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Computing the Solution of the Shrödinger Equation with High-Order Dispersion Term on Type-2 Turing Machines

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Abstract. We study computability of the solution operators of the initial-value problem for the nonlinear fourth-Schrödinger equationwith both high-order dispersion term. The computayions are performed on Type-2 turing machines.

Keywords: Computability; Type-2theory of effectivity; Initial value problem; Fourth-order Schrödinger equation

1. Introduction

This paper is concerned with the following Cauchy problem of the fourth-order nonlinear Schrödinger equation

$$i\frac{du}{dt} = \Delta^2 u + |u|^2 u\ ,\ \ u(0) = \varphi \tag{1}$$

where φ is a given function.

The equation (1) is a natural extension of the nonlinear Schrödinger equation

$$i\frac{du}{dt} = -\Delta u + mu + |u|^2 u\ . \tag{2}$$

The equation (1) arises in the study of solitons in magnetic materials. It has been used as a model to investigate the role played by higher-order dispersion terms in formation and propagation of solitary waves in magnetic materials where the effective quasi-particle mass becomes infinite (see [4, 5, 6, 8]). Fibich et al. have investigated global existence of solutions of this equation in the class $C(\mathbb{R};H^2(\mathbb{R}))$ by using the conservation laws of this equation (see [3]). Local and global well-posedness of the Cauchy problem of this equation in Sobolev spaces $H^s(\mathbb{R}^n)$ was recently made by Cui and Guo [2]. Klaus Weihraucha and Ning Zhong [10] have proved that the solution operators of the initial-value problems for the nonlinear Schroinger equation (2) are computable if the initial data are Sobolev functions.

In this paper we want to consider Turing computability of the solution operators of (1). Although the approach used in this article is similar with [10], the construction is more intricate because the order of (1) is higher than (2). In addition, the domain of the initial-value of the two equations are different, the initial-value of (2) is defined in $H^1(\mathbb{R})$ and (1) in $H^2(\mathbb{R})$, while $H^2(\mathbb{R}) \subset H^1(\mathbb{R})$. From [10], the solution operator of the initial-value for the (2) is computable in the $H^1(\mathbb{R})$-setting, but in the article we shall prove that the solution operator of the initial-value for the (1) is computable in the $H^2(\mathbb{R})$-setting. Based on this, the computability of the solution operator of (1) seems to be "weaker" than (2).

In order to prove the main theorem, we first obtain a proposition of the solution of (1) by using some technical lemmas, which is devoted to study contraction operators. Second, by the contraction mappings and the computable functions constructed, we extend the computability of the solution operator from the internal to the entire space.

As for the computational model, we use Type-2 theory of effectivity (TTE for short), developed by

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Weihrauch [9] and others. TTE is a Turing machine-based framework for the computability theory of the spaces with up to continuum cardinality. The main idea of TTE is the representation of any objects by the finite or infinite sequences and then reduces the computability of the objective space to the computability of the Cantor space which is well defined by means of (type-2) Turing machine.

2. Preliminaries

We now give a brief introduction to TTE. For details the reader is referred to the textbook [9]. It is well known that, the computability on the set §¤ of all finite strings can be defined by Turing machine, where§ is an arbitrary alphabet containing at least 0 and 1. By a coding $v:\Sigma^* \to X$ the computability on any countable set X can be defined naturally. For example, for the set $\mathbb{N}$ of all natural numbers, we can apply the standard coding $v_{\mathbb{N}}$ defined by $v_{\mathbb{N}}(1^{(n+1)}) = n$.

Notice that, the classical Turing machine accepts finite strings as inputs, and, if it halts in finitely many steps, outputs a finite strings as well. By allowing the infinite sequences as inputs and outputs, the classic Turing machine is extended into so-called type-2 Turing machine by Weihrauch [9]. In this way, the computability on the Cantor space (Σ^{∞},τ) which consists of infinite sequences can be introduced naturally. For convenience, Type-2 Turing machines are simply called Turing machines.

For any set X which has at most continuum cardinality, the computability on the objects of X can be defined by means of the representations. Here a representation of X is simply a surjective map $\delta:\Sigma^{\omega} \to X$, where Σ^{ω} is the set of infinite sequences over Σ with the product topology (also called the Cantor topology). The subset sign "$\subseteq$"indicates that the map might be partial. For any $x \in X$, if $\delta(p) = x$, then p is called a δ-name (or δ-code) of x . The pair (X,δ) is called a represented space. Through a representation computations on X can be defined by means of computations on Σ^{ω} , which are explicitly executable on Turing machines: A function $\psi :\subseteq \Sigma^{\omega} \to \Sigma^{\omega}$ is called computable if there exists a type-2 Turing machine that computes and transforms each sequence p in the domain of ψ , written on the input tape, into the corresponding sequence $\psi(p)$, written on the one-way output tape. A fundamental fact regarding the Turing computability on Σ^{ω} is that if is computable, then it is continuous with respect to the Cantor topology. Based on Turing computability on ψ , the notion of computable functions on represented spaces can be defined formally as follows.

Definition 2.1 (Weihrauch [9]). Let (X,δ) and (Y,δ') be represented spaces. A function $f :\subseteq X \to Y$ is called (δ,δ') –computable if there exists a computable function $\psi :\subseteq \Sigma^{\omega} \to \Sigma^{\omega}$ such that $f \circ \delta(p) = \delta' \circ \psi(p)$ for all $p \in dom(f \circ \delta)$.

If ψ is merely a continuous function, then f is called (δ,δ') –continuous. As a fundamental fact, f is (δ,δ') –continuous if it is (δ,δ') –computable. When X and Y are topological T_0 -spaces with countable bases, f is continuous if and only if f is (δ,δ') –continuous, provided that δ and δ' are admissible representations. For details on admissible representations, the reader is referred to [9, 7].

For any represented spaces (X,δ) and (Y,δ') with admissible δ and δ' , there is a canonical admissible representation $[\delta \to \delta']$ of $C(X;Y)$, the set of all continuous functions from X to Y (see[9, 7]). This function space representation $[\delta \to \delta']$ admits evaluation and type conversion.

Lemma 2.2 (Evaluation and type conversion, Weihrauch [9]).

1. (Evaluation): The evaluation function $(f,x) \mapsto f(x)$ is $([\delta \to \delta'],\delta,\delta')$ –computable.

2. (Type conversion): Let $\delta_i :\subseteq \Sigma^{\omega} \to X_i$ be a representation of the set $X_i, 0 \le i \le k$ .Let $f: X_0 \times \cdots \times X_0 \to X_0$ and define $F(x_1,\cdots,x_{k-1})(x_k) := f(x_1,\cdots,x_k)$ .Then f is $(\delta_1,\cdots,\delta_k,\delta_0)$ –computable (–continuous), iff F is $(\delta_1,\cdots,\delta_{k-1},[\delta_k \to \delta_0])$ –computable (–continuous).

For the represented space (X,δ) , the choice of the representation δ is very important. Different representations induce different notion of "computability" of which are not always natural. If we consider only the metric spaces, there is a widely accepted and natural representation which is called Cauchy representation. In order to define this representation precisely, let's recall the computable metric spaces at first.

Definition 2.3. (Weihrauch [9]). A computable metric space is a tuple (X,d,D,α) such that

1. $d: X\times X\to\mathbb{R}$ is a metric on X and D is a dense set in X,
2. $\alpha :\subseteq \Sigma^*\to D$ is a notation with recursive domain,
3. the restriction of d to $D\times D$ is (α,α,ρ)–computable.

For a computable metric space (X,d,D,α), the canonical Cauchy representation $\delta_X :\subseteq \Sigma^\omega\to X$ is defined as follows: $\delta_X(p)=x \Leftrightarrow p=\langle\omega_0,\omega_1,\omega_2,\cdots\rangle$, $\omega_i\in dom(\alpha)$ such that

$$d(x,\alpha(\omega_i))\le 2^{-i} \text{ (for all } i\in\mathbb{N}\text{)}.$$

Thus, from the definition, a Cauchy name of an element x in a computable metric space is a coded sequence over a chosen countable dense set that converges to x rapidly. If (X,δ) and (Y,δ) are two Cauchy represented spaces, then a function $f: X\to Y$ is (δ,δ')–computable if there is a Turing machine that transforms a sequence of approximations to x to a sequence of approximations to $f(x)$. In this sense, algorithms in TTE are"approximating" algorithms. The Cauchy representation is admissible.

For example, $(L^2(\mathbb{R}),d_{L^2},\sigma,v_{L^2})$ is a computable metric space associated with the set $L^2(\mathbb{R})$ of all L^2-function, where $d_{L^2}=\|f-g\|=(\int_{\mathbb{R}}|f-g|^2dx)^{1/2}$, v_{L^2} is a canonical notation of σ consisting of all rational finite step functions. Then the Cauchy representation δ_{L^2} is defined as follows: for any $g\in L^2(\mathbb{R})$ and $p=\omega_0\#\omega_1\#\omega_2\#\cdots$, $\delta_{L^2}(p)=g$ if the sequence $\{v_{L^2}(\omega_k)\}$ converges to g rapidly. Then, an admissible representation δ_{H^s} of the Sobolev space $H^s(\mathbb{R})$ consisted of all functions $f\in L^2(\mathbb{R})$ such that $T_s(f)\in L^2(\mathbb{R})$ equipped with the norm $\|f\|'_{H^s}=\|T_s(f)\|$,where $T_s(f)(\xi):=(1+|\xi|^2)^{s/2}F(f)(\xi)$ is a weighted Fourier transform of f, is defined by $\delta_{H^s}(p):=T_s^{-1}\circ\delta_{L^2}(p)$. When $s=k$ is a non-negative integer, the norm $\|\cdot\|'_{H^k}$ is equivalent to the norm $\|\cdot\|_{H^k}$ defined as

$$\|f\|_{H^k}=(\|f\|^2_{L^2}+\|f'\|^2_{L^2}+\cdots+\|f^{(k)}\|^2_{L^2})^{1/2}.$$

Another example of a computable metric space is $(C^\infty(\mathbb{R}),d_c,P,v^P)$, where $C^\infty(\mathbb{R})$ denote the space of complex-valued function of class $C^\infty(\mathbb{R})$ equipped with the strong topology and the metric defined by

$$d_c(\phi,\varphi)=\sum_{\alpha,\beta=0}^{\infty}2^{-<\alpha,\beta>}\frac{\|\phi-\varphi\|_{\alpha,\beta}}{1+\|\phi-\varphi\|_{\alpha,\beta}}\qquad\forall\alpha,\beta\in S(\mathbb{R}),$$

P is the set of polynomials with rational coefficients, and v^P is a canonical notation of P. $v_{\mathbb{N}}$ is a canonical notation of $\mathbb{N}$. $<\alpha,\beta>:=\beta+(\alpha+\beta)(\alpha+\beta+1)/2$ is the bijective Cantor pairing function. We use δ^P_∞ to denote its Cauchy representation. Thus, we can obtain an admissible representation δ_s defined by

$\delta_s(<q,p>)=\phi\Leftrightarrow\delta^P_\infty(p)=\phi$ and $q=u_0\#u_1\#u_2\#\cdots$,where $u_k\in dom(v_{\mathbb{N}})$ and $\sup_{|x|\ge v_{\mathbb{N}}(u_{<i,j,n>})}\left|x^i\phi^{(j)}\right|\le 2^{-n}$

of the Schwardtz space $S(\mathbb{R})$ of all functions $\phi\in C^\infty(\mathbb{R})$, such that $\sup_{x\in\mathbb{R}}\left|x^i\phi^{(j)}\right|<\infty$, although it is not a Cauchy representation. Therefore, we can define another representation $\tilde{\delta}_{H^s}$ of $H^s(\mathbb{R})$ as follows : for any $f\in H^s(\mathbb{R})$ and any infinite tuple $p=<p_0,p_1,p_2,\cdots>\in\Sigma^\omega$, $\tilde{\delta}_{H^s}(p)=f\Leftrightarrow p_i\in dom(\delta_s)$ and $\|\delta_s(p_i)-f\|_{H^s}\le 1/2^i$. Moreover, if $s\in\mathbb{R}$ is computable, the $\delta_{H^s}=\tilde{\delta}_{H^s}(p)$ (Lemma 2.12 of [11]).

However, we need to list some Lemmas (Lemma 5.7 of [12], Lemma 3.7 [10]) for convenience.

Lemma 2.4. (Weihrauch and Zhong)

1. On $S(\mathbb{R})$, the function $(a,\psi)\mapsto a\psi$ is (ρ,δ_S,δ_S)–computable; the absolute evaluation $(\psi,t)\mapsto|\psi(t)|$ is (δ_S,ρ,ρ)–computable; the addition $(\phi,\psi)\mapsto\phi+\psi$ and the multiplication $(\phi,\psi)\mapsto\phi\cdot\psi$ are $(\delta_S,\delta_S,\delta_S)$–computable.

2. The function $(\psi,t)\mapsto E(t)\cdot\psi$, $E(t)\xi:=e^{-i\xi^2(\xi^2+1)t}$ is (δ_S,ρ,δ_S)–computable.

3.The Fourier transform $F: S(\mathbb{R})\to S(\mathbb{R})$, $\phi\mapsto(2\pi)^{-1/2}\int_{\mathbb{R}}e^{-i\xi x}\phi(x)dx$ and the inverse Fourier transform $F^{-1}: S(\mathbb{R})\to S(\mathbb{R})$, $\phi\mapsto(2\pi)^{-1/2}\int_{\mathbb{R}}e^{i\xi x}\phi(\xi)d\xi$ are both (δ_S,δ_S)–computable.

Lemma 2.5. (Weihrauch and Zhong[12,10]) The function $H:C(\mathbb{R};S(\mathbb{R}))\times\mathbb{R}\times\mathbb{R}\to S(\mathbb{R})$, $H(u,a,b)=\int_a^b u(t)dt$ is $([\rho\to\delta_S],\rho,\rho,\delta_S)$–computable.

3. Some Technical Lemmas

In this section, we will recall some technical lemmas and also prove some new lemmas related to the the solution of the fourth-order nonlinear Schrödinger equation. Let $\|\cdot\|$ denote the L_2-norm and $\|\cdot\|_\infty$ the L_1-norm.

If $v\in H^1(\mathbb{R})$, then we can obtain the following two lemmas.

Lemma 3.1. (The Sobolev inequality). For any $v\in H^1(\mathbb{R})$,

$$\|v\|_\infty\le\frac{\sqrt{2}}{2}(\|v\|\cdot\|v'\|)^{1/2}.$$

Proof. See [3, Theorem 1, p. 167].

Lemma3.2. For any $v\in H^1(\mathbb{R})$, $\|v\|_\infty\le\frac{\sqrt{2}}{2}\|v\|_{H^1}$.

Proof. See [10, Fact 30].

If *v 2 H*2(R), by use of the energy function of the equation (1), then the two inequalities are obtained as follows.

Lemma 3.3. For any $v\in H^2(\mathbb{R})$, $\|v\|_{H^2}^2\le\|v\|^2+\|v'\|^2+E(v)$, where $E(v)=\|v''\|^2+\int_{\mathbb{R}}V(v)dx$ is the energy function and $V(z)=z^2\bar{z}^2/2$ is a real valued function.

Proof. Since $E(v)=\|v''\|^2+\int_{\mathbb{R}}V(v)dx$, $V(v)=(v^2\cdot\bar{v}^2)/2$, we obtain that $E(v)\ge\|v''\|^2$, which in turn implies that $\|v\|_{H^2}^2=\|v\|^2+\|v'\|^2+\|v''\|^2\le\|v\|^2+\|v'\|^2+E(v)$. □

Lemma 3.4. For any $v\in H^2(\mathbb{R})$, then $E(v)\le\|v'\|^2+\|v''\|^2+\|v\|_{H^2}^4$.

Poof. Since

$$\begin{aligned}\int_{\mathbb{R}}V(v(x))dx&=\int_{\mathbb{R}}\frac{|v(x)|^4}{2}dx=\frac{1}{2}\int_{\mathbb{R}}|v(x)|^2\cdot|v(x)|^2dx\\&\le\|v\|_\infty^2\int_{\mathbb{R}}|v(x)|^2dx=\|v\|_\infty^2\cdot\|v\|^2\\&\le(\frac{\sqrt{2}}{2})^2\|v\|_{H^1}^2\cdot\|v\|^2\\&\le\frac{1}{2}\|v\|_{H^2}^2\cdot\|v\|^2{}_{H^2}\le\|v\|^4{}_{H^2}\end{aligned}$$

then, $E(v)\le\|v'\|^2+\|v''\|^2+\|v\|_{H^2}^4$.

Lemma 3.5. For any $v_1,v_2\in H^2(\mathbb{R})$, if $\|v_1\|=\|v_2\|$ and $E(v_1)=E(v_2)$, then $\|v_2\|_{H^2}\le(1+\|v_1\|^2{}_{H^2})^{\frac{1}{2}}\|v_1\|_{H^2}$.

Proof.

$$\begin{aligned}\|v_2\|_{H^2}^2&\le\|v_2\|^2+E(v_2)\\&=\|v_1\|^2+E(v_1)\\&\le\|v_1\|^2+\|v_1'\|^2+\|v_1''\|^2+\|v_1\|_{H^2}^4\\&=\|v_1\|_{H^2}^2+\|v_1\|_{H^2}^4\\&=(1+\|v_1\|_{H^2}^2)\|v_1\|_{H^2}^2.\end{aligned}$$
□

Lemma 3.6. For any $u,v\in H^2(\mathbb{R})$, then $\|uv\|_{H^2}\le3\|u\|_{H^2}\|v\|_{H^2}$.

Proof. It is easy to obtain by Lemma 3.2 and the Minkowski inequality $\|f+g\|\le\|f\|+\|g\|$.

Next we prove the proposition of the solution for the equation (1). Two Facts are needed.

Lemma 3.7. If u is the solution of $iu_t = \Delta^2 u + |u|^2 u$, $u(0) = u_0$, then for any $s,t \in \mathbb{R}$, $\|u(t)\| = \|u(s)\|$ and $E(u(t)) = E(u(s))$.

Proof. See [2,3].

Lemma 3.8. If u is the solution of $iu_t = \Delta^2 u + |u|^2 u$, $t, x \in \mathbb{R}$ with $u(0) = u_0$, then for any s,t and $t_0 \in \mathbb{R}$,

$$\|U(t-s,u(s))\|_{H^2} \le (1 + \|u(t_0)\|_{H^2}^2)^{1/2} \|u(t_0)\|_{H^2} ,$$

where $U : \mathbb{R} \times S(\mathbb{R}) \to S(\mathbb{R})$ is the so-called free evolution defined by

$$U(t,\psi)(x) = (2\pi)^{-1/2} \int_R e^{i\xi x - i\xi^4 t} \cdot F(\psi)(\xi) d\xi \ . \tag{3}$$

Proof. First, by Lemma 3.7., $\|u(s)\| = \|u(t_0)\|$ and $E(u(s)) = E(u(t_0))$ .Secondly, from the definition of $U(t,\psi)$, it follows that the Fourier transform, $\hat{U}(t,\psi)$ of $U(t,\psi)$ is equal to $e^{i\xi x - i\xi^4 t} \cdot \hat{\psi}$ .Then

$$\begin{aligned} \|U(t-s,u(s))\|_{H^2} &= \left\|(1+|\xi|^2)\hat{U}(t-s,u(s))\right\|_{L^2} \\ &= \left\|(1+|\xi|^2) e^{i\xi x - i\xi^4} \hat{u}(s)\right\|_{L^2} \\ &= \left\|e^{i\xi x - i\xi^4} (1+|\xi|^2)\hat{u}(s)\right\|_{L^2} \\ &= \left\|(1+|\xi|^2)\hat{u}(s)\right\|_{L^2} \\ &= \|u(s)\|_{H^2} . \end{aligned}$$

By applying Lemma 3.5. to $u(s)$ and $u(t_0)$, we obtain the desired proposition. □

Therefore, the proposition of the solution is easy to be obtained as follows. It mainly be used to investigate the contraction mappings.

Proposition 3.9. If $u : \mathbb{R} \to H^2(\mathbb{R})$ is a solution of (1), then for any $t, t', t_0 \in \mathbb{R}$,

$$\|u(t)\|_{H^2} \le f(\|u(t_0)\|_{H^2}) \tag{4}$$

$$\|U(t-t',u(t'))\|_{H^2} \le f(\|u(t_0)\|_{H^2}) , \tag{5}$$

In particular, for any $\phi \in H^2(\mathbb{R})$ and any $t \in \mathbb{R}$, $\|U(t,\phi)\|_{H^2} \le f(\|\phi\|_{H^2})$, where $f(x) = x \cdot (1 + 4\sqrt{2}x^2)^{1/2}$.

Proof . First it follows from Lemma 3.5 and 3.7 that for any $t, t', t_0 \in \mathbb{R}$,

$$\|u(t)\|_{H^2} \le (1 + \|u(t_0)\|_{H^2}^2)^{1/2} \|u(t_0)\|_{H^2} \le (1 + 4\sqrt{2}\|u(t_0)\|_{H^2}^2)^{1/2} \|u(t_0)\|_{H^2} .$$

Then Lemma 3.8 leads to the following inequality

$$\begin{aligned} \|U(t-t',u(t'))\|_{H^2} &\le (1 + \|u(t_0)\|_{H^2}^2)^{1/2} \|u(t_0)\|_{H^2} \\ &\le (1 + 4\sqrt{2}\|u(t_0)\|_{H^2}^2)^{1/2} \|u(t_0)\|_{H^2} . \end{aligned}$$

Let $f(x) = x \cdot (1 + 4\sqrt{2}x^2)^{1/2}$, then the desired proposition is obtained. □

Please acknowledge collaborators or anyone who has helped with the paper at the end of the text.

4. The computability of the solution of the Schrödinger equation

In this section, we will show our main theorem. In the first, we consider an fixed initial condition $\varphi \in H^2(\mathbb{R})$ of problem (1)

$$i\frac{du}{dt} = \Delta^2 u + |u|^2 u , \quad u(0) = \varphi .$$

Next we shall costruct the solution of the initial-value problem (1), in terms of t_0 and $u(t_0)$, locally in a neighborhood of t_0 by virtue of the contraction-mapping principle, starting from $t_0 = 0$ .Assume that $u(t)$ has been constructed over the time interval $[0, t_0]$ .We show in the following how to extend the construction ,let us consider the following problem with the value $\omega(t_0)$ given:

$$i\frac{d\overline{\omega}}{dt} = \Delta^2 \overline{\omega} + |\overline{\omega}|^2 \overline{\omega} , \quad \omega(t_0) = \psi , \quad \psi \in H^2(\mathbb{R}) . \tag{6}$$

We use the equivalent integral equation

$$\omega(t) = U(t - t_0, \psi) - i\int_t^{t_0} U(t-\tau, |\omega(\tau)|^2 \omega(\tau)) d\tau . \tag{7}$$

We note that if $\psi = \omega(t_0)$ in (6), then by the uniqueness of the solution of (1), the solution of (7) is also the solution of (1). We will define two maps A and G . For $t_0 \in R$, the map $G(t_0)$:

$C(\mathbb{R};H^2(\mathbb{R})) \to C(\mathbb{R};H^2(\mathbb{R}))$ is defined by

$$G(t_0)(v)(t) = -i\int_t^{t_0} U(t-\tau, |v(\tau)|^2 v(\tau))d\tau \ , \tag{8}$$

and for $t_0 \in R$ and $\psi \in H^2(\mathbb{R})$, the map $A(t_0): C(\mathbb{R};H^2(\mathbb{R})) \to C(\mathbb{R};H^2(\mathbb{R}))$ is defined by

$$A(t_0,\psi)(v)(t) = U(t-t_0,\psi) + G(t_0)(v)(t) \ . \tag{9}$$

Both A and G are contraction mappings in a neighborhood of t_0 , which will be shown in the following lemma.

Lemma 4.1. Let r_φ be a rational number such that $r_\varphi - 1 \le \|\varphi\|_{H^2} \le r_\varphi$, let R_φ be the least integer upper bound of $f(r_\varphi)+1$ and let $I := [t_0, t_0 + T_\varphi]$,where $T_\varphi = 1/(100R_\varphi^2)$ .Then ,for any $\psi \in H^2(\mathbb{R})$ such that $\|\varphi\|_{H^2} \le R_\varphi$ and any $v_1, v_2 \in \{v \in C(\mathbb{R};H^2(\mathbb{R})): \|v\|_I \le \frac{4}{3}R_\varphi^2\}$,

$$\|A(t_0,\psi)(v_1) - A(t_0,\psi)(v_2)\|_I = \|G(t_0)(v_1) - G(t_0)(v_2)\|_I \le \frac{1}{2}\|v_1 - v_2\|_I \ , \tag{10}$$

where $\|v\|_I := \|v\|_{C(I;H^2(\mathbb{R}))} = \sup_{t\in\mathbb{R}} \|v(t)\|_{H^2}$ for $v \in C(\mathbb{R};H^2(\mathbb{R}))$.

Proof. Let $C_0 = 3$, then for any $v_1, v_2 \in C(\mathbb{R};H^2(\mathbb{R})$ satisfying $\|v_1\|_I \le \frac{4}{3}R_\varphi^2$ and $\|v_2\|_I \le \frac{4}{3}R_\varphi^2$, we have

$$\begin{aligned}
&\|G(t_0)(v_1)(t) - G(t_0)(v_2)(t)\|_I \\
&= \sup_{t\in I}\left\|\int_{t_0}^{t} U(t-\tau, |v_1(\tau)|^2 v_1(\tau) - |v_2(\tau)|^2 v_2(\tau))\right\|_{H^2} d\tau \\
&\le T_\varphi \sup_{t\in I, t_0\le\tau\le t} \left\|U(t-\tau, |v_1(\tau)|^2 v_1(\tau) - |v_2(\tau)|^2 v_2(\tau))\right\|_{H^2} \\
&= T_\varphi \sup_{\tau\in I}\left\||v_1(\tau)|^2 v_1(\tau) - |v_2(\tau)|^2 v_2(\tau)\right\|_{H^2} \\
&= T_\varphi \|v_1^2\overline{v}_1 - v_2^2\overline{v}_2\|_I \ ,
\end{aligned}$$

where $\overline{v}$ is the complex conjugate of v . Since

$$\begin{aligned}
&v_1^2\overline{v}_1 - v_2^2\overline{v}_2 = v_1^2\overline{v}_1 - v_2^2\overline{v}_1 + v_2^2\overline{v}_1 - v_2^2\overline{v}_2 \\
&= (v_1^2 - v_2^2)\overline{v}_1 + v_2^2(\overline{v}_1 - \overline{v}_2) \\
&= (v_1 + v_2)\overline{v}_1(v_1 - v_2) + v_2^2(\overline{v}_1 - \overline{v}_2) \ ,
\end{aligned}$$

then

$$\begin{aligned}
\|v_1^2\overline{v}_1 - v_2^2\overline{v}_2\|_I &\le C_0^2((\|v_1\|_I + \|v_2\|_I)\|\overline{v}_1\|_I\|v_1 - v_2\|_I + \|v_2\|_I^2\|\overline{v}_1 - \overline{v}_2\|) \\
&\le C_0^2(\frac{32}{9}R_\varphi^2 + \frac{16}{9}R_\varphi^2)\|v_1 - v_2\|_I = \frac{48}{9}C_0^2R_\varphi^2\|v_1 - v_2\|_I
\end{aligned}$$

Thus,

$$\begin{aligned}
\|G(t_0)(v_1)(t) - G(t_0)(v_2)(t)\|_I &\le T_\varphi\frac{48}{9}C_0^2R_\varphi^2\|v_1 - v_2\|_I \\
&= \frac{1}{100R_\varphi^2}\cdot\frac{48}{9}\cdot 9R_\varphi^2\|v_1 - v_2\|_I \\
&= \frac{12}{25}\|v_1 - v_2\|_I \prec \frac{1}{2}\|v_1 - v_2\|_I \ . \quad \square
\end{aligned}$$

Notice that the constant T_φ defined in Lemma 4.1 depends only on the size of the initial value φ of the problem (1). By the equations (7) and (10), the fixed point of the contraction $A(t_0,\psi)$ is the solution of (1) over the time interval I satisfying $\omega(t_0) = \psi$. Thus, we can compute the solution with the initial data from $H^2(\mathbb{R})$ as long as we can compute A on $H^2(\mathbb{R})$. The following lemma shows that the restriction of A on the Schwartz space $S(\mathbb{R})$, a subset of $H^2(\mathbb{R})$, is computable. This restriction will also be denoted as A .

Lemma 4.2. The restriction of the operator A to $S(\mathbb{R})$:

$$\mathbb{R}\times S(\mathbb{R})\times C(\mathbb{R};S(\mathbb{R})) \to C(\mathbb{R};S(\mathbb{R})) \ , t_0,\psi,v \mapsto A(t_0,\psi)(v) \ ,$$

is $(\rho,\delta_S,[\rho\to\delta_S],[\rho\to\delta_S])$ – computable.

Proof. By Lemmas 2.4 and 2.5, the function $(t_0,\psi,v,t)\mapsto A(t_0,\psi)(v)(t)$ is $(\rho,\delta_S,[\rho\to\delta_S],\rho,\delta_S)$ -computable. Applying Lemma 2.2, $t_0,\psi,v\mapsto A(t_0,\psi)(v)$ is then $(\rho,\delta_S,[\rho\to\delta_S],[\rho\to\delta_S])$ – computable.

Corollary 4.3. The map

$$F_1:(t_0,\psi,n)\mapsto (A(t_0,\psi))^n(0),$$

where $(A(t_0,\psi))^n(v)=A(t_0,\psi)((A(t_0,\psi))^{n-1}(v))$ is the nth iteration of $A(t_0,\psi)$, is $(\rho,\delta_S,v_N,[\rho\to\delta_S])$ – computble.

Proof. This is true because F_1 is a primitive recursion of the computable operator A (Lemma 4.2.). □

As known $S(\mathbb{R})$ is dense in $H^2(\mathbb{R})$, it is possible to approximate $\psi\in H^2(\mathbb{R})$ by a sequence ψ_k of $S(\mathbb{R})$-function such that $\|\psi_k-\psi\|_{H^2}\le 2^{-k}$ .The next lemma presents an algorithm to choose approximations among $A(t_0,\psi)^n(0)$ that converge to the fixed point of $A(t_0,\psi)$ rapidly.

Lemma 4.4. There are two computable functions $g_1,g_2:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that the following holds: for $\psi=u(t_0)\in H^2(\mathbb{R})$ and any sequence $\{\psi_k\}\subset S(\mathbb{R})$ satisfying $\|\psi-\psi_k\|_{H^2}\le 2^{-k}$ for $k=0,1,\cdots$,

$$\left\|v_\psi-(A(t_0,\psi_{g_2(R_\varphi,j)}))^{g_1(R_\varphi,j)}(0)\right\|_I\le 2^{-j},$$

where v_ψ is the fixed point of $A(t_0,\psi)$ in $\{v\in C(I;H^2(\mathbb{R})):\|v\|_I\le\frac{4}{3}R_\varphi^2\}$ with $I=[t_0,t_0+T_\varphi]$.

In particular, for any $t\in I$,

$$\left\|v_\psi(t)-(A(t_0,\psi_{g_2(R_\varphi,j)}))^{g_1(R_\varphi,j)}(0)(t)\right\|_{H^2}\le 2^{-j}.$$

Proof. First we prove that for any $\phi\in H^2(\mathbb{R})$ satisfying $\|\phi\|_{H^2}\le R_\varphi$, $\|(A(t_0,\phi))^n(0)\|_I\le\frac{4}{3}R_\varphi$ for all $n\in\mathbb{N}$ .For $n=1$, $\|A(t_0,\phi)(0)\|_I=\|\phi\|_{H^2}\le R_\varphi$ by assumption. Assume that $\|(A(t_0,\phi))^n(0)\|_I\le\frac{4}{3}R_\varphi$ .Denote $(A(t_0,\phi))^n(0)$ by v .Then

$$\begin{aligned}\|(A(t_0,\phi))^{n+1}(0)\|_I&=\|A(t_0,\phi)(0)\|_I\le\|\phi\|_{H^2}+T_\varphi\|v^2\bar v\|_I\\&\le R_\varphi+T_\varphi C_0^2\|v\|_I^3\le R_\varphi+\frac{1}{100R_\varphi^2}\cdot 9\cdot\frac{64}{27}R_\varphi^3\\&\prec R_\varphi+\frac{1}{3}R_\varphi=\frac{4}{3}R_\varphi.\end{aligned}$$

Thus, $\|(A(t_0,\phi))^n(0)\|_I\le\frac{4}{3}R_\varphi$ for all $n\in\mathbb{N}$, provided $\|\phi\|_{H^2}\le R_\varphi$.

Since $\psi=u(t_0)$, it follows from (5) that $\|\psi\|_{H^2}\le f(\|u(0)\|_{H^2})=f(\|\varphi\|_{H^2})\le R_\varphi$ .Then, by Lemma 13, $A(t_0,\psi)$ is a contraction on and therefore has a fixed point v_ψ.

Let $v^n:=(A(t_0,\psi))^n(0)$ .Then $\|v^n\|_I\le\frac{4}{3}R_\varphi$ for all $n\in\mathbb{N}$ .Let $v_k^n:=(A(t_0,\psi_k))^n(0)$ .Since $\|\psi-\psi_k\|_{H^2}\le 2^{-k}$,

$\|\psi_k\|_{H^2}\le\|\psi\|_{H^2}+\|\psi-\psi_k\|_{H^2}\le f(\|u(0)\|_{H^2})+2^{-k}\le f(\|u(0)\|_{H^2})+1\le R_\varphi$, and consequently $\|v_k^n\|_I\le\frac{4}{3}R_\varphi$ for all $n\in\mathbb{N}$.

We observe that

$$\|v_\psi-v_k^n\|_I\le\|v_\psi-v^n\|_I+\|v^n-v_k^n\|_I. \tag{11}$$

Also we recall that for any function h on a Banach space with $\|h(x)-h(y)\|\le\frac{1}{2}\|x-y\|$, the fixed point χ_h of h satisfies $\|\chi_h-h^n(0)\|\le\|h(0)\|\cdot 2^{-n+1}$. Thus, by applying (5) we obtain

$$\begin{aligned}\|v_\psi-v^n\|_I&\le 2^{-n+1}\|A(t_0,\psi)(0)\|_I\\&\le 2^{-n+1}\|U(t-t_0,\psi)\|_I\\&\le 2^{-n+1}f(\|\varphi\|_{H^2})\le 2^{-n+1}R_\varphi.\end{aligned}$$

Next we show by induction that $\|v^n-v_k^n\|_I\le n\cdot f(2^{-k})$. This is true for $n=0$ .Applying the special case of (5) with $\phi=\psi-\psi_k$ and Lemma 13, we obtain

$$\begin{aligned}\left\|v^{n+1}-v_k^{n+1}\right\|_I &= \left\|A(t_0,\psi)(v^n)-A(t_0,\psi_k)(v_k^n)\right\|_I \\ &\le \left\|U(t-t_0,\psi)-U(t-t_0,\psi_k)\right\|_I+\left\|G(t_0,v^n)-G(t_0,v_k^n)\right\|_I \\ &\le \left\|U(t-t_0,\psi-\psi_k)\right\|_I+\frac{1}{2}\left\|v^{n+1}-v_k^{n+1}\right\|_I \\ &\le f(\left\|\psi-\psi_k\right\|_{H^2})+\frac{1}{2}n\cdot f(2^{-k}) \\ &\le f(2^{-k})+n\cdot f(2^{-k}) \\ &\le (n+1)\cdot f(2^{-k})\,.\end{aligned}$$

Now define

$$g_1(R_\varphi,j):=\mu n[2^{-n+1}R_\varphi\le 2^{-j-1}]\,,$$

$$g_2(R_\varphi,j):=\mu k[g_1(R_\varphi,j)\cdot f(2^{-k})\le 2^{-j-1}]\,.$$

The statement of the lemma follows from (11). □

We remark that both Lemma 4.1 and 4.4 hold true on the interval $[t_0-T_\varphi,t_0]$.We also recall that v_ψ is the solution of (7) satisfying $\psi=u(t_0)$;namely, $v_\varphi(t)=\omega(t)$ for any $t\in I$ .A byproduct of Lemma 4.3 is that $\{(A(t_0,\psi_{g_2(R_\varphi,j)})^{g_1(R_\varphi,j)}(0)(t)\}$ is a "$\tilde{\delta}_{H^2}$ –name" of $\omega(t)$ for all $t\in I$ because $(A(t_0,\psi_{g_2(R_\varphi,j)}))^{g_1(R_\varphi,j)}(0)(t)$ are Schwartz function and $\left\|v_\psi(t)-(A(t_0,\psi_{g_2(R_\varphi,j)}))^{g_1(R_\varphi,j)}(0)(t)\right\|_{H^2}\le 2^{-j}$ .Moreover, since g_1 and g_2 are computable, this $\tilde{\delta}_{H^2}$ – name of $\omega(t)$ is computable from t_0,ψ and t .More precisely

Corollary 4.5. The maps

$$F_+:(t_0,\psi,t)\mapsto\omega(t),t\in[t_0,t_0+T_\varphi]\,,$$

and

$$F_-:(t_0,\psi,t)\mapsto\omega(t),t\in[t_0-T_\varphi,t_0]$$

are $(\rho,\tilde{\delta}_{H^2},\rho,\tilde{\delta}_{H^2})$ –computable, where $\omega(t)$ is the solution of (8) with $\psi=u(t_0)$.

Since $\psi=u(t_0)$, $\omega(t)$ is also the solution of (1) over the time interval $[t_0,t_0+T_\varphi]$ .Thus the solution of (1) is extended from $[0,t_0]$ to $[t_0,t_0+T_\varphi]$.

Now we are ready to lay down the proof of our main result.

Theorem 4.6. The solution $S:H^2(\mathbb{R})\to C(\mathbb{R};H^2(\mathbb{R}))$, $\varphi\mapsto u$, of the initial-value problem for the nonlinear fourth-order Schrödinger equation (1) is $(\delta_{H^2},[\rho\to\delta_{H^2}])$ –computable.

Proof . We need to show how to compute the solution $u(t)$ of the initial-value problem (1) from the initial condition $\varphi\in H^2(\mathbb{R})$ and arbitrary time $t\in\mathbb{R}$ .The proof consists of the following parts.

(a) since $\varphi\mapsto\|\varphi\|_{H^2}$ is (δ_{H^2},ρ) –computabe, we can compute a rational number r_φ such that

$$r_\varphi-1\le\|\varphi\|_{H^2}\le r_\varphi$$

and then compute $R_\varphi=\mu n[f(r_\varphi+1)\le n]$ and $T_\varphi=1/(100R_\varphi^2)$ .Here $f(x)=x\cdot(1+4\sqrt{2}x^2)^{1/2}$ is defined in the proposition 3.7.

(b) Next we compute the solution $u(z\cdot T_\varphi)$ at time $z\cdot T_\varphi$ for integers $z\in\mathbb{Z}$ .Define

$$H_+(\varphi,0):=H_-(\varphi,0):=\varphi$$

and

$$H_+(\varphi,n+1):=F_+(n\cdot T_\varphi,H_+(\varphi,n),(n+1)\cdot T_\varphi)\,, \tag{12}$$

$$H_-(\varphi,n+1):=F_-(-n\cdot T_\varphi,H_-(\varphi,n),-(n+1)\cdot T_\varphi)\,, \tag{13}$$

By corollary 4.5, $H_+(\varphi,n)=u(n\cdot T_\varphi)$ and $H_-(\varphi,n)=u(-n\cdot T_\varphi)$, and in addition, $u(n\cdot T_\varphi)$ and $u(-n\cdot T_\varphi)$ are computable from n and φ because both H_+ and H_- are primitive recursions of either the function F_+ or the function F_- , which are computable according to Corollary 4.5.

(c) Finally we show how to compute $u(t)$ from φ and t .We begin by computing an integer $z\in\mathbb{Z}$ from t and T_φ such that $z\cdot T_\varphi\le t\le(z+1)\cdot T_\varphi$.

Then we compute $u(z \cdot T_\varphi)$, and further compute $F_+(z \cdot T_\varphi, u(z \cdot T_\varphi), t)$ .By Corollary 4.5, $F_+(z \cdot T_\varphi, u(z \cdot T_\varphi), t) = u(t)$. This completes the proof. □

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