ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 3, No. 1, 2008, *pp*. 62-68

Projection Algorithm with Line Search for Solving the Convex Feasibility Problem [1]

Li Li and Yan Gao

School of Management, University of Shanghai for Science and Technology, Shanghai, 200093, China

(*Received June 3 2007, accepted January 20 2008*)

Abstract. An iterative projection algorithm by adopting Armijo-like line search to solve the convex feasibility problem (CFP) is presented and the convergence is shown under some conditions. Moreover, as a by-product, the unfixed stepsize factor is not confined to the interval (0, 2). A numerical test is listed and the results generated are really impressive, which indicate the line search method is promising.

Keywords: Convex feasibility; Iterative projection algorithm; Armijo-like line search

1. Introduction

A very common problem in diverse areas of mathematics and physical sciences consists of trying to find a point in the intersection of convex sets which is referred to as the convex feasibility problem (CFP); There are many applications, especially in the field of image restoration (see, for instance, [9,14-15]). The convex inequality problem, is to find solution of the set:

$$X = \{x \in R^n : g_i(x) \le 0, i = 1, \dots, m\},$$

where g_i are convex functions and X is nonempty. It is a certain instance of CFP. It is well known that some convex programming problems can be transformed into an equivalent system of convex inequalities and linear equations through the use of the Karush Kuhn Tucker conditions. Iterative projection algorithms have been highly recommended for solving this problem, and many well-known iterative algorithms for solving it were established; see [1-3, 6, 8,10, 12-13].

Denoted by P_C, the orthogonal projection onto C, that is, $P_C(x)$ minimizes $\|c-x\|$ over all $c \in C$. Typically one iteration for solving the CFP of finding $x \in X$, where X is a nonempty closed convex set, is given by

$$x^{k+1} = x^k + \omega_k (P_{S_k}(x^k) - x^k), \tag{1}$$

where $0 < \eta \le \omega_k \le 2-\eta$,with $0 < \eta \le 1$ and $P_{S_k}(x^k)$ is the projection of x^k on the closed set $S_k \supseteq X$. At each iteration we define a working set of inequalities $Q_k \subseteq \{1, \dots, m\}$ and perform a projection on

$$S_k = \{ y \in R^n : g_i(x^k) + \langle \xi^k, y - x^k \rangle \le 0, i \in Q_k \}.$$

The projection $P_{S_k}(.)$ is a quadratic programming problem that can be solved efficiently.

We notice some other previous algorithms use a fixed stepsize, which sometimes affects convergence of the algorithms. In this paper, we modify the projection algorithm by adopting Armijo-like line search, which is popular in iterative algorithms for solving nonlinear programming problems, variational inequality problems and so on [5,11,16]. The proposed algorithm makes an accelerated convergence to the solution of CFP. Moreover, the unfixed stepsize factor is not confined to the interval (0, 2). We also show convergence of the proposed algorithm under reasonable assumptions.

The rest of this paper is organized as follows. Section 2 reviews some concepts and exiting results. Section 3 gives a modification on the iterative projection algorithm and shows its convergence. Section 4

[1] This work was supported by National Science foundation of China (under grant:10671126) and Shanghai leading academic discipline project(under grant:T0502)}.

gives numerical test. Section 5 gives some concluding remarks.

2. Preliminaries

In this section, we review some definitions and basic results which will be used later on.

Definition 2.1. Let F be a mapping from set $X \subset R^n$ into R^n. Then

(a) F is said to be Lipschitz continuous on X with constant $\lambda > 0$, if

$$\|F(x) - F(y)\| \le \lambda \|x - y\|, \quad \forall x, y \in X ;$$

(b) F is said to be uniformly monotone on X with modulus $\delta > 0$, if

$$\langle F(x) - F(y), x - y \rangle \ge \delta \|F(x) - F(y)\|^2, \quad \forall x, y \in X .$$

Definition 2.2. For any $x \in R^n$, subdifferential of g at x is defined as follows:

$$\partial g(x) = \{ \xi \in R^n : g(z) \ge g(x) + \langle \xi, z - x \rangle , \ \forall z \in R^n \}.$$

It is true that subdifferential is (uniformly) bounded on R^n.

For the given nonempty closed convex subset in R^n, the orthogonal projection from R^n onto X is defined by

$$P_X(y) = \arg\min\{\|x - y\| x \in X\}, \quad y \in R^n .$$

It has the following well-known properties.

Lemma 2.1 (see [17]**).** Let X be a nonempty closed convex subset in R^n, then for any $y, z \in R^n$ and $x \in X$,

(1) $\langle y - P_X(y), P_X(y) - x \rangle \ge 0$;

(2) $\|P_X(y) - P_X(z)\|^2 \le \langle P_X(y) - P_X(z), y - z \rangle$;

(3) $\|P_X(z) - x\|^2 \le \|z - x\|^2 - \|P_X(z) - z\|^2$.

Lemma 2.2 (see [1]**).** Let X be a nonempty closed convex subset in R^n, for some certain $x \in X$ and $d \in R^n$, define

$$H(a) := P_X(x - ad), \quad a \ge 0,$$

we have

(1) $\langle H(a) - x + ad, y - H(a) \rangle \ge 0, \quad \forall y \in X, a \ge 0$;

(2) $\langle d, x - H(a) \rangle \ge \dfrac{\|x - H(a)\|^2}{a}$.

Toint in [7], Gafni and Bertsekas in [4] give the following projection properties, respectively:

Lemma 2.3. Let X be a nonempty closed convex subset in R^n, for any $x \in X$ and $d \in R^n$,

(1) $\|x - H(a)\|$ on $a \ge 0$ is nondecreasing;

(2) $\dfrac{\|x - H(a)\|}{a}$ on $a > 0$ is nonincreasing.

From Lemma 2.1, we know that P_X is Lipschitz continuous (with constant 1) (i.e., $\|P_X(y) - P_X(z)\| \le \|y - z\|$) (see [17]) and uniformly monotone (with modulus 1).

Let F be a mapping from R^n into R^n. For any $x \in R^n$ and $a > 0$, define

$$x(a) = P_X(x - aF(x)), \quad e(x, a) = x - x(a), \quad r(x, a) = \|e(x, a)\|.$$

Remark 2.1. $\|e(x,a)\|$ on $a>0$ is nondecreasing and $\frac{\|e(x,a)\|}{a}$ on $a>0$ is nonincreasing.

Lemma 2.4. Let F be a mapping from R^n into R^n. For any $x\in R^n$ and $a>0$, we have

$$\min\{1,a\}\|e(x,1)\|\le\|e(x,a)\|\le\max\{1,a\}\|e(x,1)\|.$$

3. A Modified Iterative Projection Algorithm and its Convergence

In this section, we establish a projection algorithm. First, we assume the following conditions are satisfied:

(1) The set X is given by:

$$X=\{x\in R^n : g(x)\le 0\},$$

where $g(x)=\max_{1\le i\le m}\{g_i(x)\}$ and g: $R^n\to R$ is convex and X is nonempty. Obviously, where $g_i(x)$ are convex and $\{1,\dots,m\}$ is an index set, may be regarded as equivalent to the single inequality $g(x)\le 0$ with $g(x)=\max_{1\le i\le m}\{g_i(x)\}$.

(2) For any $x\in R^n$, at least one subgradient $\xi\in\partial g(x)$ can be calculated.

Now we give the algorithm.

Let x^0 be arbitrary. For $k=0,1,\dots,$ calculate (1),

$$S_k=\{x\in R^n : g(x^k)+\langle\xi^k,x-x^k\rangle\le 0\},$$

where ξ^k is an element in $\partial g(x^k)$.

By the definition of subdifferential, it is clear that half space S_k contains X. From the expression of S_k, the orthogonal projection onto S_k can be calculated efficiently.

For every k, using S_k we define the function F_k: $R^n\to R^n$ by

$$F_k(x)=x-P_{s_k}(x)=(I-P_{s_k})x.$$

Remark 3.1. F_k is Lipschitz continuous on R^n with constant 1 and uniformly monotone on R^n with modulus 1, where I denotes the identity operator.

Algorithm 3.1. Let x^0 be arbitrary. For $k=0,1,\dots,$ if x^k is not in the solution of CFP, let

$$x^{k+1}=x^k-a_kF_k(x^k), \tag{2}$$

where $a_k=\gamma l^{m_k}$ with $\gamma>0$, $l\in(0,1)$ and m_k is the smallest noninteger such that

$$\|F_k(x^{k+1})-F_k(x^k)\|\le\mu\frac{\|x^{k+1}-x^k\|}{a_k},\ \mu\in(0,1). \tag{3}$$

By Algorithm 4.1, S_k depend on k, and from Remark 4.1, Armijo-like line search rule (3) is well defined.

Lemma 3.1. let x^k is given by Algorithm 4.1. For any $x^*\in X$, we have

$$\langle F_k(x^k),x^{k+1}-x^*\rangle\ge\langle F_k(x^k),x^{k+1}-x^k\rangle\ge\frac{(1-\mu)}{a_k}\|x^{k+1}-x^k\|^2. \tag{4}$$

Proof. Obviously that $x^*\in S_k$ and $F_k(x^*)$=0 for all $k=0,1,\dots$.

By the monotonicity of F_k (Remark 4.1), we have for all $k=0,1,\dots,$

$$\langle F_k(x^k)-F_k(x^*),x^k-x^*\rangle\ge 0,$$

this implies

$$\langle F_k(x^k), x^k - x^* \rangle \geq 0 . \quad (5)$$

Also,

$$\langle F_k(x^k), x^k - x^* \rangle = \langle F_k(x^k), x^{k+1} - x^* + x^k - x^{k+1} \rangle$$
$$= \langle F_k(x^k), x^{k+1} - x^* \rangle - \langle F_k(x^k), x^{k+1} - x^k \rangle \geq 0 ,$$

Therefore, we have

$$\langle F_k(x^k), x^{k+1} - x^* \rangle \geq \langle F_k(x^k), x^{k+1} - x^k \rangle .$$

Moreover, by Lemma 2.2 and (3), we have

$$\langle F_k(x^k), x^{k+1} - x^* \rangle \geq \langle F_k(x^k), x^{k+1} - x^k \rangle$$
$$= \langle F_k(x^{k+1}), x^{k+1} - x^k \rangle - \langle F_k(x^{k+1}) - F_k(x^k), x^{k+1} - x^k \rangle$$
$$\geq \frac{\|x^{k+1} - x^k\|^2}{a_k} - \|x^{k+1} - x^k\| . \mu . \frac{\|x^{k+1} - x^k\|}{a_k}$$
$$= (1-\mu) . \frac{\|x^{k+1} - x^k\|^2}{a_k} . \qquad \square$$

Lemma 3.2. $\mu l \leq a_k \leq \gamma$, for all $k = 0,1,\ldots$.

Proof. Obviously, $a_k = \gamma l^{m_k} \leq \gamma$ for all $k = 0,1,\ldots$. If $a_k = \gamma$, the lemma is proved. If $a_k < \gamma$, from the search rule (3), we know that $\frac{a_k}{l}$ must violate this inequality, i.e.,

$$\left\| F_k(x^k) - F_k\left(x^k - \frac{a_k}{l} F_k(x^k)\right) \right\| > \mu \frac{\left\| x^k - \left(x^k - \frac{a_k}{l} F_k(x^k)\right) \right\|}{\frac{a_k}{l}} .$$

By virtue of Remark 4.1, we have

$$\mu l \leq a_k . \qquad \square$$

Now, we establish global convergence of Algorithm 4.1.

Theorem 3.1. Let $\{x^k\}$ be a sequence generated by Algorithm 4.1. If the solution set of the CFP is nonempty, then $\{x^k\}$ converges to a solution of the CIP.

Proof. Let x^* be a solution of the CIP. By (2), (4) and (5),

$$\|x^{k+1} - x^*\|^2 = \|x^k - a_k F_k(x^k) - x^*\|^2$$
$$= \|x^k - a_k F_k(x^k) - x^*\|^2 - \|x^{k+1} - x^k + a_k F_k(x^k)\|^2$$
$$= \|x^k - x^*\|^2 - 2a_k \langle F_k(x^k), x^k - x^* \rangle - 2a_k \langle F_k(x^k), x^{k+1} - x^k \rangle - \|x^{k+1} - x^k\|^2$$
$$\leq \|x^k - x^*\|^2 - 2a_k \langle F_k(x^k), x^{k+1} - x^k \rangle - \|x^{k+1} - x^k\|^2$$
$$\leq \|x^k - x^*\|^2 - 2a_k . \frac{(1-\mu)}{a_k} \|x^{k+1} - x^k\|^2 - \|x^{k+1} - x^k\|^2$$
$$= \|x^k - x^*\|^2 - (3 - 2\mu) \|x^{k+1} - x^k\|^2 .$$

Then, we have

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - (3 - 2\mu) \|x^{k+1} - x^k\|^2 , \text{ for all } k = 0,1,\ldots, \quad (6)$$

which implies the sequence $\{\|x^k - x^*\|^2\}$ is monotonically decreasing and hence $\{x^k\}$ is bounded.

Sometimes we conclude

$$(3-2\mu)\sum_{k=0}^{\infty}\left\|x^{k+1}-x^{k}\right\|^{2} \le \sum_{k=0}^{\infty}\{\left\|x^{k}-x^{*}\right\|^{2}-\left\|x^{k+1}-x^{*}\right\|^{2}\} < +\infty .$$

Consequently, we get

$$\lim_{k\to\infty}\left\|x^{k+1}-x^{k}\right\| = 0 . \tag{7}$$

Assume that $\tilde{x}$ is an accumulation point of $\{x^k\}$ and $\lim_{k_i\to\infty} x^{k_i} = \tilde{x}$, where $\{x^k\}$ is a subsequence of $\{x^k\}$.The main purpose of the remaining part of the proof is to show that $\tilde{x}$ is a solution of CFP.

First we show that $\tilde{x}\in X$. If $a_{k_i} \ge a_{\min} > 0$, by Lemma 2.4 and (7), we have

$$\left\|e(\tilde{x},1)\right\| = \lim_{k_i\to\infty}\left\|e(x^{k_i},1)\right\| \le \lim_{k_i\to\infty}\frac{\left\|x^{k_i}-x^{k_i+1}\right\|}{\min\{1,a_{\min}\}} = 0 .$$

On the other hand, if $\{a_{k_i}\}\to 0$, by Lemma 2.3 and the monotonicity of F_k (Remark 4.1), according to Armijo-like line search, for sufficient large k_i ,

$$\left\|e(x^{k_i},1)\right\| \le \frac{\left\|e(x^{k_i},\frac{1}{\gamma}a_{k_i})\right\|}{\frac{1}{\gamma}a_{k_i}} = \left\|F_{k_i}(x^{k_i})\right\| = \left\|F_{k_i}(x^{k_i})-F_{k_i}(x^{*})\right\|$$
$$\le \left\|x^{k_i}-x^{*}\right\| .$$

Then, it follows

$$\left\|e(\tilde{x},1)\right\| = \lim_{k_i\to\infty}\left\|e(x^{k_i},1)\right\| \le \lim_{k_i\to\infty}\left\|x^{k_i}-x^{*}\right\| =0.$$

Therefore, we conclude that $\tilde{x}\in X$.

Thus, we may use $\tilde{x}$ in place of x^* in (6), and obtain that $\{\left\|x^k-\tilde{x}\right\|^2\}$ is convergent. Because there is a subsequence $\{\left\|x^{k_i}-\tilde{x}\right\|^2\}$ of $\{\left\|x^k-\tilde{x}\right\|^2\}$ converging to 0, then

$$\lim_{k\to\infty} x^k = \tilde{x} . \qquad \square$$

4. Numerical Test.

Example. Let:

$$g_1(x) = -2x_1 - 4x_2 - 5x_3 - x_4 \quad \le 0,$$
$$g_2(x) = 3x_1 - x_2 + 7x_3 - 2x_4 + 1 \quad \le 0,$$
$$g_3(x) = -5x_1 - 2x_2 - x_3 - 6x_4 + 15 \le 0,$$
$$g_4(x) = (x_2 - 2) + (x_1 - 1)^2 \quad \le 0,$$
$$g_5(x) = (x_1 - 2)^4 - (x_2 + 2) \quad \le 0.$$

Initial point $x^0 = (100, 100, 100, 100)$.

Table 1 Iterative projection algorithm $(\omega = \frac{1}{2})$

Number of processors	x^k	$\max_{i=1,\ldots,5}\{g_i(x)\}$
0	(100, 100, 100, 100)	92236714.0000
1	(100.0000, 170.1000, 100.0000, 240.2000)	92236643.3000
2	(99.1416, 169.2415, 99.1416, 239.3417)	89047180.7819
3	(94.4027, 164.5027, 94.4027, 234.6028)	72901699.6898
4	(87.0768, 157.1768, 87.0768, 227.2769)	52389380.0000
5	(78.0262, 148.1262, 78.0262, 218.2263)	33408054.3533
6	(63.6591, 133.7592, 63.6591, 203.8593)	14453886.7008
7	(45.7477, 115.8478, 45.7477, 185.9479)	3662747.0567
8	(25.4735, 95.5736, 25.4735, 165.6737)	303509.1593
9	(3.6240, 73.7241, 3.6240, 143.8242)	75.9855
10	(1.3132, 71.4133, 1.3132, 141.5134)	69.1982

10 runs.

Table 2 Iterative projection algorithm (Armijo-like line search)

Number of processors	x^k	$\max_{l=1,\ldots,5}\{g^l(x)\}$
0	(100, 100, 100, 100)	92236714.0000
1	(100.0000, 107.0100, 100.0000, 114.0200)	92236706.9900
2	(82.9900, 90.0000, 82.9900, 97.0100)	43025375.2963
3	(65.3266, 72.3366, 65.3266, 79.3466)	16082097.0359
4	(47.8812, 54.8912, 47.8812, 61.9012)	4431323.9175
5	(30.3630, 37.3730, 30.3630, 44.3830)	647116.1469
6	(4.3288, 11.3388, 4.3288, 18.3487)	16.0735
7	(0.8515, 7.8618, 0.8585, 14.8717)	6.0324
8	(1.3773, 7.8618, 1.3773, 14.8717)	5.6269
9	(7.8289, 7.8618, 7.8289, 14.8717)	1144.5119
10	(0.7311, 0.7640, 0.7311, 7.7738)	–0.1716

10 runs.

Tables 1 and 2 show the $g_i(x)$ $(i = 1,\ldots,5)$ can attain the maximal values when the number of processors increased. After 10 processors, Table 1 gives the results obtained still larger than zero, which indicate the iterative point is not in the solution of CIP; Table 2 shows after 10 processors the $\max_{i=1,\ldots,5}\{g_i(x)\}$ is smaller than zero, which implies the iterative point has been in the solution of CIP. The results obtained in Table 2 with Armijo-like line search are really impressive.

5. Concluding Remarks

In this paper, a modified iterative projection algorithm with Armijo-like line search for solving the convex inequality problem has been presented. It uses an unfixed stepsize factor related to Armijo-like line search to force an accelerated convergence to the solution of CFP. The corresponding convergence properties have been established. Lemma 4.2 shows that the unfixed stepsize factor is not confined to interval (0, 2), which is rather a surprising property. Also in forthcoming papers there are better algorithms for solving the convex feasibility problem.

6. Reference

[1] H. H. Bauschke and J. M. Borwein. On projection algorithms for solving convex feasibility problems, *SIAM Rev*. 1996, **38**: 367-426.

[2] U. M. Garc´ıa-Palomares and F. J. Gonz´alez-Casta˜no, Incomplete projection algorithms for solving the convex feasibility problem, *Numerical Algorithms*. 1998, **18**: 177-193.

[3] Y. Censor, Iterative methods for the convex feasibility problem, *Ann. Discrete Math*. 1984, **20**: 83-91.

[4] E. M. Gafni and D. P. Bertsekas, Two-metric projection problems and methods for asymmetric variational inequality problems, *Math program*. 1984, **53**: 99-110.

[5] N. H. Xiu and J. Z. Zhang, Some recent advances in projection-type methods for variational inequalities, *J. Comput*. 2003, **152**: 559-85.

[6] R. Aharoni, A. Berman and Y. Censor, An interior points algorithm for the convex feasibility problem, *Adv. Appl. Math.* 1983, **4**: 479-489.

[7] PH.L. Toint, Global convergence of a class of trust region methods for nonconvex minimization in Hilbert space, *SIMA J. Numer. Anal*. 1998, **8**: 231-52.

[8] N. Echebest, M. T. Guardarucci, H. Scolink and M. C. Vacchino, An acceleration scheme for solving convex feasibility problems using incomplete projection algorithms, *Numerical Algorithms*. 2004, **35**: 331-350.

[9] Y. Censor, M. D. Altschuler and W. D. Powlis, On the use of Cimmino's simultaneous projections method for computing a solution of the inverse problem in radiation therapy treatment planning, *Inverse Problems*. 1998, **4**: 607-623.

[10] A. R. De Pierro and A. N. Iusem, A Finitely Convergent " Row-Action " Method for the Convex Feasibility Problem, *Math. Optim.Appl*. 1998, **17**: 225-235.

[11] M. Fukushima, A relaxed projection method for variational inequalities, *Math. Program*. 1986, **35**: 58-70.

[12] L. T. Dos Santos, A parallel subgradient method for the convex feasibility problem, *J. Compute. Appl. Math*. 1987, **18**: 226-235.

[13] R. Aharoni and Y. Censor, Block-iterative projection methods for parallel computation of solutions to convex feasibility problems, *Linear Algebra Appl*. 1989, **120**: 165-175.

[14] Y. Censor and G. T. Herman, On some optimization techniques in image reconstruction from projections, *Appl. Numer. Math*. 1987, **3**: 365-391.

[15] G. T. Herman, *Image Reconstruction from projections*, Academic Press, New York, 1980.

[16] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and complememtarity Problems*, New York: Springer.

[17] E. H. Zarantonello, *Projections on convex sets in Hilbert space and spectral theory Contributions to Nonlinear Functional Analysis*, Academic Press, New York, 1971.