1/92$, the maximum absolute errors associated with y_i for the system (21) are summarized in Table 2 and compared with [14].

Example 3. We consider the following boundary-value problem

$$y^{(5)} + y(x) = -4e^{x}(x^2 Cos(x) - 2xCos(x) - 9Cos(x)) -$$
$$e^{x}(3x^2 \sin(x) + 34xSin(x) + 3Sin(x)), 0 \le x \le 1 \tag{22}$$
$$y(0) = y(1) = 0, y'(0) = 1, y'(1) = 0, y''(0) = -2$$

The exact solution for this problem is $y(x) = (1-x^2)e^{x}Sin(x)$. We solved this example by different values of $h = 1/12, \ldots, 1/92$, the maximum absolute errors associated with y_i for the system (22) are tabulated in Table 3 and compared with [14].

Table1: Observed maximum absolute errors for example 1.

h	Our method : $\alpha = 0$, $\beta = \frac{1}{24}, \gamma = \frac{11}{24}$	Our method : $\alpha = \frac{1}{72}$, $\beta = 0, \gamma = \frac{35}{72}$	Our method : $\alpha = \frac{1}{720}, \beta = \frac{57}{720}$, $\gamma = \frac{302}{720}$	Caglar and Caglar [10]	Siddiqi and Akram [11]	Siraj-ul-islam[12]
$\frac{1}{10}$	3.164×10^{-11}	7.631×10^{-8}	1.329×10^{-5}	1.57×10^{-1}	1.29×10^{-4}	2.76×10^{-3}
$\frac{1}{20}$	2.285×10^{-13}	5.475×10^{-9}	3.814×10^{-6}	7.47×10^{-2}	2.79×10^{-5}	2.45×10^{-4}
$\frac{1}{40}$	3.837×10^{-12}	3.501×10^{-10}	9.685×10^{-7}	2.08×10^{-2}	9.40×10^{-6}	2.01×10^{-5}

Table2: Observed maximum absolute errors for example 2.

h	Our method : $a = 0,$ $b = \frac{1}{24}, g = \frac{11}{24}$	Our method : $a = \frac{1}{72},$ $b = 0, g = \frac{35}{72}$	Our method : $a = \frac{1}{720}, b = \frac{57}{720},$ $g = \frac{302}{720}$	Method [14]
$\frac{1}{12}$	3.146×10^{-13}	1.193×10^{-9}	3.847×10^{-7}	9.9638×10^{-5}
$\frac{1}{24}$	1.364×10^{-13}	7.719×10^{-11}	9.920×10^{-8}	1.3881×10^{-5}
$\frac{1}{48}$	1.656×10^{-13}	3.617×10^{-12}	2.488×10^{-8}	1.8200×10^{-6}
$\frac{1}{92}$	4.218×10^{-11}	1.209×10^{-11}	7.067×10^{-9}	2.3269×10^{-7}

Table3: Observed maximum absolute errors for example 3.

h	Our method : $a = 0,$ $b = \frac{1}{24}, g = \frac{11}{24}$	Our method : $a = \frac{1}{72},$ $b = 0, g = \frac{35}{72}$	Our method : $a = \frac{1}{720}, b = \frac{57}{720},$ $g = \frac{302}{720}$	Method [14]
$\frac{1}{12}$	1.547×10^{-10}	1.645×10^{-7}	1.151×10^{-5}	9.0155×10^{-4}
$\frac{1}{24}$	1.004×10^{-12}	1.143×10^{-8}	1.096×10^{-5}	7.0710×10^{-5}
$\frac{1}{48}$	1.719×10^{-11}	7.133×10^{-10}	2.761×10^{-6}	5.4810×10^{-6}
$\frac{1}{92}$	1.881×10^{-10}	2.172×10^{-10}	7.525×10^{-7}	4.4069×10^{-7}

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