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Bounds for the extreme eigenvalues using the trace and determinant

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Abstract. Bounds for the extreme eigenvalues involving trace and determinant are presented. Also, we give the upper bounds for the Perron root of a nonnegative symmetric matrix under certain conditions.

Keywords: Eigenvalue; Trace; Determinant; Nonnegative symmetric matrix; Perron root

1. Introduction

Let A be an $n \times n$ complex matrix with singular values $\sigma_1(A) \ge \sigma_2(A) \ge \cdots \ge \sigma_n(A)$.

The properties

$$\sigma_1(A) + \sigma_2(A) + \cdots + \sigma_n(A) = \|A\|_F^2,$$

$$\sigma_1(A)\sigma_2(A)\cdots\sigma_n(A) = |\det A|$$

are well known, where $\|A\|_F$ and $\det A$ denote the Frobenius norm of A and the determinant of A, respectively. In [1], Rojo presents monotonic sequences of bounds for $\sigma_k(A)\ (1 \le k \le n)$:

$$\alpha^{(n-1)/2} \le \sigma_k(A) \le \beta^{(n-1)/2},$$

where α and β are the positive roots of the equation

$$x^n - \|A\|_F^2 x + (n-1)|\det A|^{2/n-1} = 0.$$

And if $x_0 = 0$, $\left\{x_k^{(n-1)/2}\right\}$ is an increasing sequence of lower bounds for $\sigma_n(A)$; if $x_0 = \|A\|_F^{2/(n-1)}$, $\left\{x_k^{(n-1)/2}\right\}$ is a decreasing sequence of upper bounds for $\sigma_1(A)$ where $\{x_k\}$ is a sequence defined by

$$x_{k+1} = \frac{n-1}{\|A\|_F^2} \cdot \frac{|\det A|^{2/(n-1)} - x_k^n}{1 - \left(n / \|A\|_F^2\right) x_k^{n-1}}.$$

Similarly, Let A be an $n \times n$ complex matrix with real and positive eigenvalues

$$\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A) > 0.$$

The properties

$$\lambda_1(A) + \lambda_2(A) + \cdots + \lambda_n(A) = trA,$$

$$\lambda_1(A)\lambda_2(A)\cdots\lambda_n(A) = \det A$$

motivate one to estimate the bounds for eigenvalues of A where trA denotes the trace of A. In [2, p.

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20-21], monotonic sequences of bounds for $\lambda_k(A)$ $(1 \le k \le n)$ are presented using the same technique in [1] as follows:

$$\alpha^{n-1} \le \lambda_k(A) \le \beta^{n-1},$$

where α and β are the positive roots of the equation

$$x^n - trAx + (n-1)(\det A)^{1/n-1} = 0. \quad (1)$$

And if $x_0 = 0$, $\{x_k{}^{n-1}\}$ is an increasing sequence of lower bounds for $\lambda_n(A)$ if $x_0 = (trA)^{1/n-1}$, $\{x_k{}^{n-1}\}$ is a decreasing sequence of upper bounds for $\lambda_1(A)$ where $\{x_k\}$ is a sequence defined by

$$x_{k+1} = \frac{n-1}{trA} \cdot \frac{(\det A)^{1/n-1} - x_k^n}{1 - (n/trA)x_k^{n-1}}. \quad (2)$$

In this paper, let $A \in C^{n\times n}$ $(n \ge 2)$ be the matrices with real and positive eigenvalues $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A) > 0$. We use this symbol throughout. We give bounds for the extreme eigenvalues using trace and determinant.

The paper is organized as follows. In Section 2, two short proofs of lower bound for the smallest eigenvalue of A are given. In Section 3, we obtain another lower bound for the smallest eigenvalue, which is sharper than the result in Section 2. Examples are presented in Section 4 which give comparisons with results in the related literatures. Finally, in Section 5, we consider the upper bounds for a nonnegative symmetric matrix under certain conditions.

2. Two simple lower bounds for the smallest eigenvalue

In [3], Yu and Gu give lower bounds for the smallest singular value using arithmetic-geometric-mean inequality. Here we utilize the similar technique and give the following theorems.

First, we prove the following weaker version of lower bound for $\lambda_n(A)$.

Theorem 2.1. Let $A \in C^{n\times n}$ $(n \ge 2)$ be a matrix with real and positive eigenvalues ordered in decreasing sequence and $trA = n$. Then

$$\lambda_n(A) > \left(\frac{n-1}{n}\right)^{n-1} \det A. \quad (3)$$

Proof. Considering the fact that the geometric mean of positive number doer not exceed their arithmetic mean and the identity

$$n = trA = \lambda_1(A) + \lambda_2(A) + \cdots + \lambda_n(A),$$

We have

$$\lambda_1(A)\lambda_2(A)\cdots\lambda_{n-1}(A) \le \left(\frac{\lambda_1(A) + \lambda_2(A) + \cdots + \lambda_{n-1}(A)}{n-1}\right)^{n-1} = \left(\frac{n - \lambda_n(A)}{n-1}\right)^{n-1} < \left(\frac{n}{n-1}\right)^{n-1}.$$

Multiply this inequality by $\lambda_n(A)$ and solve for $\lambda_n(A)$ we obtain the result.

Theorem 2.2. Let $A \in C^{n\times n}$ $(n \ge 2)$ be a matrix with real and positive eigenvalues ordered in decreasing sequence and $trA = n$. Then

$$\lambda_n(A) > \left(\frac{n-1}{n}\right)^{n-1} \det A \left[1 + \left(\frac{n-1}{n}\right)^n \det A\right]. \quad (4)$$

Proof. The arithmetic-geometric-mean inequality and the identity

$$n = trA = \lambda_1(A) + \lambda_2(A) + \cdots + \lambda_n(A)$$

give

$$\lambda_1(A)\lambda_2(A)\cdots\lambda_{n-1}(A) \le \left(\frac{\lambda_1(A)+\lambda_2(A)+\cdots+\lambda_{n-1}(A)}{n-1}\right)^{n-1} = \left(\frac{n-\lambda_n(A)}{n-1}\right)^{n-1} \tag{5}$$

Multiplying both sides of this inequality by $\lambda_n(A)$, we have

$$\det A = \lambda_1(A)\lambda_2(A)\cdots\lambda_n(A) \le \left(\frac{n-\lambda_n(A)}{n}\right)^{n-1}\lambda_n(A).$$

Hence,

$$\lambda_n(A)^{1/n-1} \ge \frac{n-1}{n}(\det A)^{1/n-1} + \frac{1}{n}\left[\lambda_n(A)^{1/n-1}\right]^n$$
$$> \frac{n-1}{n}(\det A)^{1/n-1}.$$

Because $\lambda_n(A) > 0$, we obtain

$$\lambda_n(A)^{1/n-1} > \frac{n-1}{n}(\det A)^{1/n-1} + \frac{1}{n}\left(\frac{n-1}{n}\right)^n(\det A)^{n/n-1}.$$

Solving this inequality for $\lambda_n(A)$ gives

$$\lambda_n(A) > \left(\frac{n-1}{n}\right)^{n-1}\det A\left[1+\frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1}\det A\right]^{n-1}.$$

Since $n \ge 2$, it follows that

$$\lambda_n(A) > \left(\frac{n-1}{n}\right)^{n-1}\det A\left[1+\left(\frac{n-1}{n}\right)^{n}\det A\right].$$

Remark 1. From the above proof we know that the lower bound for $\lambda_n(A)$ in (4) is an improvement of the result in (3).

3. Further lower bound for the smallest eigenvalue

First, we give the following corollary which follows from (2) immediately.

Corollary 3.1. *Let $B \in C^{n\times n}\ (n \ge 2)$ be a matrix with real and positive eigenvalues ordered in decreasing sequence and $trB = n$. Let $\{x_k\}$ be a sequence defined by*

$$x_{k+1} = \frac{n-1}{n}\cdot\frac{(\det B)^{1/n-1} - x_k^n}{1-x_k^{n-1}}. \tag{6}$$

Then,

1. if $x_0 = 0$, $\{x_k{}^{n-1}\}$ is an increasing sequence of lower bounds for $\lambda_n(B)$;
2. if $x_0 = n^{1/n-1}$, $\{x_k{}^{n-1}\}$ is a decreasing sequence of upper bound for $\lambda_1(B)$.

Let $B \in C^{n\times n}\ (n \ge 2)$ be a matrix with $trB = n$. Let $x_0 = 0$. Then, from (6),

$$x_1 = \frac{n-1}{n}(\det B)^{1/n-1}.$$

Hence,

$$x_1^{n-1} = \left(\frac{n-1}{n}\right)^{n-1} \det B < \lambda_n(B)$$

This is the result of Theorem 2.1.

Again from (6), we have

$$x_2 = \frac{n-1}{n} \frac{(\det B)^{1/n-1} - x_1^{\ n}}{1 - x_1^{\ n-1}}$$

$$= \frac{n-1}{n} (\det B)^{1/n-1} \frac{1 - (n-1/n)^n \det B}{1 - (n-1/n)^{n-1} \det B}$$

$$= \frac{n-1}{n} (\det B)^{1/n-1} \left[1 + \frac{1}{n} \left(\frac{n-1}{n}\right)^{n-1} \det B \frac{1}{1 - (n-1/n)^{n-1} \det B} \right].$$

We know that x_2^{n-1} is a lower bound for $\lambda_n(B)$. Then,

$$\lambda_n(B) > x_2^{n-1}$$

$$= \left(\frac{n-1}{n}\right)^{n-1} \det B \left[1 + \frac{1}{n} \left(\frac{n-1}{n}\right)^{n-1} \det B \frac{1}{1 - (n-1/n)^{n-1} \det B} \right]^{n-1} > \left(\frac{n-1}{n}\right)^{n-1} \det B \left[1 + \left(\frac{n-1}{n}\right)^{n} \det B \frac{1}{1 - (n-1/n)^{n-1}} \right.$$

.

Thus, we have presented the following lemma.

Lemma 3.2. Let $B \in C^{n \times n} (n \geq 2)$ be a matrix with real and positive eigenvalues ordered in decreasing sequence and $trB = n$. Then

$$\lambda_n(B) > \left(\frac{n-1}{n}\right)^{n-1} \det B \left[1 + \theta(B) \left(\frac{n-1}{n}\right)^{n} \det B \right],$$

where,

$$\theta(B) = \frac{1}{1 - (n-1/n)^{n-1} \det B}. \tag{8}$$

With the Lemma 3.2 we may now establish the following theorem.

Theorem 3.3. Let $A \in C^{n \times n} (n \geq 2)$ be a matrix with real and positive eigenvalues ordered in decreasing sequence. Then

$$\lambda_n(A) > \left(\frac{n-1}{trA}\right)^{n-1} \det A \left[1 + \theta(A) \left(\frac{n-1}{trA}\right)^{n} \det A \right], \tag{9}$$

where

$$\theta(A) = \frac{1}{1 - (n-1/n)^{n-1} (n/trA)^n \det A}. \tag{10}$$

Proof. Applying Lemma 3.2 to matrix $B = (n/trA)A$.

This theorem contains Lemma 3.2 as a special case.

4. Comparison with related results

Estimation of extreme eigenvalues is important in theory and practice. Bounds for eigenvlaues have been obtained by many authors. Let $A \in C^{n\times n} \left(n \geq 2\right)$ be a matrix with real and positive eigenvalues $\lambda_1\left(A\right) \geq \lambda_2\left(A\right) \geq \cdots \geq \lambda_n\left(A\right) > 0$ and let $1 \leq k \leq l \leq \mathrm{n}$. Bounds for $\lambda_k \cdots \lambda_l$ and $\lambda_k + \cdots + \lambda_l$, involving k, l, n, trA, and $\det A$ only, are presented as follows.

Theorem 4.1 [4, Theorem 1]**.** Let $1 \leq k \leq l \leq \mathrm{n}$. Then

$$\left[\left(\frac{k-1}{trA}\right)^{k-1} \det A\right]^{1/(n-k+1)} \leq \left(\lambda_k \cdots \lambda_l\right)^{1/(l-k+1)} \leq \frac{\lambda_k + \cdots + \lambda_l}{l-k+1} \leq \frac{trA}{l} - \left(\frac{n}{l} - 1\right)\left[\left(\frac{l}{trA}\right)^{l} \det A\right]^{1/(n-l)}. \quad (11)$$

Theorem 4.2 [4, Theorem 3]**.** Let $1 \leq k \leq l \leq \mathrm{n}$. Then

$$\frac{l-k+1}{n-k+1}\left[trA - \frac{1}{\det A}\frac{k^k}{(k-1)^{k-1}}\left(\frac{trA}{n+1}\right)^{n+1}\right] \leq \lambda_k + \cdots + \lambda_l \leq (l-k+1)\frac{1}{\det A}\left(\frac{l+1}{l}\right)^{l+1}\left(\frac{trA}{n+1}\right)^{n+1}. \quad (12)$$

Let us recall another possible eigenvalue bounds using k, l, n, trA, and trA^2 only. Wolkowicz and Styan derive the following theorem. It is worth noting that the eigenvalues are real; their positivity is not needed.

Theorem 4.3 [5, Theorem 2.2]**.** Let $1 \leq k \leq l \leq \mathrm{n}$. Then

$$\frac{trA}{n} - \sqrt{\frac{k-1}{n-k+1}\frac{1}{n}\left(trA^2 - \frac{(trA)^2}{n}\right)} \leq \frac{\lambda_k + \cdots + \lambda_l}{l-k+1} \leq \frac{trA}{n} + \sqrt{\frac{n-l}{l}\frac{1}{n}\left(trA^2 - \frac{(trA)^2}{n}\right)}. \quad (13)$$

As special cases, bounds for individual eigenvalues, especially for the smallest eigenvalue, can be obtained by the above theorems. We conclude the section with two examples to compare the lower bounds for the smallest eigenvalue and give some remarks.

Example 1. Let

$$A = \begin{bmatrix} 3 & & \\ & 2 & \\ & & 2 \end{bmatrix}.$$

This matrix was used in [4] to compare the lower bounds for $\lambda_3\left(A\right)$ and they were $\lambda_3\left(A\right) \geq 0.980$ by (11), $\lambda_3\left(A\right) \geq 1.667$ by (13) and $\lambda_3\left(A\right) \geq 1.724$ by (12). In this note, the bound (9) gives $\lambda_3\left(A\right) \geq 1.4522$, only prior to the result (11).

To further illustrate our bounds we consider the following example.

Example 2. Let

$$B = \begin{bmatrix} 1 & 0 & -0.2 \\ 0 & 1 & 0.5 \\ -0.2 & 0.5 & 1 \end{bmatrix}.$$

For this matrix with $trB = 3$, we have the comparison results of lower bounds for $\lambda_n(B)$ in Table 1.

	(3)	(4)	(7)
$\lambda_3(B) >$	0.3156	0.3819	0.4125
	(11) $(k=l=n)$	(12) $(k=l=n)$	(13) $(k=l=n)$
$\lambda_3(B) \geq$	0.3156	-0.0081	0.3782

Table 1: Lower bounds for $\lambda_n(B)$.

However, the exact smallest eigenvalue is $\lambda_3(B) = 0.4615$.

Remark 2. From the Example 2 we see that the lower bound for $\lambda_n(B)$ is accurate by (7). Theorem 4.2 in [4] fails to provide nontrivial lower bound for $\lambda_n(B)$ in Example 2.

Remark 3. The bound (7) is always at least as large as the bound (4). Since for any matrix B with $trB = n$,

$$\det B = \prod_{i=1}^{n} \lambda_i(B) \leq \left(\frac{\sum_{i=1}^{n} \lambda_i(B)}{n} \right)^n = \left(\frac{trB}{n} \right)^n = 1.$$

Hence,

$$\left(\frac{n-1}{n} \right)^{n-1} \det B < 1.$$

This implies $\theta(B) > 1$ and thus the lower bound for $\lambda_n(B)$ in (4) has been improved by Lemma 3.2.

5. Upper bounds for the Perron root of nonnegative symmetric matrices

Nonnegative matrix has applications in many areas [6]. Let A be a matrix with all entries nonnegative. By the Perron-Frobenius theorem, A has a characteristic root equal to its spectral radius, which is called the Perron root of A and is usually denoted by $\rho(A)$. Bounds for the Perron root have been surveyed by many authors. In this section, we give upper bounds for the Perron root of a nonnegative symmetric matrix satisfied some certain conditions.

We have the following result.

Theorem 5. Let A be a nonnegative symmetric matrix which is strictly diagonally dominant. Let $\{x_k\}$ be the sequence defined by (2). Then $\{x_k^{\,n-1}\}$ is a decreasing sequence of upper bounds for $\rho(A)$ if $x_0 = (trA)^{1/n-1}$.

The result is obvious and hence its proof is omitted.

Example 3. Let

$$A=\begin{bmatrix}4&1&0&2\\1&5&1&2\\0&1&6&3\\2&2&3&8\end{bmatrix}.$$

Clearly, A is positive definite and $\rho(A)=11.4698$. For this matrix equation (1) is

$$x^4-23x+25.0330=0.$$

The application of Theorem 5 gives the following upper bounds for $\rho(A)$ in Table 2.

k	x_k	$x_k^{\ n-1}$
0	2.8439	23.0008
1	2.4811	15.2733
2	2.3272	12.6098
3	2.2966	12.1131
4	2.2954	12.0944
5	2.2954	12.0944

Table 2: Upper bounds for $\rho(A)$.

Remark 4. In Theorem 5, the strictly diagonal dominance is sufficient to guarantee nonnegative symmetric matrix A is positive definite. See the following matrix

$$A=\begin{bmatrix}4&1&0&2\\1&5&1&2\\0&1&6&3\\2&2&3&3\end{bmatrix},$$

which is not strictly diagonally dominant in the last row and we can also apply Theorem 5 to estimate the upper bounds for the Perron root of A. Actually the above matrix A is positive definite with eigenvalues $\lambda_1(A)=0.1907$, $\lambda_2(A)=3.4724$, $\lambda_3(A)=5.0316$, $\lambda_4(A)=9.3053$.

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