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A Numerical Scheme to Solve Nonlinear Volterra Integral Equations System of the Second Kind

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Abstract. In this study, we use a recursive method based upon power series to solve nonlinear Volterra integral equations system of the second kind. This method gives an approximate solution as the Taylor expansion for the solution of the system via some simple computations. Numerical examples illustrate the pertinent features of the method.

Keywords: Nonlinear integral equations system, Numerical method, Taylor expansion.

1. Introduction

In recent years, many different methods have been used to approximate the solution of linear or nonlinear Volterra integral equations system [1, 2, 3, 4, 5, and 7]. Tricomi, in his book [6], introduced the classical method of successive approximations for nonlinear Volterra integral equations. In [2], Brunner applied a collocation-type method to nonlinear Volterra equations and integro-differential equations and discussed its connection with the iterated collocation method. For Volterra-Hammerstein equations, the asymptotic error expansion of a collocation method was introduced in [3]. In general, most of numerical methods transform the integral equation to a linear or nonlinear system of algebraic equations which can be solved by direct or iterative methods. Yousefi and Razzaghi in [7], and also Maleknejad et al in [4] used Legendre wavelets to numerical solution of linear and nonlinear Volterra integral equations. Recently, In [5], Chebyshev polynomials are applied for solving of nonlinear Volterra integral equations of the second kind.

In the present article, we consider the second kind Volterra integral equations system of the form:

$$\begin{cases} y_1(s) = g_1(s) + \sum_{j=1}^{n} \int_0^s \lambda_{1j} k_{1j}(s,t) \left(y_j(t) \right)^{P_{1j}} dt, \\ y_2(s) = g_2(s) + \sum_{j=1}^{n} \int_0^s \lambda_{2j} k_{2j}(s,t) \left(y_j(t) \right)^{P_{2j}} dt, \\ \vdots \\ y_n(s) = g_n(s) + \sum_{j=1}^{n} \int_0^s \lambda_{nj} k_{nj}(s,t) \left(y_j(t) \right)^{P_{nj}} dt, \end{cases} \tag{1}$$

where, $0 \le t, s \le 1$, $\lambda_{ij} \ne 0, i, j = 1,2,...,n$ is a real constant, and $P_{i,j}, i.j = 1,2,...,n$ is a nonnegative integer. Moreover, in Eq.(1) the function $g_i(s)$ and the kernel $k_{i,j}(s,t)$ are given and assumed to be sufficiently differentiable with respect to all their arguments on the interval $0 \le t, s \le 1$, for all $i, j = 1,2,...,n$. Also, $Y(s) = [y_1(s), y_2(s),..., y_n(s)]^T$ is the solution to be determined.

To avoid of complesity, we simplify Eq.(1) by using the following notations,

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$$\begin{cases} G(s) = [g_1(s), g_2(s), ..., g_n(s)]^T, \\ K(s,t) = [k_{ij}(s,t)], \ i, j = 1, 2, ..., n, \\ \Lambda = [\lambda_{ij}], \ i, j = 1, 2, ..., n, \\ P = [P_{ij}], \ i, j = 1, 2, ..., n, \\ [Y(s)]^P = [y_i^{P_{ij}}(s)], \ i, j = 1, 2, ..., n. \end{cases}$$

So, we can rewrite Eq.(1) as

$$Y(s) = G(s) + \Lambda \int_0^s K(s,t)[Y(t)]^P dt. \tag{2}$$

If we set $s = 0$ in Eq.(2) then we have $Y(0) = G(0)$, as the initial condition. Hence, the solution of Eq.(2) can be assume that,

$$Y = E_0 + E_1 s, \tag{3}$$

where, $E_1 = [e_{11}, e_{12}, ..., e_{1n}]^T$ is an unknown vector and,

$$E_0 = G(0) = [g_1(0), g_2(0), ..., g_n(0)]^T. \tag{4}$$

Substitute Eq.(3) into Eq.(2) and neglect higher order terms, we have linear equation of E_1 in the form,

$$AE_1 = b, \tag{5}$$

where, $A = [a_{i,j}]_{n\times n}$ is a known constant matrix and $b_{n\times 1}$ is a known constant vector. By solving Eq.(5), the coefficient of in Eq.(3) can be determined.

Repeating above procedure for higher order terms, we can get an arbitrary order power series of the solution for Eq.(2).

2. Statement of the Scheme

Suppose the solution of Eq.(2) with $E_0 = Y(0) = G(0)$ as the initial condition to be as follows,

$$Y(x) = E_0 + E_1 s, \tag{6}$$

where, E_1 is an unknown vector.

If we substitute Eq.(6) into Eq.(2) we obtain the following linear algebraic equation,

$$AE_1 - b + Q(s) = 0, \tag{7}$$

where, A and b are known constant value and $Q(s) = [q_1(s), q_2(s), ..., q_n(s)]^T$ is a vector function which $q_i(s),\ i = 1, 2, ..., n$ is a polynomial with the order greater than one.

Neglecting $Q(s)$ in Eq.(7) and solving the system of $AE_1 = b$, the unknown vector E_1 and therefore the coefficient of s in Eq.(6) is obtained.

In the next step, we assume that the solution of Eq.(2) to be,

$$Y(s) = E_0 + E_1 s + E_2 s^2, \tag{8}$$

here, E_0 and E_1 both are known vectors and E_2 is an unknown vector. By substituting Eq.(8) into Eq.(2), we have following system,

$$\left(AE_2 - b\right)s + Q(s^2) = 0. \tag{9}$$

Again, by neglecting and solving the system of $AE_2 = b$, the unknown vector E_2 and therefore the coefficient of s^2 in Eq.(8) is obtained.

By repeating the above procedure for m iterations, a Power series of the following form is derived,

$$Y(s)=E_0+E_1s+E_2s^2+\cdots+E_ms^m. \tag{10}$$

Eq.(10) is an approximation for the exact solution $Y(s)$ of the integral equation (2).

Theorem 1. Suppose all of components of functions $G(s)$ and $K(s,t)$ in (2) be analytic functions with respect to all their arguments. Also, let $F(s)=[f_1(s),...,f_n(s)]^T$ be the exact solution of the following Volterra integral equation

$$F(s)=G(s)+\Lambda\int_0^s K(s,t)[F(t)]^P dt. \tag{11}$$

Futhermore, suppose all elements of $F(s)$ be infinitely differentiable functions in a neighborhood of $s=0$. Then, the method obtains the Taylor expansion of $F(s)$.

Proof. As it was showed, in the presented method, we assume that the approximate solution to Eq.(11) be as follows,

$$\tilde{F}(s)=E_0+E_1s+E_2s^2+\cdots. \tag{12}$$

Hence, it is sufficient that we prove,

$$E_m=\frac{1}{m!}F^{(m)}(0), \qquad m=1,2,3,..., \tag{13}$$

where, $E_m=[e_{m1},e_{m2},...,e_{mn}]^T$ and $F^{(m)}(0)=\left[\frac{df_1(s)}{ds},\frac{df_2(s)}{ds},...,\frac{df_n(s)}{ds}\right]^T_{s=0}$.

Note that for $m=0$, we always have

$$F(0)=G(0)=E_0=[e_{01},e_{02},...,e_{0n}]^T. \tag{14}$$

For $m=1$, if we set $Y_i=F_i(s)$ and then derivative from Eq.(11), we obtain

$$f_i'(s)=g_i'(s)+\sum_{j=1}^n \lambda_{ij}k_{ij}(s,s)[f_j(s)]^{P_{ij}}, \quad i=1,2,...,n. \tag{15}$$

Setting $s=0$ in Eq.(15), then we get

$$f_i'(0)=g_i'(0)+\sum_{j=1}^n \lambda_{ij}k_{ij}(0,0)[f_j(0)]^{P_{ij}}, \quad i=1,2,...,n. \tag{16}$$

In other hand, from Eq.(12) and Eq.(14), we can derive

$$\tilde{f}_i(s)=e_{0i}+e_{1i}s+\cdots, \qquad i=1,2,...,n. \tag{17}$$

So, by substituting Eq.(17) into Eq.(15) and setting $s=0$, for any $i=1,2,...,n$, we get

$$e_{1i}=g_i'(0)+\sum_{j=1}^n \lambda_{ij}k_{ij}(0,0)(e_{0j})^{P_{ij}}=g_i'(0)+\sum_{j=1}^n \lambda_{ij}k_{ij}(0,0)[f_j(0)]^{P_{ij}} \tag{18}$$

therefore, with comparison Eq.(16) and Eq.(18), we conclude that,

$$e_{1i}=f_i'(0), \qquad i=1,2,...,n. \tag{19}$$

For $m=2$, similar to the last step, this time we derivative from (15), we have

$$f_i''(s)=g_i''(s)+\sum_{j=1}^n \lambda_{ij}\left(\frac{d}{ds}k_{ij}(s,s)[f_j(s)]^{P_{ij}}+P_{ij}k_{ij}(s,s)\frac{df_j(s)}{ds}[f_j(s)]^{P_{ij}-1}\right) \tag{20}$$

again, we set $s=0$ in Eq.(20) and we get

$$f_i''(0)=g_i''(0)+\sum_{j=1}^n \lambda_{ij}\left(\frac{d}{ds}k_{ij}(0,0)[f_j(0)]^{P_{ij}}+P_{ij}k_{ij}(0,0)\frac{df_j(0)}{ds}[f_j(0)]^{P_{ij}-1}\right). \tag{21}$$

According to Eq.(12), Eq.(14) and Eq.(19), we can suppose that

$$\tilde{f}_i(s)=f_i(0)+f_i'(0)+e_{2i}s^2+\cdots, \qquad i=1,2,...,n, \tag{22}$$

by substituting Eq.(22) into Eq.(20), and setting $s=0$, we have

$$2e_{2i}=g_i''(0)+\sum_{j=1}^{n}\lambda_{ij}\left(\frac{d}{ds}k_{ij}(0,0)[f_j(0)]^{P_{ij}}+P_{ij}k_{ij}(0,0)\frac{df_j(0)}{ds}[f_j(0)]^{P_{ij}-1}\right). \qquad (23)$$

So, with comparison Eq.(21) and Eq.(23), we conclude that

$$2e_{2i}=f_i''(0)\ , \quad \text{or} \quad e_{2i}=\frac{1}{2!}f_i''(0), \qquad i=1,2,\dots,n.$$

By continuing the above procedure, we can easily prove Eq.(13) for $m=3,4,\dots$.

Corollary 2. If the exact solution to Eq.(11) be a polynomial, then the method will obtain it.

3. Numerical Results

In the first example, we try to illustrate the proposed method, analytically.

Example 1. The test problem, consider the following nonlinear integral equation

$$\begin{cases} y_1(s)=g_1(s)+\int_0^s k_{11}(s,t)y_1^2(t)dt+\int_0^s k_{12}(s,t)y_2(t)dt \\ y_2(s)=g_2(s)+\int_0^s k_{21}(s,t)y_1(t)dt+\int_0^s k_{22}(s,t)y_2^2(t)dt \end{cases} \qquad (24)$$

where,

$$\begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix}=\begin{pmatrix} 1-\frac{7}{2}s^3-\frac{7}{12}s^4-\frac{4}{15}s^6-\frac{9}{8}s^9 \\ -\frac{1}{2}s^2+4s^4+\frac{27}{20}s^5-\frac{1}{6}s^7-s^9-\frac{8}{5}s^{11} \end{pmatrix},$$

and

$$\begin{pmatrix} k_{11}(s,t) & k_{12}(s,t) \\ k_{21}(s,t) & k_{22}(s,t) \end{pmatrix}=\begin{pmatrix} st & s+t \\ s+t & st \end{pmatrix}.$$

lso, $\begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix}=\begin{pmatrix} 1-3s^3 \\ s^2+4s^4 \end{pmatrix}$ is the exact solution.

If we set $s=0$ in Eq.(24), we get $\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as the initial condition to Eq.(24).

Now, we apply the proposed method for solving Eq.(24). Let the solution of Eq.(24) to be,

$$\begin{matrix} \tilde{y}_1(s)=y_1(0)+e_{11}s & & \tilde{y}_1(s)=1+e_{11}s \\ & \Rightarrow & \\ \tilde{y}_2(s)=y_2(0)+e_{12}s & & \tilde{y}_2(s)=0+e_{12}s \end{matrix} \qquad (25)$$

Substitute Eq.(25) into Eq.(24), we have,

$$1+e_{11}s=1-\frac{7}{2}s^3-\frac{7}{12}s^4-\frac{4}{15}s^6-\frac{9}{8}s^9+\int_0^s k_{11}(s,t)\left(1+e_{11}t\right)^2dt+\int_0^s k_{12}(s,t)e_{12}tdt$$

$$e_{12}s=-\frac{1}{2}s^2+4s^4+\frac{27}{20}s^5-\frac{1}{6}s^7-s^9-\frac{8}{5}s^{11}+\int_0^s k_{21}(s,t)\left(1+e_{11}s\right)dt+\int_0^s k_{22}(s,t)\left(e_{12}t\right)^2dt$$

After simplifying, we get,

$$\left(e_{11}-0\right)s+q_1(s^2)=0$$
$$\left(e_{12}-0\right)s+q_2(s^2)=0 \tag{26}$$

where,

$$q_1(s^2)=(-3+\frac{5}{6}e_{12})s^3+(\frac{2}{3}e_{11}-\frac{7}{12})s^4+\frac{1}{4}e_{11}^2s^5-\frac{4}{15}s^6-\frac{9}{8}s^9,$$
$$q_2(s^2)=s^2+\frac{5}{6}e_{11}s^3+4s^4+(\frac{1}{4}e_{11}^2+\frac{27}{20})s^5-\frac{1}{6}s^7-s^9-\frac{8}{5}s^{11}.$$

Neglecting $q_1(s^2)$ and $q_2(s^2)$ in Eq.(26), we obtain $e_{11}=0$ and $e_{12}=0$, hence

$$\tilde{y}_1(s)=1$$
$$\tilde{y}_2(s)=0 \tag{27}$$

is the first approximation for the exact solution to Eq.(24).

Now, from Eq.(27), the solution of Eq.(24) can be supposed as:

$$\tilde{y}_1(s)=1+e_{21}s^2,$$
$$\tilde{y}_2(s)=e_{22}s^2. \tag{28}$$

Substituting Eq.(28) into Eq.(24), gives,

$$(e_{21}-0)s^2+q_1(s^3)=0,$$
$$(e_{22}-1)s^2+q_2(s^3)=0, \tag{29}$$

where,

$$q_1(s^3)=-3s^3+(\frac{7}{12}e_{22}-\frac{7}{12})s^4+\frac{1}{2}e_{21}s^5-\frac{4}{15}s^6+\frac{1}{6}e_{21}^2s^7-\frac{9}{8}s^9,$$
$$q_2(s^3)=(\frac{7}{12}e_{21}+4)s^4+\frac{27}{20}s^5+(\frac{1}{6}e_{22}^2-\frac{1}{6})s^7-s^9-\frac{8}{5}s^{11}.$$

By neglecting $q_1(s^3)$ and $q_2(s^3)$ in Eq.(29), we have $e_{21}=0$ and $e_{22}=1$. Therefore

$$\tilde{y}_1(s)=1,$$
$$\tilde{y}_2(s)=s^2. \tag{30}$$

From Eq.(30) the solution of Eq.(24) can be supposed as

$$\tilde{y}_1(s)=1+e_{31}s^3,$$
$$\tilde{y}_2(s)=s^2+e_{32}s^3. \tag{31}$$

Substituting Eq.(31) into Eq.(24), we obtain

$$(e_{31}+3)s^3+q_1(s^4)=0,$$
$$(e_{32}-0)s^3+q_2(s^4)=0, \tag{32}$$

where,

$$q_1(s^4)=\frac{9}{20}e_{32}s^5-(\frac{2}{5}e_{31}-\frac{4}{15})s^6+(\frac{1}{8}e_{31}^2-\frac{9}{8})s^9,$$
$$q_2(s^4)=4s^4+(\frac{9}{20}e_{31}+\frac{27}{20})s^5+\frac{2}{7}e_{32}s^8+(\frac{1}{8}e_{32}^2-1)s^9-\frac{8}{5}s^{11}.$$

By ignoring $q_1(s^4)$ and $q_2(s^4)$ in Eq.(32), we obtain $e_{31}=-3$ and $e_{32}=0$. Therefore

$$\tilde{y}_1(s) = 1 - 3s^3, \quad \tilde{y}_2(s) = s^2. \tag{33}$$

Proceeding in this way, we get,

$$\tilde{y}_1(s) = 1 - 3s^3, \quad \tilde{y}_2(s) = s^2 + 4s^4. \tag{34}$$

Indeed, according to Corollary 2, since the solutions are polynomials, the method gives them exactly.

Example 2. Consider the following nonlinear Volterra integral equation with logarithmic singularity,

$$\begin{aligned} y_1(s) &= g_1(s) + \int_0^s \ln|s-t+1| y_1^2(t)dt + \int_0^s (s^2+t) y_2(t)dt, \\ y_2(s) &= g_2(s) + \int_0^s \sqrt{s-t+2}\, y_1^4(t)dt + \int_0^s (s-t^2) y_2^3(t)dt, \end{aligned} \tag{35}$$

where, $0 \le t \le s \le 1$, and

$$g_1(s) = \sqrt{s+1} + \frac{3}{2}s - \frac{1}{2}s^2\ln(s+1) + \frac{7}{4}s^2 - 2s\ln(s+1) - \frac{3}{2}\ln(s+1) - e^s(s^2-s+1) - 1,$$

$$g_2(s) = e^s + (\frac{24}{5}\sqrt{2} + \frac{1}{3})s + \frac{4}{3}\sqrt{2}s^2 - (2+s)^{\frac{3}{2}}(\frac{82}{35} - \frac{8}{7}s - \frac{16}{105}s^2) - e^{3s}(\frac{1}{3}s^2 + \frac{5}{9}s + \frac{2}{27}) + \frac{164}{35}\sqrt{2} - \frac{2}{27}.$$

Here, the exact solution is

$$y_1(s) = \sqrt{s+1}$$
$$y_2(s) = \exp(s).$$

To solve Eq.(35) via the method, first we need to calculate the Taylor expansion of the functions $g_1(s)$, $g_2(s)$, $\ln|s-t+1|$ and $\sqrt{s-t+2}$, with respect to all their arguments and then substitute them in Eq.(35). We applied the method (with $m = 4$) to Eq.(35), and the following approximate solutions have been found

$$\tilde{y}_1(s) = 1 + \frac{1}{2}s - \frac{1}{8}s^2 + \frac{1}{16}s^3 - \frac{5}{128}s^4,$$
$$\tilde{y}_2(s) = 1 + s + \frac{1}{2}s^2 + \frac{1}{6}s^3 + \frac{1}{24}s^4.$$

The numerical results are illustrated in Fig. 1 and Table 1.

s_i	$\tilde{y}_1(s_i)$	$y_1(s_i)$	$\tilde{y}_2(s_i)$	$y_2(s_i)$
0	1.000000	1.000000	1.000000	1.000000
0.1	1.048808	1.408808	1.105170	1.105170
0.2	1.095437	1.095445	1.221400	1.221402
0.3	1.140121	1.140175	1.349837	1.349858
0.4	1.183000	1.183259	1.491733	1.491824
0.5	1.224121	1.224744	1.648437	1.648721
0.6	1.263437	1.264911	1.821400	1.822118
0.7	1.300808	1.303840	2.012170	2.013752
0.8	1.336000	1.341640	2.222400	2.225540
0.9	1.368683	1.378404	2.453875	2.459603
1	1.398437	1.414217	2.708333	2.718281

Table 1. Numerical results for Example 2.

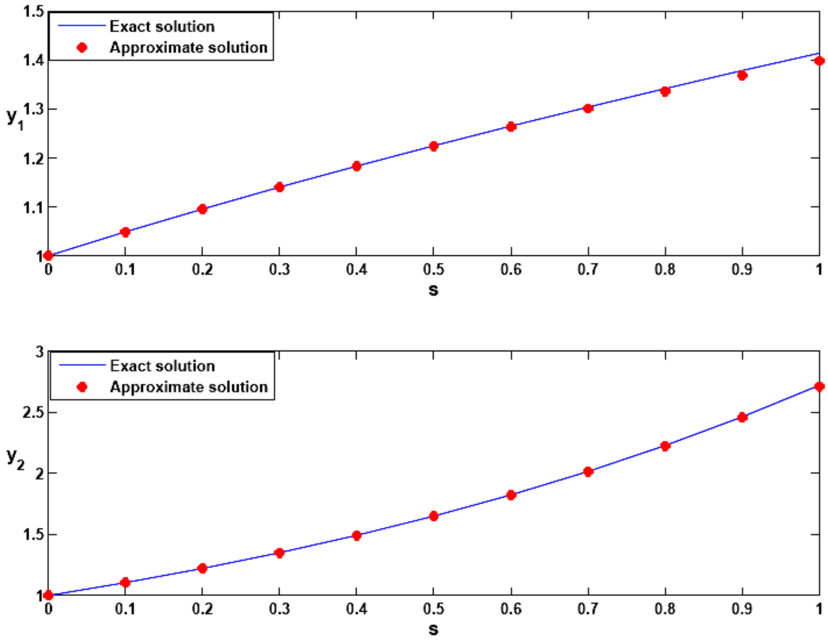

Fig. 1: The results of Example 2 .

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