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A stable numerical algorithm for solving an inverse parabolic problem

R. Pourgholi [1] and M. Rostamian [2]

School of Mathematics and Computer Sciences,
Damghan University of Basic Science, Damghan, P.O.Box 36715-364, Iran.

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Abstract. In this paper we consider a numerical approach for the determination of an unknown boundary condition in the inverse heat conduction problem (IHCP). The given heat conduction equation, the boundary condition, and the initial condition are presented in a dimensionless form. The numerical algorithm based on finite-difference method and the least-squares scheme for solving the inverse problem. To regularize the resultant ill-conditioned linear system of equations, we apply the Tikhonov regularization method with L-curve scheme to obtain the stable numerical approximation to the solution.

Keywords: Inverse heat conduction problem, Finite difference method, Consistency, Stability, Least-square method, Regularization method.

1. Introduction

To date, various methods have been developed for the analysis of the inverse problems and inverse heat conduction problems involving the estimation of temperature and heat flux by measuring temperature inside the material [3-8].

This paper seeks to determine an unknown function in the IHCP. By using a sensor located at a point inside the body and measuring the temperature at a point $x = x_1$, $0 < x_1 < 1$, and applying finite difference method to the IHCP, we determine a stable numerical solution to the problem.

The plan of this paper is as follows: In section 2, we formulate a one-dimensional IHCP. In section 3, The finite difference method is used to discretize IHCP. The least-squares method and the Tikhonov regularization method with L-curve scheme will be discussed in section 4. Finally a numerical experiment will be given in section 5.

2. Formulation of an IHCP

In this section, let us consider the following IHCP

$$U_t(x,t) = U_{xx}(x,t), \quad 0 < x < 1, \ 0 < t < T, \tag{1}$$

$$U(x,0) = f(x), \quad 0 < x < 1, \tag{2}$$

$$U(0,t) = p(t), \quad 0 < t < T, \tag{3}$$

$$U(1,t) = q(t), \quad 0 < t < T, \tag{4}$$

and the over-specified condition

$$U(x_1,t) = \phi(t), \quad 0 < x_1 < 1, \ \ 0 < t < T. \tag{5}$$

where T is a given positive constant, $p(t)$ and $\phi(t)$ are infinitely differentiable, while the temperature $U(1,t) = q(t)$ is unknown which remain to be determined.

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[1] pourgholi@dubs.ac.ir
[2] rostamiyan@dubs.ac.ir

For unknown function $q(t)$ we use an additional information (5) to provide a solution to the inverse problem (1)-(5).

3. Overview of the numerical method

In order to solve the problem (1)-(5) numerically, let $q(t)$ is known then we use O'Brien et al. [7] implicit finite difference formula for the equation (1) which is in the form

$$-ru_{i-1,j+1}+(1+2r)u_{i,j+1}-ru_{i+1,j+1}=u_{i,j},\ i=1,2,...,N-1,\ j=1,...,M, \tag{6}$$

where $x=i\Delta x, t=j\Delta t, r=\frac{\Delta t}{(\Delta x)^2}$, $N\Delta x=1$, and $M\Delta t=T$. The equation (6) for $i=1,2,...,N-1$, can be written as

$$AX=D, \tag{7}$$

where

$$A=\begin{pmatrix} (1+2r) & -r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -r & (1+2r) & -r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -r & (1+2r) & -r & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & -r & (1+2r) & -r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -r & (1+2r) & -r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r & (1+2r) \end{pmatrix},$$

$$X^T=\begin{pmatrix} u_{1,j+1} & u_{2,j+1} & \dots & u_{N-2,j+1} & u_{N-1,j+1} \end{pmatrix},$$

$$D^T=\begin{pmatrix} u_{1,j}+rp((j+1)\Delta t) & u_{2,j} & \dots & u_{N-2,j} & u_{N-1,j}+rq((j+1)\Delta t) \end{pmatrix}.$$

If there are $(N-1)$ internal mesh point along each time then for $i = 1,...,N-1$, equation (6) given $(N-1)$ unknown pivotal values along the $(j+1)$th row in terms of known initial and boundary values.

Theorem 1. If U is the exact solution of problem (1)-(4) and u is the exact solution of finite-difference equation (6), then the discretization error $e_{i,j}=U_{i,j}-u_{i,j}$ tends to zero as $N\to\infty$.

Proof. Putting $e=U-u$, then by substituting $u_{i,j}=U_{i,j}-e_{i,j}$ into (6) and Taylor's theorem we obtain,

$$re_{i-1,j+1}-(1+2r)e_{i,j+1}+re_{i+1,j+1}+$$
$$k\left[\left(\frac{\partial U}{\partial t}\right)(x_i,t_j+\theta_3 k)-\left(\frac{\partial^2 U}{\partial x^2}\right)(x_i+\theta_4 h,t_{j+1})\right]=-e_{i,j}, \tag{8}$$

where $0<\theta_1<1$, $0<\theta_2<1$, $0<\theta_3<1$ and $-1<\theta_4<1$. Now let E_j denote the maximum value of $|e_{i,j}|$ along the jth time-row and $\hat{M}$ is the maximum modulus of the expression in the bracts for all i and j. Then, from (8) we conclude that

$$(1+2r)|e_{i,j+1}|-r|e_{i-1,j+1}|-r|e_{i+1,j+1}|\le|e_{i,j}|+k\hat{M}.$$

As this is true for all values of i it is true for $\max|e_{i,j}|$. Hence

$$E_{j+1} \le E_0 + jk\tilde{M} = jk\tilde{M},$$

Because the initial values for u and U are the same, i.e. , $E_0 = 0$. when h tends to zero, then E_j is zero. □

Therefore the difference scheme (6) for $i = 1, 2, ..., N-1,$ is consistent.

Theorem 2. The finite difference scheme (6); $i = 1, 2, ..., N-1,$ unconditionally stable.

Proof. From equation (6) for $i = 1, 2, ..., N-1,$ we obtain

$$X = A^{-1}D. \qquad (9)$$

Therefore difference scheme (6) for $i = 1, 2, ..., N-1,$ will be stable when the modulus of every eigenvalue of A^{-1} does not exceed one. The eigenvalues of A^{-1} is,

$$\lambda = \left(1 + 4r\sin^2\left(\frac{s\pi}{2N}\right)\right)^{-1}, \; s = 1, 2, ..., N-1.$$

Hence the modulus of every eigenvalue of A^{-1} does not exceed one. □

The LU-Decomposition algorithm is used to solving equation (7), and the solution

$$X^T = \left(u_{1,j+1} \quad u_{2,j+1} \quad \ldots \quad u_{N-1,j+1}\right),$$

will be obtained. In this work the polynomial form proposed for the unknown $q(t)$ before performing the inverse calculation. Therefore $q(t)$ approximated as

$$q(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_\gamma t^\gamma, \qquad (10)$$

where $\{a_0, a_1, ..., a_\gamma\}$ are constants which remain to be determined simultaneously for each interval.

4. Least-squares minimization technique and the Tikhonov regularization method with L-curve scheme

The estimated coefficients a_i can be determined by using least squares method when the sum of the squares of the deviation between the calculated $u_{v,j+1}$ and the measured $\phi((j+1)k)$ at $x = x_1 = v\Delta x$ is less than a small number such as 0.001. The error in the estimates $E(a_0, a_1, ..., a_\gamma)$ can be expressed as

$$E(a_0, a_1, ..., a_\gamma) = \sum_{j=0}^{M-1}\left(u_{v,j+1} - \phi((j+1)k)\right)^2, \qquad (11)$$

which is to be minimized.

To obtain the minimum value of E with respect to a_i, differentiation of E with respect to a_i will be performed. Thus to minimize E one has to solve the following system

$$\begin{cases} \dfrac{\partial E}{\partial a_0} = 0 \\ \vdots \\ \dfrac{\partial E}{\partial a_\gamma} = 0. \end{cases} \qquad (12)$$

In matrix form, the values of a_i, can be obtained from solving the following matrix equation

$$\Lambda\theta = B, \qquad (13)$$

where Λ is a $(\gamma+1)\times(\gamma+1)$ square matrix, and

$$\theta^T = \begin{pmatrix} a_0 & a_1 & \dots & a_\gamma \end{pmatrix}.$$

Mathematically, IHCPs belong to the class of ill-posed problems, i.e. small error in measured data can lead to large deviations in the estimated quantities. The physical reason for the ill-posedness of the estimation problem is that variations in the surface conditions of the solid body are damped towards the interior because of the diffusive nature of heat conduction. As a consequence, large-amplitude changes at the surface have to be inferred from small-amplitude changes in the measurements data. Errors and noise in the data can therefore be mistaken as significant variations of the surface state by the estimation procedure. Since the matrix Λ is ill-conditioned, the solution of equation (13) can be corrupted by an amplified propagation of the data noise, so that regularization methods must be used for controlling this noise propagation. In our computation we adapt the Tikhonov regularization method [1] to solve the matrix equation (13). The Tikhonov regularized solution θ_α is defined to be the solution to the following least square problem

$$\min_\theta \left\{ \|\Lambda\theta - B\|^2 + \alpha^2 \|\theta\|^2 \right\}, \tag{14}$$

where $\|.\|$ denotes the usual Euclidean norm and α is called the regularization parameter.

Note. The Tikhonov regularized solution to the system of linear algebraic equation $\Lambda\theta = B$ is given by

$$\theta_\alpha : f_\alpha(\theta_\alpha) = \min_\theta f_\alpha(\theta),$$

Where f_α represents the zeroth order Tikhonov functional given by

$$f_\alpha(\theta) = \|\Lambda\theta - B\|^2 + \alpha^2 \|\theta\|^2.$$

Solving $\nabla f_\alpha(\theta) = 0$ with respect to θ, then we obtain, the Tikhonov regularized solution of the regularized equation

$$\left(\Lambda^T\Lambda + \alpha^2 I\right)\theta_\alpha = \Lambda^T B.$$

Definition. Let $A \in R^{m\times n}$ be a matrix and let $m \geq n$, then the singular value decomposition (SVD) of A is defined by

$$A = U\begin{pmatrix} S \\ 0 \end{pmatrix}V^T,$$

where $U = \begin{pmatrix} u_1 & \dots & u_m \end{pmatrix} \in R^{m\times m}$, and $V = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \in R^{n\times n}$, are orthogonal matrix with orthogonal columns, and u_i, v_i are the left and right singular vectors of A respectively, and $S \in R^{n\times n}$ is a diagonal matrix that diagonal elements appearing in non-increasing order of nonnegative singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$ In case $m < n$ then define the SVD from A^T.

Now for $\Lambda\theta = B$,

$$\Lambda = \sum_{i=1}^{\gamma+1} u_i \sigma_i v_i^T,$$

$$\theta = \sum_{i=1}^{\gamma+1} \left(v_i^T\theta\right) v_i,$$

and

$$B = \sum_{i=1}^{\gamma+1} \left(u_i^T B \right) u_i .$$

Therefore the Tikhonov solution can be formulated as

$$\theta_\alpha = \left(\Lambda^T \Lambda + \alpha^2 I \right)^{-1} \Lambda^T B = \sum_{i=1}^{\gamma+1} \frac{\sigma_i^2}{\sigma_i^2 + \alpha^2} \frac{u_i^T B}{\sigma_i} v_i .$$

In our computation we use the L-curve scheme to determine a suitable value of α [2]. The L-curve method is sketched in the following form,

$$L = \left\{ \left(\log\left(\|\theta_\alpha\|^2 \right) \right), \left(\log\left(\|\Lambda \theta_\alpha - B\|^2 \right) \right),\ \alpha > 0 \right\} \tag{15}$$

The curve is known as L-curve and a suitable regularization parameter α corresponds to a regularized solution near the, corner, of the L-curve [9].

5. Numerical results and discussion

In this section, we are going to illustrate numerically, some of the results for unknown boundary condition in the inverse problem (1)-(5). All the computation are performed on the PC (Pentium(R) 4 CPU 3.20 GHz).

Example 1. In this example let us consider the following inverse problem

$$U_t(x,t) = U_{xx}(x,t), \quad 0 < x < 1,\ 0 < t < T , \tag{16}$$

$$U(x,0) = \cos x , \quad 0 \le x \le 1, \tag{17}$$

$$U(0,t) = \exp(-t), \quad 0 \le t \le T , \tag{18}$$

$$U(1,t) = q(t), \quad 0 \le t \le T , \tag{19}$$

with the overspecified conditions

$$U(0.5,t) = \exp(-t)\cos(0.5), \quad 0 \le t \le T . \tag{20}$$

The exact solution of this problem is

$$U(x,t) = \exp(-t)\cos(x),$$

and

$$q(t) = \exp(-t)\cos(1).$$

In this example $q(t)$ approximated by

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 , \tag{21}$$

where $\{a_0, \ldots, a_5\}$ are constants, $A \in R^{6\times 6}$ and the singular values of A are $\sigma_1 = 6.4871,\ \sigma_2 = 0.6919,\ \sigma_3 = 0.0300,\ \sigma_4 = 0.0006,\ \sigma_5 = 0.0000,\ \sigma_6 = 0.0000.$

Table 1, shows the values of $q(j\Delta t)$ and $U(0.7, j\Delta t)$ in $t = j\Delta t$ when $\Delta t = \frac{1}{10}$ and $j = 1,\ldots,10.$

Table 1. The values of $q(j\Delta t)$ and $U(0.7, j\Delta t)$ in $t = j\Delta t$.

j	Numerical $q(j\Delta t)$	Exact $q(j\Delta t)$	Numerical $U(0.7, j\Delta t)$	Exact $U(0.7, j\Delta t)$
1	0.4768	0.4889	0.6894	0.6921
2	0.4329	0.4424	0.6233	0.6262
3	0.3923	0.4003	0.5641	0.5666
4	0.3549	0.3622	0.5105	0.5127
5	0.3210	0.3277	0.4619	0.4639
6	0.2903	0.2965	0.4179	0.4198
7	0.2627	0.2683	0.3781	0.3798
8	0.2379	0.2428	0.3421	0.3437
9	0.2153	0.2197	0.3096	0.3110
10	0.1946	0.1988	0.2801	0.2814

Figure 1 and 2 show the comparison between the exact result and the present numerical results for $q(j\Delta t)$ and $U(0.7, j\Delta t)$ respectively where $j = 1,...,10$.

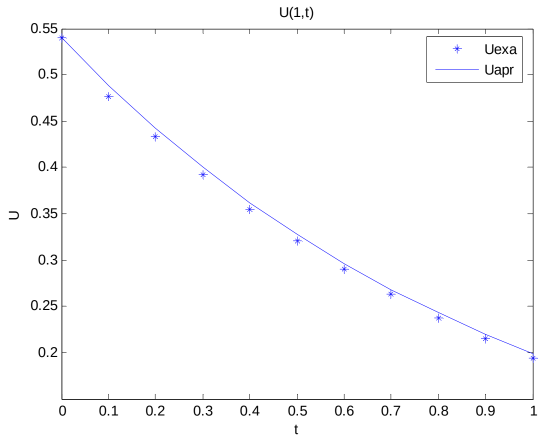

Figure 1. Comparison between the exact results and the present numerical results.

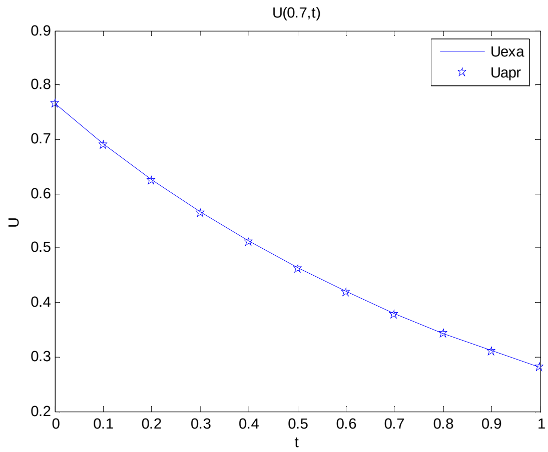

Figure 2. Comparison between the exact results and the present numerical results.

Example 2. In this example let us consider the following inverse problem

$$U_t(x,t) = U_{xx}(x,t), \quad 0 < x < 1,\ 0 < t < T, \tag{22}$$

$$U(x,0) = x^2 + \sin x, \quad 0 \le x \le 1, \tag{23}$$

$$U(0,t) = 2t, \quad 0 \le t \le T, \tag{24}$$

$$U(1,t) = q(t), \quad 0 \le t \le T, \tag{25}$$

with the overspecified conditions

$$U(0.4,t) = 0.16 + 2t + \exp(-t)\sin(0.4), \quad 0 \le t \le T. \tag{26}$$

The exact solution of this problem is

$$U(x,t) = x^2 + 2t + \exp(-t)\sin(x),$$

$$q(t) = 1 + 2t + \exp(-t)\sin(1).$$

In this example $q(t)$ approximated by

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5, \tag{27}$$

where $\{a_0, \ldots, a_5\}$ are constants, $A \in R^{6\times 6}$ and the singular values of A are $\sigma_1 = 3.9834$, $\sigma_2 = 0.4045$, $\sigma_3 = 0.0169$, $\sigma_4 = 0.0003$, $\sigma_5 = 0.0000$, and $\sigma_6 = 0.0000$.

Table 2, shows the values of $q(j\Delta t)$ and $U(0.7, j\Delta t)$ in $t = j\Delta t$ when $\Delta t = \frac{1}{10}$ and $j = 1, \ldots, 10$.

Table 2. The values of $q(j\Delta t)$ and $U(0.7, j\Delta t)$ in $t = j\Delta t$.

	Numerical	Exact	Numerical	Exact
j	$q(j\Delta t)$	$q(j\Delta t)$	$U(0.7, j\Delta t)$	$U(0.7, j\Delta t)$
1	1.9533	1.9614	1.2712	1.2729
2	2.0831	2.0899	1.4157	1.4174
3	2.2186	2.2234	1.5658	1.5672
4	2.3597	2.3641	1.7206	1.7218
5	2.5063	2.5104	1.8796	1.8807
6	2.6580	2.6618	2.0425	2.2099
8	2.9751	2.9781	2.3786	2.3795
9	3.1395	3.1421	2.5512	2.5519
10	3.3069	3.3096	2.7262	2.7270

Figure 3 and 4 show the comparison between the exact result and the present numerical results for $q(j\Delta t)$ and $U(0.7, j\Delta t)$ respectively where $j = 1,...,10.$

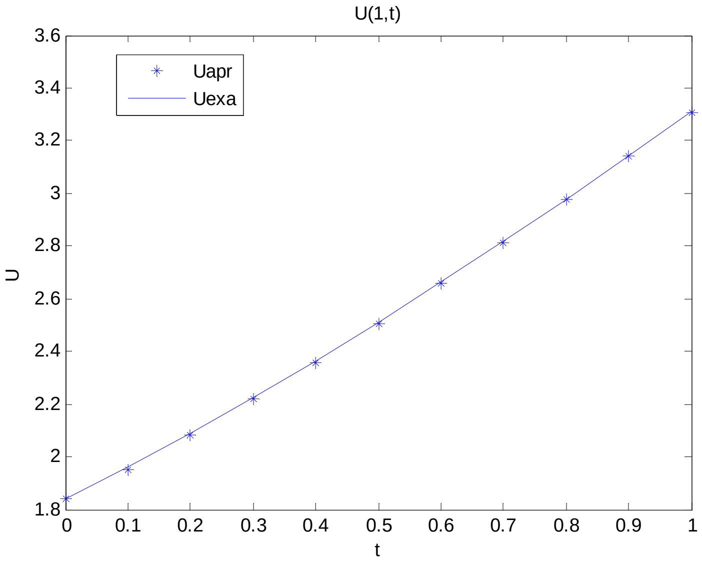

Figure 3. Comparison between the exact results and the present numerical results.

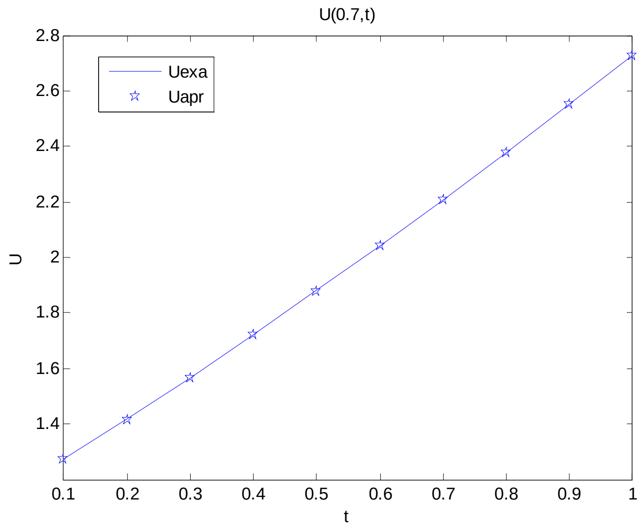

Figure 4. Comparison between the exact results and the present numerical results.

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