ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 4, No. 1, 2009, *pp*. 026-032

Approximate Stabilization of Switched Nonlinear Systems +

Zhi-E Lou [1, 2], Yan Gao [1]

[1]School of Management, University of Shanghai for Science and Technology,200093 Shanghai,China

[2]Department of Science, Bengbu College, 233030, Bengbu, China

(*Received June 20, 2008, accepted October 20, 2008*)

Abstract. In this paper, we consider the approximate stabilization of a class of switched nonlinear system composed of a finite family of subsystems. We show that there exists a piecewise constant feedback controller such that the system can achieve the approximate stabilization property under arbitrary switching signal. We also discuss the numerical computation problem in determining the control input value at the sampling instant or switching instant and present a method to solve the problem.

Keywords: switched control systems, approximate stabilization, switched nonlinear systems

1. Introduction

Recently, switched control systems has received a great deal of attention in the control community. Informally, a switched system is a family of continuous-time dynamical subsystems and a rule that determines the switching between them. Many engineering systems, such as robot manipulators[1], traffic management[2],power systems[3,4],etc. are essentially switched systems. The reader can refer to [5,6]for a detailed study for this system.

One important problem in the theory of switched system may be stabilization. The interest in the problem is reflected by numerous works, mainly for linear systems[7,8,9,10].However, characterizing the stability and stabilization of switching among families of nonlinear systems presents a much more challenging task. Recently, the stabilization of switched nonlinear systems has also been investigated[11,12].The switched technique is implemented for the stabilization of some typical kinds of nonlinear systems[13,14]. Most existing studying results on the switched system mainly focus on seeking for conditions under which the switched system is stable for any switching signal. In this paper, however, our problem does not lie in seeking these conditions but in the construction of the feedback controller that guarantees the system is approximate stable for any switching signal.

The rest paper is organized as follows: Section 2 provides the class of switched nonlinear system we consider and presents some assumption and definition used throughout this paper. Section 3 is our main result, in this section we develop the approximate stabilization conditions and prove that there exists a piecewise constant feedback controller such that the system can be stabilized. In section 4 we discuss the numerical computation of the control input value. Concluding remarks are then followed in section 5.

2. Problem statement and preliminaries

Given the family of locally Lipschitz vector fields $D=\left\{ f_i:\mathbb{R}^n\times U\to\mathbb{R}^n : i\in\wedge\right\}$ parameterized by the index set $\wedge=\left\{1,2,...,N\right\}$, we consider the switched nonlinear control system

$$\dot{x}=f_{\sigma(t)}\left(x,u\right) \tag{1}$$

Where $x\in\mathbb{R}^n$ is the state, $u\in U\subseteq\mathbb{R}^m$ is the control input, $\sigma\left(t\right):[t_0,+\infty)\to\wedge$ is the switching signal, which is a piecewise constant function of time, taking value from the finite index set $\wedge=\left\{1,2,...,N\right\}$,and $\sigma\left(t\right)=i\ (i=1,2,\cdots N)$, means that the i-th subsystem $\dot{x}=f_i\left(x,u\right)$ works. For an

+ This work was supported by National Science foundation of China (under grant:10671126) and Shanghai leading academic discipline project(under grant:S30501).

arbitrary switching path $\sigma(t) = i_m \in \wedge$ $(t \in [t_m, t_{m+1})$, $m = 0,1,2..., s-1)$, $\{t_m\}_{m=0}^{s}$ is called the switching time sequence, which is assumed to satisfy

$$t_0 < t_1 < t_2 < \cdots < t_m < \cdots < t_s$$

If $s < \infty$, then possibly $t_s = \infty$. To facilitate the analysis, we let

$$\{x_m\}_{m=0}^{s-1} : x_0, x_1, x_2, \cdots, x_m, \cdots, x_{s-1}.$$

denote the switching state sequence corresponding to the switching time sequence $\{t_m\}_{m=0}^{s}$,i.e $\{x_m\}_{m=0}^{s-1}$ is the solution $x(\cdot)$ of system(1) .We assume here the state of system (1) does not jump at the switching instants, i.e. the solution $x(\cdot)$ is everywhere continuous. The switching state sequence, along with system (1), completely describes the trajectory of the system according to the following rule: x_m means that the system evolves according to $\dot{x} = f_{i_m}(x(t), u)$ for $t_m < t < t_{m+1}$. Throughout, we assume that $i_m \neq i_{m+1}$.

From now on we assume that D satisfies the following hypotheses.

Assumption: $f_i : \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous and has linear growth with respect to x.i.e. there exist $\lambda_i \geq 0$ such that $|f_i(x,u)| \leq \lambda_i (1 + |x|)$ for all $(x,u) \in \mathbb{R}^n \times U$. $f_i(x,u)$ is Lipschitz with the constant k_i .i.e. there exists a constant k_i such that for all $x, y \in \mathbb{R}^n$, $|f_i(x,u) - f_i(y,u)| \leq k_i |x-y|$. And for all $u \in U$, the set valued map $x \to \{f_i(x,u)\}$ is Marchaud .The set U is compact.

Let us recall that the set valued map is called Marchaud if and only if it has convex and compact values and is upper-semicontinous. A set valued map $F(\cdot) : \mathbb{R}^n \to \mathbb{R}^{2^n}$ is called upper-semicontinuous if for all $x \in \mathbb{R}^n$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that for all x' with $|x - x'| < \delta$, $F(x') \subseteq F(x) + \varepsilon B$ .Where B denote the closed unit ball in $\mathbb{R}^n$ centered at the origin.

Throughout this paper, we shall use the following definition of approximate exponential stability for the switched nonlinear systems.

Definition Given a Lyapunov function V, a time instant $t_s > t_0$, an arbitrary accuracy parameter $\varepsilon > 0$, a switching signal $\sigma(t) : [t_0, t_s] \to \wedge$ and input $u : [t_0, t_s] \to \mathbb{R}^m$,if all finite runs $x_m(t)(m = 0,1,\cdots, s-1)$ of the system(1)starting at some initial state $x_0(t_0)$ satisfy

$$V(x_m(t)) \leq e^{-c(t-t_0)} V(x_0(t_0)) + \varepsilon$$

for all $t \in [t_m, t_{m+1})$, $m = 0,1,\cdots s-1$. Then the switched nonlinear system(1) is approximate exponentially stable.

3. Main Result

In this section we derive a piecewise constant feedback controller that ensure the system (1) achieves the approximate stability property. In order to prove the subsequent main result, we first present the following proposition whose proof is postponed to the Appendix:

Proposition: Consider the switched nonlinear system of the form (1) with the switching time sequence $\{t_m\}_{m=0}^{s}$ ($t_s < \infty$), a nonempty closed set $K \subseteq \mathbb{R}^n$ and an accuracy parameter $\varepsilon > 0$. Assume that, in addition to Assumption 1, we have $\inf_{u \in U} \left(f_{\sigma(t)}(x,u)\right)^{\mathrm{T}} p \leq 0$, for all $x \in K$ and all $p \in NP_K(x)$.

Then there exists a sampling time h_m over each time horizon $[t_m, t_{m+1})$ $(m = 0,1\cdots, s-1)$ and a piecewise constant controller $g(\cdot) : \mathbb{R}^n \to U$ such that all finite runs $\{x_m\}_{m=0}^{s-1}$ of system (1) starting at some initial state $x_0\left(t_0\right)\in K$ satisfy

$$x_m\left(t\right)\in K+\varepsilon B \quad (m=0,1,\cdots s-1)\text{, for all } t\in\left[t_m,t_{m+1}\right). \tag{2}$$

where $NP_K(x)$ denotes the set of proximal normals to the set K at x : $NP_K(x)=\left\{y\in\mathbb{R}^n \mid d_K(y+x)=|y|\right\}$, recall that $d_K\left(x\right)$ denotes the distance of the point x to the set K defined by $d_K\left(x\right)=\inf_{y\in K}\left|x-y\right|$.

Remark. Proposition shows that there exists a piecewise constant controller such that the set K is viable on the system (1). The reader can refer to [15] for a detailed study of the viability property.

The following theorem is the main result of this paper.

Theorem. Consider a system of the form(1) with the switching time sequence $\left\{t_m\right\}_{m=0}^{s}$ ($t_s<\infty$), a Lyapunov function $V\left(\cdot\right)$: $\mathbb{R}^n\to\mathbb{R}$ and an accuracy parameter $\eta>0$. Assume that in addition to Assumption the following hold:

1. $V\left(\cdot\right)$ is $l-$Lipschitz continuous.

2. There exists $c\in\left[0,\infty\right)$ such that $\inf_{u\in U}\left(f_{\sigma(t)}\left(x,u\right)\right)^{\mathrm{T}}p-cqV(x)\le 0$ for all $x\in\mathbb{R}^n$ and all $\left(p,q\right)\in NP_{Epi_V}\left(x,V\left(x\right)\right)$.

Then there exists a piecewise constant controller such that all runs of the system (1) satisfy

$$V\left(x_m(t)\right)\le e^{-c(t-t_0)}V(x_0(t_0))+\eta$$

for all $m=0,1,\cdots s-1$ and all $t\in\left[t_m,t_{m+1}\right)$.

Proof : we introduce an auxiliary state variable $y\in\mathbb{R}$ into the system(1) and define the following extended system

$$\dot{\hat{x}}=\hat{f}_{\sigma(t)}\left(\hat{x},u\right)=\begin{bmatrix} f_{\sigma(t)}\left(x,u\right) \\ -cy \end{bmatrix} \tag{3}$$

where $\hat{x}=(x,y)\in\mathbb{R}^{n+1}$ is the state, the control $u\in U$, for a run starting at $x_0\left(t_0\right)$ we initialize the auxiliary variable to $y_0\left(t_0\right)=V\left(x_0\left(t_0\right)\right)$ .Then by construction $\left(x_0\left(t_0\right),y_0\left(t_0\right)\right)\in\mathrm{Epi}_V$ .Where $\mathrm{Epi}_V=\left\{\left(x,y\right)\in\mathbb{R}^{n+1}\mid V\left(x\right)\le y\right\}$ is the epigraph of V .

Since the original system(1) satisfies the Assumption ,so it is not hard to know that the extended system(3) also satisfies the assumption.

Consider arbitrary $\hat{x}=(x,y)\in\mathbb{R}^{n+1}$ and $\left(p,q\right)\in NP_{Epi_V}\left(\hat{x}\right)$. If $\hat{x}$ is on the boundary of Epi_V , then $\inf_{u\in U}\left(\hat{f}_{\sigma(t)}\left(\hat{x},u\right)\right)^{\mathrm{T}}(p,q)=\inf_{u\in U}\left(f_{\sigma(t)}\left(x,u\right)\right)^{\mathrm{T}}p-cqV(x)\le 0$. If on the other hand, $\hat{x}$ is in the interior of Epi_V then $\left(p,q\right)=0$ and the equation holds trivially. Therefore, the extended system satisfies condition of Proposition (Epi_V playing the role of K). Thanks to Proposition, we infer that there exists a piecewise constant feedback controller such that all runs of the extended system starting from $\left(x_0\left(t_0\right),y_0\left(t_0\right)\right)$ satisfy $\hat{x}_m\left(t\right)\in\mathrm{Epi}_V+\varepsilon B$, or, equivalently $d_{\mathrm{Epi}_V}\left(x_m\left(t\right),y_m\left(t\right)\right)\le\varepsilon$ for all $m=0,1,\cdots s-1$ and all $t\in\left[t_m,t_{m+1}\right)$. Therefore, there exists x_m' with $\left|x_m\left(t\right)-x'\right|\le\varepsilon$ such that $\left(V\left(x_m'\right)-y_m\left(t\right)\right)^2+\left|x_m\left(t\right)-x_m'\right|^2\le\varepsilon^2$. In particular, $V\left(x_m'\right)\le y_m\left(t\right)+\varepsilon=e^{-c(t-t_0)}V\left(x_0\left(t_0\right)\right)+\varepsilon$. Since V is $l-$Lipschitz , $V\left(x_m'\right)\ge V\left(x_m\left(t\right)\right)-l\varepsilon$. Hence, we have $V\left(x_m(t)\right)\le e^{-c(t-t_0)}V(x_0(t_0))+\varepsilon(1+l)$ .Taking $\varepsilon<(1/(1+l))\eta$, completes the proof. □

4. Numerical computation problem in determining the control input

In the previous section, we have proved the existence of a piecewise constant feedback controller in stabilization of the system(1), but how to determine the control input value at the sampling instant or switching instant? In the following, we will solve numerically the problem.

From the proof of the Theorem, we know that the stability of the system(1) is equivalent to the viability of the extended system (3) on Epi_V , so in order to stabilize the system(1),we only need to design a feedback controller such that Epi_V is viable on the extended system . Based on the proof of Proposition and the viability theory, we derive an approach to compute the control input value:

- If $\hat{x}=(x,y)\in \mathrm{Epi}_V$, then we pick an arbitrary element $\bar{u}_{mj}\in U$ and define

$$\forall t\in[t_{mj},t_{mj+1}],\ \ u=\bar{u}_{mj}$$

- If $\hat{x}=(x,y)\notin \mathrm{Epi}_V$, let us consider its unique projection z onto Epi_V ,then there exists $\tilde{u}_{mj}\in U$ such that

$$\left(\hat{f}_{im}\left(z,\tilde{u}_{mj}\right)\right)^{\mathrm{T}}\left(\hat{x}-z\right)\le 0$$

then we define

$$\forall t\in[t_{mj},t_{mj+1}],\ \ u=\tilde{u}_{mj}$$

the first case is obvious, but the second one is more complex because we need to find the projection of $\hat{x}$ onto Epi_V . In the following ,by the optimization theory, we can establish an optimization model to find the projection of $\hat{x}$.

Suppose $\left(a,b\right)\notin \mathrm{Epi}_V$ is the state of system (3) at a certain sampling instant or the switching instant, and $\left(x^*,V(x^*)\right)$ is its projection onto Epi_V . Then it is not hard to see that $\left(x^*,V(x^*)\right)$ is the solution to the following optimization problem:

$$\text{(P)}\ \min G(x)=\left\|x-a\right\|^2+\left(V(x)-b\right)^2$$

most optimization algorithms can be applied directly to solve the problem (P).

5. Concluding remark

The paper has investigated the approximate stabilization of switched nonlinear systems under arbitrary switching time sequence. Under a realistic assumption, it has been shown that there exists a piecewise constant feedback controller such that the switched nonlinear system is approximate stable. Finally we have presented an approach to determining the control input value at the sampling instant or the switching instant and an optimization model has also been established to find the projection of the state onto Epi_V .

6. Appendix: proof of proposition

The following lemmas will be used in the proof of the proposition.

Lemma 1 consider the time sequence $\left\{t_m\right\}_{m=0}^{s}$ and the state sequence $\left\{x_m\right\}_{m=0}^{s-1}$ in system (1), where $x_m(\cdot):\left[t_m,t_{m+1}\right)\to\mathbb{R}^n$ denote the solution of $\dot{x}=f_{i_m}\left(x,u\right)$ starting at $x_m(t_m)$ .Then for all $s\in\left[t_m,t_{m+1}\right)$

$$\left|f_{i_m}\left(x_m\left(s\right),u\left(s\right)\right)\right|\le\lambda_{i_m}\left(1+\left|x_m(t_m)\right|\right)e^{\lambda_{i_m}(t_{m+1}-t_m)}\quad m=0,1,\cdots,s-1.$$

Proof : Note that for $x_m\left(s\right)\neq 0$

$$\left|f_{i_m}\left(x_m(s),u(s)\right)\right|=\frac{d}{ds}\left|x_m(s)\right|\le\left|\frac{d}{ds}x_m(s)\right|\le\lambda_{i_m}\left(1+\left|x_m(s)\right|\right)$$

Then by the Gronwall Lemma[16]

$$\left|x_m(s)\right|\le\left(\left|x_m(t_m)\right|+1\right)e^{\lambda_{i_m}s}-1$$

We have $\left|f_{i_m}\left(x_m(s),u(s)\right)\right|\le\lambda_{i_m}\left(1+\left|x_m(t_m)\right|\right)e^{\lambda_{i_m}s}\le\lambda_{i_m}\left(1+\left|x_m(t_m)\right|\right)e^{\lambda_{i_m}(t_{m+1}-t_m)}$.□

Lemma 2 Let us suppose that $K\subseteq\mathbb{R}^n$ is a nonempty closed set and consider a point $\bar{x}\notin K$ and fix $\bar{y}\in\prod_K(\bar{x})$. If there exists a control $\bar{u}\in U$ such that

$$\left(f\left(\bar{y},\bar{u}\right)\right)^{\mathrm{T}}\left(\bar{x}-\bar{y}\right)\le 0,$$

and that for all $t\in\left[0,\frac{1}{2k_{i_m}}\right]$, $\left\|f_{i_m}(x,\bar{u})\right\|\le M_{i_m}$,then

$$\forall t\in\left[0,\frac{1}{2k_{i_m}}\right],\quad d_K^2\left(x_m(t)\right)\le eM_{i_m}{}^2t^2+e^{2k_{i_m}t}d_K^2\left(\bar{x}\right).$$

Recall that k_{i_m} is the Lipschitz constant of the subsystem $\dot{x}=f_{i_m}\left(x,u\right)$. The proof is a straight forward modification of Lemma 5.1 of [17].

Lemma 3[18] Suppose that for all $i=0,1,\cdots,N,\quad a_i,b_i,r_i\ge 0$ and $r_{i+1}\le a_ir_i+b_i$. Then

$$r_{N+1}\le\max_{0\le i\le N}\left\{r_0,b_i\right\}\left\{\sum_{i=0}^{N}\left(\prod_{j=i}^{N}a_j\right)+1\right\}\text{ and } r_{N+1}\le r_0\prod_{i=0}^{N}a_i+\max_{0\le i\le N}\left\{b_i\right\}\left\{\sum_{i=1}^{N}\left(\prod_{j=i}^{N}a_j\right)+1\right\}.$$

Proof of Proposition. The proof consists in building a feedback controller which satisfies (2).For this purpose, We defined the piecewise constant controller by a feedback map $g(\cdot):\mathbb{R}^n\to U$, used to set u either after a switching, or whenever a sampling time elapses from the last time u was set.

For $g(x)$, we distinguish two cases. If $x\in K$, choose an arbitrary $u\in U$ and set $g(x)=u$. If $x\in\mathbb{R}^n\setminus K$, choose its projection y onto K, by definition, $x-y\in NP_K(y)$, then by condition of the proposition and the compactness of U such that $\left(f_{\sigma(t)}\left(y,\tilde{u}\right)\right)^{\mathrm{T}}\left(x-y\right)\le 0$.We define $g(x)=\tilde{u}$.

Let $\left\{x_m\right\}_{m=0}^{s-1}$ be an arbitrary finite run of the system(1) with its corresponding switching time sequence $\left\{t_m\right\}_{m=0}^{s}$: $t_0<t_1<t_2<\cdots<t_m<\cdots<t_s$,starting at some initial state $x_0(t_0)\in K$. For $[t_m,t_{m+1}]$ let h_m denote the sampling time used over time interval $[t_m,t_{m+1}]$ and assume that $0<h_m<\frac{1}{2k_{i_m}}$.Let $\left\{t_{mj}\right\}_{j=0}^{N_m}$ denote the set of times in $[t_m,t_{m+1}]$ at which u is set by the controller. Clearly, $0\le N_m<\infty$ （recall that $t_s<\infty$, $h_m>0$）, $t_{m0}=t_m$ and for all $0\le j<N_m$, $t_{m(j+1)}-t_{mj}\le h_m$ （refer to figure）.

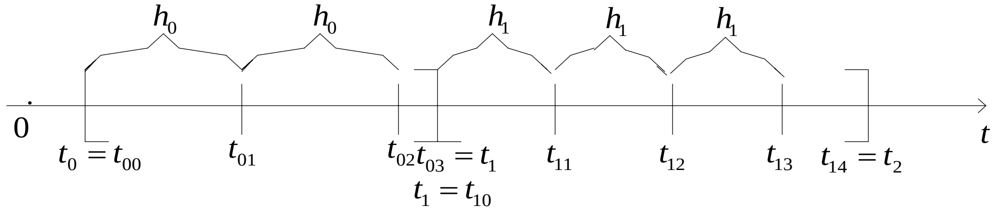

Figure Illustration of sampling times. In this case $s=2,\ N_0=3,\ N_1=4$

In particular, $t_{m(j+1)} - t_{mj} = h_m$ for all $0 \le j < N_m - 1$. Moreover, if $m < s-1, \quad t_{mN_m} = t_{m+1}$.

Finally, let $M_{im} = \lambda_{im}\left(1+\left|x_m\left(t_m\right)\right|\right)e^{\lambda_{im}\left(t_{m+1}-t_m\right)}$

By Lemma 1 and Lemma 2

$$d_K^2\left(x_m\left(t\right)\right) \le eM_{i_m}{}^2(t-t_{mj})^2 + e^{2k_{i_m}(t-t_{mj})}d_K^2\left(x_m\left(t_{mj}\right)\right)$$

for all $t \in \left[t_{mj}, t_{m(j+1)}\right]$. Therefore

$$d_K^2\left(x_m\left(t_{m(j+1)}\right)\right) \le eM_{i_m}{}^2 h_m{}^2 + e^{2k_{i_m}(t_{m(j+1)}-t_{mj})}d_K^2\left(x_m(t_{mj})\right)$$

By Lemma 3, for all $t \in \left[t_{mj}, t_{m(j+1)}\right]$

$$\begin{aligned}
d_K^2\left(x_m\left(t\right)\right) &\le d_K^2\left(x_m\left(t_{m0}\right)\right)e^{2k_{i_m}(t-t_{mj})}\prod_{\alpha=0}^{j-1}e^{2k_{i_m}(t_{m(\alpha+1)}-t_{m\alpha})} + eM_{i_m}{}^2 h_m{}^2 \\
&\quad \times(e^{2k_{i_m}(t-t_{mj})} + e^{2k_{i_m}(t-t_{mj})}\sum_{\alpha=1}^{j-1}\sum_{\beta=\alpha}^{j-1}e^{2k_{i_m}(t_{m(\beta+1)}-t_{m\beta})}+1) \\
&= d_K^2\left(x_m\left(t_m\right)\right)e^{2k_{i_m}(t-t_m)} + eM_{i_m}{}^2 h_m{}^2 \times (e^{2k_{i_m}(t-t_{mj})} + e^{2k_{i_m}(t-t_{mj})}\sum_{\alpha=1}^{j-1}\sum_{\beta=\alpha}^{j-1}e^{2k_{i_m}h_m}+1) \\
&\le d_K^2\left(x_m\left(t_m\right)\right)e^{2k_{i_m}(t-t_m)} + eM_{i_m}{}^2 h_m{}^2 \times (e^{2k_{i_m}h_m}\sum_{\alpha=0}^{j-1}e^{2k_{i_m}h_m\alpha}+1) \\
&= d_K^2\left(x_m\left(t_m\right)\right)e^{2k_{i_m}(t-t_m)} + M_{i_m}{}^2 h_m{}^2 e^{2k_{i_m}h_m+1}\left(\frac{e^{2k_{i_m}h_m j}-1}{e^{2k_{i_m}h_m}-1}+1\right) \\
&= d_K^2\left(x_m\left(t_m\right)\right)e^{2k_{i_m}(t-t_m)} + M_{i_m}{}^2 h_m e^{2k_{i_m}h_m+1}\left(\frac{2k_{i_m}h_m}{e^{2k_{i_m}h_m}-1}\frac{e^{2k_{i_m}h_m j}-1}{2k_{i_m}}+h_m\right) \\
&\le d_K^2\left(x_m\left(t_m\right)\right)e^{2k_{i_m}(t-t_m)} + M_{i_m}{}^2 h_m e^{2k_{i_m}h_m+1}\left(\frac{e^{2k_{i_m}h_m j}-1}{2k_{i_m}}+h_m\right) \\
&\le d_K^2\left(x_m\left(t_m\right)\right)e^{2k_{i_m}(t-t_m)} + M_{i_m}{}^2 h_m e^{2k_{i_m}h_m+1}\left(\frac{e^{2k_{i_m}(t_{(m+1)}-t_m)}-1}{2k_{i_m}}+h_m\right)
\end{aligned}$$

The second term decreases to zero as h_m decreases. Therefore, for any $\delta > 0$ we can choose $h_m \in \left(0, \frac{1}{2k_{i_m}}\right]$ small enough to ensure that for all $t \in \left[t_m, t_{m+1}\right)$

$$d_K^2\left(x_m\left(t\right)\right) \le \delta + e^{2k_{i_m}(t-t_m)}d_K^2\left(x_m\left(t_m\right)\right) \tag{4}$$

Then due to $x_m\left(t_m\right) = x_{m-1}\left(t_m\right)$ we have

$$d_K^2\left(x_m\left(t_m\right)\right) = d_K^2\left(x_{m-1}\left(t_m\right)\right) \le \delta + e^{2k_{i_{m-1}}(t-t_{m-1})}d_K^2\left(x_{m-1}\left(t_{m-1}\right)\right)$$

Recall that $x_0\left(t_0\right) \in K$, therefore $d_K\left(x_0\left(t_0\right)\right) = 0$. By Lemma 3

$$\begin{aligned}
d_K^2\left(x_m\left(t_m\right)\right) &\le d_K^2\left(x_0\left(t_0\right)\right)\prod_{\alpha=0}^{m-1}e^{2k_{i_\alpha}(t-t_0)} + \delta\left(\sum_{\alpha=1}^{m-1}\prod_{\beta=\alpha}^{m-1}e^{2k_{i\beta}(t-t_\beta)}\right) \\
&= \delta\left(\sum_{\alpha=1}^{m-1}\prod_{\beta=\alpha}^{m-1}e^{2k_{i\beta}(t-t_\beta)}\right)
\end{aligned}$$

Let $\gamma = \max_{1\le\beta\le m-1}\left\{k_{i_\beta}\right\}$, then

$$d_K^2\left(x_m\left(t_m\right)\right) \le \delta\left(\sum_{\alpha=1}^{m-1}\prod_{\beta=\alpha}^{m-1}e^{2\gamma\left(t_s-t_0\right)}\right) \tag{5}$$

Hence, it follows from (4) and (5) that

$$d_K^2\left(x_m\left(t\right)\right) \le \delta + \delta\left(\sum_{\alpha=1}^{m-1}\prod_{\beta=\alpha}^{m-1}e^{2\gamma\left(t_s-t_0\right)}\right) = \delta\left(1+\sum_{\alpha=1}^{m-1}\prod_{\beta=\alpha}^{m-1}e^{2\gamma\left(t_s-t_0\right)}\right)$$

Let $\delta = \dfrac{\varepsilon^2}{1+\sum_{\alpha=1}^{m-1}\prod_{\beta=\alpha}^{m-1}e^{2\gamma\left(t_s-t_0\right)}}$, then we obtain $d_K^2\left(x_m\left(t\right)\right)\le\varepsilon^2$, i.e. $x_m\left(t\right)\in K+\varepsilon B$, $m=0,1,\cdots s-1$, and the claim of the proposition follows. □

7. References

[1] Jeon.D, Tomizuka.M.. Learning hybrid force and position control of robot manipulators[J]. *IEEE Trans on Robotics and Automation.* 1993, **9**(4): 423-431.

[2] Varaiya.P.P.. Smart car on smart roads:problems of control[J]. *IEEE Trans.on Automatic Control.*1993, **38**(2): 195-207.

[3] Sira-Ranirez.H.. Nonlinear P-I controller design for switch mode dc-to-dc power converts[J]. *IEEE Trans. Circuits and Systems.* 1991, **38**(4): 410-417.

[4] Williams.S.M., Hoft.R.G.. Adaptive frequency domain control of PM switched power line conditioner[J]. *IEEE Trans. Power Electronics.* 1991, **6**(10): 665-670.

[5] Decarlo.R., Branicky. M., Pettersson.S.,and Lennartson B.. Perspectives and results on the stability and stabilizability of Hybrid Systems[J]. *Proceedings of the IEEE.* 2000, **88**(7): 1069-1082.

[6] Agrachev.A.A., Liberzon.D.. Lie-algebraic stability criteria for switched systems[J]. *SIAM J.of control optimization.* 2001, **40**(1): 253-269.

[7] Branicky.M.S.. Multiple Lyapunov functions and other analysis tools for switched hybrid systems[J]. *IEEE Trans. On automatic control*. 1998, **43**(4): 475-482.

[8] Shortenm.R.N., Narendra.K.S.. A sufficient condition for the existence of a common Lyapunov function for two second-order linear systems[C]. *Proc.36th conf. on decision and control*. New York: IEEE Press. 1997, pp.3521-3522.

[9] Sun Z.D., Ge.S.S.. Analysis and synthesis of switched linear control systems[J]. *Automatica*. 2005, **41**:181-195.

[10] Mancilla-Aguilar.J.L., Garcua.R.A.. A converse Lyapunov Theorem for nonlinear switched systems[J]. *Systems ＆ Control Letters.* 2000, **44**(1): 67-71.

[11] Mancilla-Aguilar.J.L.. A condition for the stability of switched nonlinear systems[J]. *IEEE Trans. on Automatic Control.* 2000, **45**(11): 2077-2079.

[12] Zhao.J. and Spong M.W.. Hybrid control for global stabilization of the cart-pendulum system[J]. *Automatic.* 2001, **37**(12): 1941-1951.

[13] Chen.W. and Balance.D.J.. On a switching control scheme for nonlinear systems with ill-defined relative degree[J]. *Systems ＆ Control Letters.* 2002, **47**(2): 159-166.

[14] Aubin J-P.. Viability Theory[M]. Boston: Birkhauser, 1991.

[15] Clarke F H.,Leda Yu S., Stern R J., et al. *Nonsmooth Analysis and Control Theory[M].* New York: Springer-Verlag, 1998.

[16] Quincampoix M., Seube N.. Stabilization of uncertain control systems through piecewise constant feedback [J]. *Journal of Mathematical Analysis and Applications.* 1998, **218**(1): 240-255.

[17] Gao Y., Lygeros J., Quincampoix M., Seube N.. On the control of uncertain impulsive systems : approximate stabilization and controlled invariance [J]. *International Journal of Control*. 2004, **77**(16): 1393-1407.