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Computability of the Solution Operator of the Generalized KdV Equation

Dianchen Lu Qingyan Wang Shengrang Cao

Nonlinear Scientific Research Center,Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu,212013

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Abstract. In this paper, we study the initial value problem of the Generalized KdV equation, define a nonlinear map $K_R : H^s(\mathbb{R}) \to C(\mathbb{R}, H^s(\mathbb{R}))$ $(s \geq 3)$, and prove K_R is Turing computable for $s \geq 3$. Therefore, the solution of the Generalized KdV equation with arbitrary precision on Turing machines can be satisfied.

Keywords. Generalized KdV equation, Sobolev space, computability, Turing machines

1. Introduction

Regardless of in daily life, or in project and scientific research, people continuously carry on computation. However, we carry out an in-depth study on the computability in recently several decades. At present, the computability of solutions of the nonlinear developed equation has become an important topic to mathematics and theory computer workers. The Generalized KdV equation $u_t + u^m u_x + u_{xxx} = 0$ is an important equation, frequently appears in physics, hydrodynamics, biological and chemical fields. When $m = 1$, the equation is the KdV equation which describes small amplitude wave in shallow water equation, the magnetic fluid in cold plasma, and the wave process in biological and physical systems. Klaus Weihrauch and Ning Zhong have studied the solution operator of the KdV equation, and provided effective method for other equations in [1]. When $m = 2$, the equation is the mKdV equation which describes the acoustic spread of non-harmonic Lattice and the Alfen wave sport of non-collision plasma in plasma physics, solid physics, atomic physics, hydrodynamics and the theory of quantum, Dianchen Lu and Qingyan Wang have proved that the solution operator of the equation is computable in [9]. The KdV equation and the mKdV equation are the general form of the Generalized KdV equation. Therefore, studing the computability of the solutions of the Generalized KdV equation is very important.

2. Preliminaries

The computability of subsets and functions on the discrete (countable) sets is usually defined by means of Turing machines. Both inputs and outputs of a Turing machine are finite words. In order to investigate the computability on uncountable sets, the Turing machines have been extended by Klaus Weihrauch so that their inputs and outputs can be infinite sequences as well. These machines are usually called Type 2 Turing machines and they can be used to define the computability on the set Σ^ω of infinite sequences in an analogous way while the (classic) Turing machines introduce the computability to the set $\sum *$. If we want to introduce the computability to other set D of a cardinality up to continuum, we can choose a representation $\delta : \Sigma^\omega \to D$ which is simply a surjective function. That is, the representation δ assigns (possibly infinite) names

(δ -names) to each element $x \in D$ and transfers the computability on Σ^ω straightforwardly to the set D. For example, an element $x \in D$ is called δ -computable if it has a computable δ -name.

In order to investigate the computability of the solution operation of various differential equations, we have to introduce the corresponding computability notion to the function spaces at first. In this section, we recall the definitions of computability on several function spaces which are necessary for our discussions. They essentially belong to Klaus Weihrauch and Ning Zhong.

Usually, we are interested in the computability on some metric spaces.

2.1.

If a metric space (M,d) has a countable dense subset, we can define its effectivization as a computable quadruple metric space (M,d,A,ν) in which (1) A is a dense subset of M ; (2) $\nu :\subseteq \Sigma^* \to A$ is a surjective function (so-called notation of A); and (3) the set $\{u,v,w,x \in \Sigma^* : \nu_Q(w) < d(\nu(u),\nu(v)) < \nu_Q(x)\}$ is a recursively enumerable set, where $\nu_Q : \Sigma^* \to \mathbb{Q}$ is the notation of the rational numbers. In a computable metric space (M,d,A,ν) we can introduce the computability to the following Cauchy representation $\delta_C : \Sigma^\omega \to M$ which is a surjective function such that $\delta_C(p) = x$ if and only if $p = w_0 \# w_1 \# w_2 \# \cdots$ with $w_i \in dom(\nu)$ and the sequence $\{\nu(w_i)\}$ converges effectively to x in the sense that $d(x,\nu(w_i)) \le 2^{-i}$ for all $i \in \mathbb{N}$.

2.2.

The Cauchy representation δ_{L^2} of the computable metric space $(L^2(\mathbb{R}), d_{L^2}, \sigma, \nu_{L^2})$ for any $p \in \Sigma^\omega$, $g \in L^2(\mathbb{R})$ is a δ_{L^2} -name of g iff $p = w_0 \# w_1 \# w_2 \# \cdots$ with $w_i \in dom(\nu_{L^2})$ and $\| \nu_{L^2}(w_i) - g \| \le 2^{-i}$ for all $i \in \mathbb{N}$,where $L^2(\mathbb{R}) = \{ f(x) | \left(\int_{\mathbb{R}} f^2(x) dx \right)^{1/2} < \infty \}$, $d_{L^2}(f,g) = \| f - g \|_{L^2}$, σ is the set of all rational finite step functions and ν_{L^2} is a notation of σ.

2.3.

The Sobolev space $H^s(\mathbb{R})$ is the set of all functions $f \in L^2(\mathbb{R})$ such that $T_s(f) \in L^2(\mathbb{R})$,where

$$T_s(f)(\xi) := \left(1 + |\xi|^2\right)^{s/2} F(f)(\xi) \in L^2(\mathbb{R})$$

is a weighted Fourier transform of f with weight $\left(1+|\xi|^2\right)^{s/2}$, $F(f)(\xi)$ denotes the Fourier transform of f . A infinite word $p \in \Sigma^\omega$ is a δ_{H^s} -name of $f \in H^s(\mathbb{R})$, iff p is a δ_{L^2} -name of the weighted Fourier transform $T_s(f) \in L^2(\mathbb{R})$.

2.4.

Let $S(\mathbb{R})$ be the Schwartz space defined by

$$S(\mathbb{R}) = \{\phi \in C^\infty(\mathbb{R}) : \forall \alpha, \beta \in \mathbb{N}, \sup_{x \in \mathbb{R}} | x^\alpha \phi^{(\beta)}(x) | < \infty\}$$

(1) The representation $\delta_s :\subseteq \Sigma^\omega \to S(\mathbb{R})$ of the Schwartz space $S(\mathbb{R})$: for any $\phi \in S(\mathbb{R})$ and $p, q \in \Sigma^\omega$, $\delta_s(\langle q, p \rangle) = \phi \Leftrightarrow \delta_\infty^p(p) = \phi$ for $q = u_0 \# u_1 \# u_2 \cdots$, where $u_k \in dom(\nu_{\mathbb{N}})$ and $\sup_{|x| \ge \nu_{\mathbb{N}}(u<i,j,n>)} | x^i \phi^{(j)}(x) | \le 2^{-n}$.

(2) $\delta_{sc} :\subseteq \Sigma^\omega \to S(\mathbb{R})$ is the Cauchy representation of the computable metric space $\left(S(\mathbb{R}), d_s, \mathrm{P}^*, \nu_\infty^p\right)$,where P^* is the set of the truncated polynomials with rational coefficients and ν_∞^p is the notation of P^*.

(3) A infinite word $p \in \Sigma^\omega$ is $\tilde{\delta}_{H^s}$ -name of f , iff $p = \langle p_0, p_1, \cdots \rangle$ with $p_i \in dom(\delta_{sc})$ and $\| \delta_{sc}(p_i) - f \|_s \le 2^{-i}$.

2.5.

$C\left(\mathbb{R}:H^{s}\left(\mathbb{R}\right)\right)$ is the set of all continuous functions from $\mathbb{R}$ to $H^{s}\left(\mathbb{R}\right)$, and $[\rho\to\delta_{H^{s}}]$ is a representation of $C\left(\mathbb{R}:H^{s}\left(\mathbb{R}\right)\right)$.

3. Main Result

For rigorous notation we occasionally write $u(x,t):=u(t)(x)$, the functions $u(t):\mathbb{R}\to C\left(\mathbb{R}:H^{s}\left(\mathbb{R}\right)\right)$.

The initial value problem (IVP for short) of the Generalized KdV equation on the real line $\mathbb{R}$,

$$u_t+u^m u_x+u_{xxx}=0 \qquad (t,x\in\mathbb{R})$$
$$u\left(x,0\right)=\varphi\left(x\right) \tag{1}$$

we establishes a nonlinear map K_R from the initial data $\varphi\in H^{s}\left(\mathbb{R}\right)$ for $s\geq 3$ to the solution $u\in C\left(\mathbb{R}:H^{s}\left(\mathbb{R}\right)\right)$, defined by $K_R\left(\varphi\right)=u$. In this section, we shall prove our main result:

Theorem3.1 The solution operator $K_R:H^{s}\left(\mathbb{R}\right)\to C\left(\mathbb{R}:H^{s}\left(\mathbb{R}\right)\right)$ of the initial value problem (1) is $\left(\delta_{H^s},\left[\rho\to\delta_{H^s}\right]\right)$-computable for any integer $s\geq 3$.

The following equivalent integral equation of the initial value problem (1)

$$u\left(t\right)=F^{-1}\left(E\left(t\right)\cdot F\left(\varphi\right)\right)-\frac{1}{m+1}\int_0^t F^{-1}\left(E\left(t-\tau\right)\cdot F\left(\frac{d}{dx}\left(u\left(\tau\right)\right)^{m+1}\right)\right)d\tau \tag{2}$$

where $\mathrm{u}\left(\mathrm{t}\right)\left(x\right):=u\left(x,t\right),E\left(t\right)\left(x\right):=e^{ix^3t}$, $F\left(\varphi\right)\left(x\right)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-ix\xi}\varphi\left(\xi\right)d\xi$.

In the iterative, we show that the following iterative sequence with the initial data φ :

$$V_0\left(t\right)=F^{-1}\left(E\left(t\right)\cdot F\left(t\right)\right)$$
$$V_{j+1}\left(t\right)=V_0\left(t\right)-\frac{1}{m+1}\int_0^t F^{-1}\left(E\left(t-\tau\right)\cdot F\left(\frac{d}{dx}\left(V_j\left(\tau\right)\right)^{m+1}\right)\right)d\tau \tag{3}$$

The iterative sequence (3) is contracting near $t=0$, thus the sequence converges to a unique limit. Since the limit satisfies the integral equation (2), it is the solution of the initial value problem (1) near $t=0$. To prove that the solution operator is computable, we need to construct a type-2 Turing machine whose structure sees [1].

Firstly, we define the operator

$$S\left(u,\varphi,t\right)=F^{-1}\left(E\left(t\right)\cdot F\left(\varphi\right)\right)-\frac{1}{m+1}\int_0^t F^{-1}\left(E\left(t-\tau\right)\cdot F\left(\frac{d}{dx}\left(u\left(\tau\right)\right)^{m+1}\right)\right)d\tau$$

which is $\left(\left[\rho\to\delta_s\right],\delta_s,\rho,\delta_s\right)$ -computable. This follows Lemma 3.2 in [8] straightforwardly. Therefore, the function $\overline{S}\left(u,\varphi\right)\left(t\right):=S\left(u,\varphi,t\right)$ is $\left(\left[\rho\to\delta_s\right],\delta_s,\left[\rho\to\delta_s\right]\right)$-computable. Then we define the function $v:S\left(\mathbb{R}\right)\times\mathbb{N}\to C\left(\mathbb{R}:S\left(\mathbb{R}\right)\right)$ by

$$v\left(\varphi,0\right)=\overline{S}\left(0,\varphi\right)$$
$$v\left(\varphi,j+1\right)=\overline{S}\left(v\left(\varphi,j\right),\varphi\right)$$

it is easy to verify that v is $\left(\delta_s,\gamma_{\mathbb{N},}\left[\rho\to\delta_s\right]\right)$-computable.

Proposition 3.2 If $u(x,t)$ is the solution of the IVP (1), then there is a computable function $e:\mathbb{N}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that

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$$\sup_{0\le t\le T} \| u(x,t) \|_s \le e_T^s(\| \varphi \|_s)$$

where $e_T^{\,s}(r) := e(s,T,r)$, s is an integer and $s \ge 3$.

Proof. see [3].

Proposition3.3 $v^0 := \overline{S}(0, \varphi)$, and $v^{j+1} := \overline{S}(v^j, \varphi)$. If

$$2^{m+1} a_T^s T^{1/2} (3+T)^{(m+1)/2} \| \varphi \|_s^m + 8(3+T) d_T^s T^{1/2} \| \varphi \|_s \le 1,$$

then we have

$$\| v^{j+1}(t) - v^j(t) \|_s \le 2^{-j-1} (3+T)^{1/2} \| \varphi \|_s$$

where

$$d_T^s = \sqrt{s}\left(e_T^s\left(\|\varphi\|_s\right)+1\right)^{m-1} \cdot \left[\left(2^{(s+1)^2}+1\right)\cdot T^{1/2}+1\right], a_T^s = \sqrt{s}\cdot 2^s \cdot T^{1/2}+1, 0\le t\le T, \varphi \in H^s(\mathbb{R}).$$

Firstly, we need to construct the space. Let T>0, the continuous function $u: Y \to H^s(\mathbb{R})$ with $[0,T] \subseteq Y$, define

$$\Lambda_{1,T}^s(u) = \sup_{0\le t\le T} \left\| u(\bullet,t) \right\|_s$$

$$\Lambda_{2,T}^s(u) = \left(\sup_{x\in\mathbb{R}} \int_0^T \left| D_x^{s+1} u(x,t) \right|^2 dt \right)^{1/2}$$

$$\Lambda_{3,T}^s(u) = \left(\int_{\mathbb{R}} \sup_{0\le t\le T} \left| u(x,t) \right|^2 dx \right)^{1/2}$$

$$\|u\|_{X_T^s} := \Lambda_T^s(u) := \left(\left(\Lambda_{1,T}^s(u)\right)^2 + \left(\Lambda_{2,T}^s(u)\right)^2 + \left(\Lambda_{3,T}^s(u)\right)^2 \right)^{1/2}$$

Then $X_T^s = \left\{ u \in C\left([0,T]; H^s(\mathbb{R}); \Lambda_T^s(u) < \infty\right) \right\}$ is a Banach space with the norm $\|u\|_{X_T^s}$.

Lemma 3.4 If $T > 0$, $u \in X_T^s$, $\sup_{0\le t\le T} \| u(x,t) \|_s \le e_T^s(\| \varphi \|_s)$, then we have

$$\int_0^T \left\| u^m u_x \right\|_s dt \le a_T^s T^{1/2} \|u\|_{X_T^s} \|u\|_{X_T^s}$$

where $d_T^s = \sqrt{s}\left(e_T^s\left(\|\varphi\|_s\right)+1\right)^{m-1} \cdot \left[\left(2^{(s+1)^2}+1\right)\cdot T^{1/2}+1\right]$,and $e_T^s(\| \varphi \|_s)$ is the same form as Proposition 3.2.

Proof. For $s \ge 3$

$$D_x^s\left(u^m u_x\right) = \sum_{i=1}^{s} \binom{s}{i} \left(u^m\right)^{(i)} \left(u_x\right)^{(s-i)} + u^m u^{(s+1)}$$

$$\left\| D_x^s\left(u^m u_x\right) \right\| \le \sum_{i=1}^{s} \binom{s}{i} Q\left(\| u \|, \| u_x \|, \cdots \| u^{(i)} \|\right) \| u^{(s-i+1)} \| + \| u^m \| \| u^{(s+1)} \|$$

where Q is the polynomials about $u, u_x, \cdots u^{(i)}$ (see [3]).

Since $\left\| f^{(k)} \right\| = \left\| Ff^{(k)} \right\| = |\xi|^k \left\| Ff \right\|$, $1 + |\xi|^2 + \ldots + |\xi|^{2s} \le s\left(1 + |\xi|^s\right)^2$

$$\left\| u^m u_x \right\|_s \le \sqrt{s}\left(\left\| u^m u_x \right\| + \left\| D_x^s\left(u^m u_x\right) \right\| \right)$$

$$\le \sqrt{s}\sum_{i=1}^{s}\binom{s}{i}Q\left(\|u\|,\|u_x\|,\cdots\|u^{(i)}\|\right)\|u^{(s-i+1)}\|+\sqrt{s}\left\|u^m u^{s+1}\right\|+\sqrt{s}\,\|u^m\|\|\|u_x\|$$

$$\le \sqrt{s}\left(\sum_{i=1}^{s}\binom{s}{i}2^{i(i+1)/2}+1\right)\|u\|_s^{m+1}+\sqrt{s}\,\|u\|_s^m\|u^{(s+1)}\|$$

$$\le \sqrt{s}\left(e_T^s\left(\|\varphi\|_s\right)+1\right)^{m-1}\cdot\left[\left(\sum_{i=1}^{s}\binom{s}{i}2^{i(i+1)/2}+1\right)\|u\|_s\bullet\|u\|_s+\|u\|_s\bullet\|u^{(s+1)}\|\right]$$

We know $\sup_{0\le t\le T}\|u(x,t)\|_s\le e_T^s\left(\|\varphi\|_s\right)$, by proposition 3.2, the lemmma follows straightforwardly.

Proof.(of Proposition 3.3) Let $W(t)\varphi=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{ix\xi}e^{i\xi^3 t}\hat{\varphi}(\xi)d\xi$

Since $v^0:=\bar{S}(0,\ \varphi)$, $v^{j+1}:=\bar{S}\left(v^j,\varphi\right)$, by Lemma 3.4 and Lemma 4.9,4.10 in [1], for $j\ge 1$,

$$\|v^j\|_{X_T^s}=\|W(t)\varphi-\frac{1}{m+1}\int_0^t W(t-\tau)\left(\left[\left(v^{j-1}\right)^{m+1}\right]_x\right)d\tau\|_{X_T^s}$$

$$\le(3+T)^{1/2}\|\varphi\|_s+(3+T)^{1/2}\int_0^T\|\left(v^{j-1}\right)^m v_x^{j-1}\|_s\,d\tau$$

$$\le(3+T)^{1/2}\|\varphi\|_s+(3+T)^{1/2}d_T^sT^{1/2}\|v^{j-1}\|_{X_T^s}^2$$

Let $T\ge 0$ such that $2^{m+1}a_T^sT^{1/2}(3+T)^{(m+1)/2}\|\varphi\|_s^m+8(3+T)d_T^sT^{1/2}\|\varphi\|_s\le 1$,

from $\|v^0\|_{X_T^s}=\|W(t)\varphi\|_{X_T^s}\le(3+T)^{1/2}\|\varphi\|_s$, we obtain by induction

$$\|v^j\|_{X_T^s}\le 2(3+T)^{1/2}\|\varphi\|_s\ \left(\forall j\in\mathbb{N}\right)$$

Then by Lemma 4.8 in [1]

$$\|v^j-v^{j-1}\|_{X_T^s}=\|\frac{1}{m+1}\int_0^t W(t-\tau)\left(\left[\left(v^{j-1}\right)^{m+1}\right]_x-\left[\left(v^{j-2}\right)^{m+1}\right]_x\right)d\tau\|_{X_T^s}$$

$$\le\frac{1}{m+1}(3+T)^{1/2}a_T^sT^{1/2}\|\sum_{i=1}^{m}\left(v^{j-1}\right)^i\left(v^{j-2}\right)^{m-i}\|_{X_T^s}\|\left(v^{j-1}-v^{j-2}\right)_x\|_{X_T^s}$$

$$\le 2^m a_T^sT^{1/2}(3+T)^{(m+1)/2}\|\varphi\|_s^m\|v^{j-1}-v^{j-2}\|_{X_T^s}$$

$$\le\frac{1}{2}\|\left[v^{j-1}-v^{j-2}\right]\|_{X_T^s}$$

If $2^{m+1}a_T^sT^{1/2}(3+T)^{(m+1)/2}\|\varphi\|_s^m+8(3+T)\alpha_T^sT^{1/2}\|\varphi\|_s\le 1,\ 0\le t\le T$, we obtain the result that

$$\|y^{j+1}(t)-y^j(t)\|_s\le\|y^{j+1}(t)-y^j(t)\|_{X_T^s}\le 2^{-j-1}(3+T)^{1/2}\|\varphi\|_s$$

Proposition 3.5 If $2^{m+1}a_T^sT^{1/2}(3+T)^{(m+1)/2}(\|\varphi\|_s+1)^m+8\alpha_T^sT^{1/2}(3+T)\left(\|\varphi\|_s+1\right)\le 1$, we have

$$\left\|v(t)-v_n(t)\right\|_s\le 2(3+T)^{1/2}\left\|\varphi-\varphi_n\right\|_s$$

where $d_T^s=\left(e_T^s\left(\|\varphi\|_s\right)+1\right)\cdot\sqrt{s}\cdot 2^s\cdot\left(2^s+1\right)\cdot T^{1/2}+1$, $a_T^s=\sqrt{s}\cdot 2^s\cdot T^{1/2}+1,0\le t\le T$.

Proof. Since

$$v(t)=\bar{S}(v,\varphi)(t),$$

$$v_n(t)=\bar{S}(v_n,\varphi_n)(t) \quad \| v-v_n \|_{X_T^s} = \| W(t)(\varphi-\varphi_n)-\frac{1}{m+1}\int_0^t W(t-\tau)\left[v^{m+1}-v_n^{m+1}\right]_x d\tau \|_{X_T^s}$$

$$\le \frac{1}{m+1}(3+T)^{1/2} a_T^s T^{1/2} \| \sum_{i=0}^{m-1}\left(v^{j-1}\right)^i \left(v_n^{\ j-2}\right)^{m-1-i} \|_{X_T^s} \| v-v_n \|_{X_T^s} + (3+T)^{1/2} \| \varphi-\varphi_n \|_s$$

By proposition3.3, if $2^{m+1}a_T^s T^{1/2}(3+T)^{(m+1)/2}(\|\varphi\|_s+1)^m+8\alpha_T^s T^{1/2}(3+T)(\|\varphi\|_s+1)\le 1$ （notice that $\|\varphi_n\|_s \le \|\varphi\|_s+1$）, then

$$\| v-v_n \|_{X_T^s} \le (3+T)^{1/2} \| \varphi-\varphi_n \|_s + \frac{1}{m+1}(3+T)^{1/2} \alpha_T^s T^{1/2} \| \sum_{i=0}^{m}\left(v^{j-1}\right)^i \left(v_n^{\ j-2}\right)^{m-i} \|_{X_T^s} \| v-v_n \|_{X_T^s}$$

$$\le (3+T)^{1/2} \| \varphi-\varphi_n \|_s + 2^m a_T^s T^{1/2}(3+T)^{(m+1)/2}(\|\varphi\|_s+1)^m \| v-v_n \|_{X_T^s}$$

$$\le (3+T)^{1/2} \| \varphi-\varphi_n \|_s + \frac{1}{2}\| v-v_n \|_{X_T^s}$$

Therefore $\| v-v_n \|_{X_T^s} \le 2(3+T)^{1/2} \| \varphi-\varphi_n \|_s$, the sequence $\{v_n\}$ is uniform convergence.

Proof. (of Theorem 3.1) For a given initial value $\varphi \in H^s(\mathbb{R})$ and a rational number $\bar{T}>0$ we will show how to compute the solution $u(t)$ of the initial value problem (1) at the time $0\le t\le \bar{T}$. For this purpose, we first find some appropriate rational number T such that $0<T<\bar{T}$ and show how to compute $u(t)$ from t' and $\psi := u(t')$ at the time interval $[t',t'+T]$, $0\le t'\le \bar{T}$ by a fixed point iteration. Using this method, we can compute the values $u(T/2m)$ successively for $m=1,2,\cdots$ and finally $u(t)$ for any $0\le t\le \bar{T}$.

If $u_t+u^m u_x+u_{xxx}=0$, $u(x,t')=\psi(x)$, and define $v(x,t):=u(x,t+t')$, then

$$v_t+v^m v_x+v_{xxx}=0 \quad x\in\mathbb{R}, t\ge 0$$

$$v(x,0)=\psi(x) \tag{4}$$

We assume that the initial value $\psi\in H^s(\mathbb{R})$ is given by a $\tilde{\delta}_{H^s}$-name, i.e., by a sequence $\psi_0,\psi_1\cdots$ of Schwartz functions such that $\|\psi-\psi_n\|\le 2^{-n}$ .For any $n\in\mathbb{N}$, we define function $v_n^{\ 0}, v_n^{\ 1},\cdots \in C(\mathbb{R}:S(\mathbb{R}))$ by

$$v_n^{\ 0} := \bar{S}(0,\psi_n) \quad v_n^{\ j+1} := \bar{S}(v_n^{\ j},\psi_n)$$

We note that the sequence $\{v_n^{\ j}\}$ can be computed from ψ_n. By Proposition 3.3, the iterative sequence $v_n^{\ 0}, v_n^{\ 1},\cdots$ converges to some v_n, then v_n is the fixed point of the iteration $\bar{S}$ and satisfies the following internal equation:

$$v_n(t)=\bar{S}(v_n,\psi_n)$$

$$=F^{-1}(E(t)\cdot F(\psi_n))-\frac{1}{m+1}\int_0^t F^{-1}\left(E(t-\tau)\cdot F\left(\frac{d}{dx}(v_n(\tau))^{m+1}\right)\right)d\tau$$

hence v_n solves the initial value problem:

$$\frac{\partial v_n}{\partial t}+v_n^{\ m}\frac{\partial v_n}{\partial x}+\frac{\partial v_n}{\partial x^3}=0 \quad v_n(x,0)=\psi_n(x)$$

By Proposition 3.5, we show that, by a contraction argument, for some sufficiently small computable real number $T>0$ (depending only on $\overline{T}$ and φ), $v_n^{\,j}(t)\to v_n(t)$ as $j\to\infty$ for all n, and $v_n(t)\to v(t)$ as $n\to\infty$, sufficiently fast and uniformly in $t\in[0,T]$. We recall that v is the solution of the initial value problem (4). Then we can effectively determine a computable subsequence of the double sequence $\{v_n^{\,j}\}$ which converge fast to v uniformly in $t\in[0,T]$.

Since v is the limit of a fast convergent computable sequence, v itself is computable. So $K_{R:}(\varphi,t)\to u(t)$ is $(\delta_{H^s},\rho,\delta_{H^s})$ - computable for $t\geq 0$, define the reflection $R:S(\mathbb{R})\to S(\mathbb{R})$, $R(\psi)(x):=\psi(-x)$ is $(\delta_{H^s},\delta_{H^s})$ - computable. Let $u'(t)(x):=u(-t)(-x)$, then $u'_t+u'^m u'_x+u'_{xxx}=0$ and for $t\geq 0$, $u(-t)=R\circ u'(t)=R\circ K_R(u'(0),t)=R\circ K_R(R(\varphi),t)$, i.e, $u(t)=R\circ K_R(R(\varphi),-t)$ for $t\leq 0$. Therefore, as the join at 0 of two computable functions, K_R is computable for $t\in\mathbb{R}$. (see e.g. Lemma 4.35 in [1]).

Thus, we prove the main result, can see that the machine searches fast approximations to $u(x,t)$, and computes the solutions of the Generalized KdV equation with arbitrary precision. This approach can be extended to other nonlinear equations.

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