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Estimating Error Bounds for Non-stationary Binary Subdivision Curves / surfaces *

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Abstract. Error bounds between non-stationary binary subdivision curves/surfaces and their control polygons after k-fold subdivision are estimated. The bounds can be evaluated without recursive subdivision. A computational formula of subdivision depth for non-stationary binary subdivision surfaces is also presented. From this formula one can predict the subdivision depth within a user specified error tolerance. Our results not only remove errors in the results of [3] but also contain the generalization of their results.

Keywords: Subdivision curve, subdivision surface, subdivision depth, error bound.

1. Introduction

In recent years subdivision schemes have been important because they provide an efficient way to describe curves, surfaces and other geometric objects. Subdivision allows to generate smooth geometric objects from a given control polygon. With this coarse polygon a sequence of refined polygon can be computed. In the limit this sequence of polygon converges to a continuous smooth object. Each refinement step can be divided into two different aspects. First, a topological operation is performed. Therefore new vertices are added to the control polygon and the rectangles/triangles are split. Then the geometry of the control polygon is changed by a smoothing operation. The question is how refined polygon approximates the limiting curve/surface [1], [2], [3], [5], [6], [7], [8], [9] and [10]. This question appears in many applications as rendering, intersection, testing and design. Existing methods only compute the error bounds for stationary subdivision schemes. To best of our knowledge no work has been done to compute the error bounds for non-stationary subdivision schemes. As non-stationary subdivision schemes are the generalization of stationary subdivision schemes. Therefore we are interested in getting error bounds estimations between non-stationary subdivision curves/surfaces and their control polygons. In this article, for simplicity we have used the notations and methodology of [3].

The rest of the paper is arranged as follows. In Section 2, we give the definition of non-stationary subdivision curves and gather some notations to set out our terminology needed in Section 3. We present our main result to estimate error bounds between non-stationary subdivision curve and its control polygon in Section 3. In Section 4, we give the definition of non-stationary subdivision surfaces and gather some new notations to set out our terminology needed in Section 5. We present our main result to estimate error bounds between non-stationary subdivision surface and its control polygon in Section 5. In this Section we also derive a computational formula of subdivision depth for non-stationary subdivision surfaces.

2. Non-stationary subdivision curves

Given a set of control points $p_i^k \in R^N$, $i \in Z, N \geq 2$, where k is a nonnegative integer. A non-stationary binary subdivision scheme is defined by

$$\begin{cases} p_{2i}^{k+1} = \sum_{j=0}^{m} a_j^k \, p_{i+j}^k, \\ p_{2i+1}^{k+1} = \sum_{j=0}^{m} b_j^k \, p_{i+j}^k, \end{cases} \tag{2.1}$$

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where m is greater than zero. The coefficients $\{a_j^k\}_{j=0}^m$ and $\{b_j^k\}_{j=0}^m$ are called the mask at the kth level of the subdivision scheme. If the mask is independent of k, then the scheme is called stationary, otherwise it is called non-stationary. A necessary condition for the uniform convergence of the subdivision scheme (2.1) on the diadic points for arbitrary initial data, is that

$$\sum_{j=0}^{m} a_j^k \approx \sum_{j=0}^{m} b_j^k \approx 1. \tag{2.2}$$

2.1. Notations

We gather here some notations to set out our terminology needed in the sequel.

$$\begin{cases} T_k^1 = \max_i \left\| p_{2i}^{k+1} - p_i^k \right\|, \\ T_k^2 = \max_i \left\| p_{2i+1}^{k+1} - \frac{1}{2}(p_i^k + p_{i+1}^k) \right\|. \end{cases} \tag{2.3}$$

$$\phi = \max\left\{ \sum_{j=0}^{m-1} \left|A_j^k\right|, \sum_{j=0}^{m-1} \left|B_j^k\right| \right\}, \tag{2.4}$$

where

$$A_j^k = \sum_{i=j+1}^{m} a_i^k,\ B_j^k = \sum_{i=j+1}^{m} b_i^k,\ j \ge 1,.\ B_0^k \sum_{i=1}^{m} b_i^k - \frac{1}{2}.$$

$$\varphi = \max\left\{ \hat{C}, \hat{D} \right\} \tag{2.5}$$

where

$$\hat{C} = \max\left\{ \sum_{j=0}^{m-1} \left|C_j^{l-1}\right|, l = 1,2,...,k \right\},\ \hat{D} = \max\left\{ \sum_{j=0}^{m} \left|D_j^{l-1}\right|, l = 1,2,...,k \right\},\ C_j^k = \sum_{i=0}^{j} (a_i^k - b_i^k) \text{ and } D_j^k = a_j^k - C_j^k.$$

3. The error bounds for non-stationary subdivision curves

The aim of this section is to estimate error bounds between limiting non-stationary subdivision curves and their control polygons. First we prove the following Lemma's needed for Theorem 3.5.

Lemma 3.1. Given initial control polygon $p_i^0 = p_i$, $i \in Z$, let the values p_i^k, $k \ge 0$ be defined recursively by non-stationary subdivision scheme (2.1) together with (2.2) then

$$T_k^1 \le \sum_{j=0}^{m-1} \left|A_j^k\right| \max_i \left\| p_{i+1}^k - p_i^k \right\|, \tag{3.1}$$

where T_k^1 and A_j^k are defined in (2.3) and (2.4) respectively.

Proof: From (2.1) and (2.2)

$$p_{2i}^{k+1} - p_i^k = \sum_{j=0}^{m} a_j^k p_{i+j}^k - \left(\sum_{j=0}^{m} a_j^k \right) p_i^k.$$

This implies

$$p_{2i}^{k+1} - p_i^k = \left(a_1^k + a_2^k + ... + a_m^k\right)\left(p_{i+1}^k - p_i^k\right) + \left(a_2^k + a_3^k + ... + a_m^k\right)\left(p_{i+2}^k - p_{i+1}^k\right) + ...$$
$$+ \left(a_{m-1}^k + a_m^k\right)\left(p_{i+m-1}^k - p_{i+m-2}^k\right) + \left(a_m^k\right)\left(p_{i+m}^k - p_{i+m-1}^k\right).$$

From this we get

$$p_{2i}^{k+1} - p_i^k = \sum_{j=0}^{m-1} A_j^k \left(p_{i+j+1}^k - p_{i+j}^k \right),$$

where $A_j^k = \sum_{i=j+1}^{m} a_i^k$. If $T_k^1 = \max_i \left\| p_{2i}^{k+1} - p_i^k \right\|$, then $T_k^1 \le \sum_{j=0}^{m-1} \left|A_j^k\right| \max_i \left\| p_{i+1}^k - p_i^k \right\|$. This completes the proof.

Lemma 3.2. Given initial control polygon $p_i^0 = p_i,\ i \in Z$, let the values $p_i^k,\ k \ge 0$ be defined recursively by non-stationary subdivision scheme (2.1) together with (2.2) then

$$T_k^2 \le \sum_{j=0}^{m-1} \left|B_j^k\right| \max_i \left\| p_{i+1}^k - p_i^k \right\|,$$

where T_k^2 and B_j^k are defined in (2.3) and (2.4) respectively.

Proof: From (2.1) and (2.2)

$$p_{2i+1}^{k+1} - \frac{1}{2}(p_i^k + p_{i+1}^k) = \sum_{j=0}^{m} b_j^k p_{i+j}^k - \frac{1}{2}\left(\sum_{j=0}^{m} b_j^k\right)\left(p_i^k + p_{i+1}^k\right).$$

$$= \left(\frac{1}{2}\right)\sum_{j=0}^{m} b_j^k \left(p_{i+j}^k + p_{i+j}^k - p_i^k - p_{i+1}^k\right),$$

$$p_{2i+1}^{k+1} - \frac{1}{2}(p_i^k + p_{i+1}^k) = \frac{1}{2}\Big\{-b_0^k\left(p_{i+1}^k - p_i^k\right) + b_1^k\left(p_{i+1}^k - p_i^k\right)$$

$$+ b_2^k\left(p_{i+2}^k - p_{i+1}^k + p_{i+1}^k - p_i^k\right) + \ldots + b_m^k\left(p_{i+m}^k - p_{i+m-1}^k + p_{i+m-1}^k \ldots - p_{i+1}^k + p_{i+1}^k - p_i^k\right)\Big\},$$

$$p_{2i+1}^{k+1} - \frac{1}{2}(p_i^k + p_{i+1}^k) = \frac{1}{2}\left(-b_0^k + b_1^k + \ldots + b_m^k\right)\left(p_{i+1}^k - p_i^k\right) + \left(b_2^k + b_3^k + \ldots + b_m^k\right)\left(p_{i+2}^k - p_{i+1}^k\right)$$

$$+ \left(b_3^k + b_4^k + \ldots + b_m^k\right)\left(p_{i+3}^k - p_{i+2}^k\right) + \ldots + \left(b_{m-1}^k + b_m^k\right)\left(p_{i+m-1}^k - p_{i+m-2}^k\right) + \left(b_m^k\right)\left(p_{i+m}^k - p_{i+m-1}^k\right).$$

Since from (2.2), $\sum_{i=1}^{m} b_i^k - 1 = -b_0^k$, then we have

$$p_{2i+1}^{k+1} - \frac{1}{2}(p_i^k + p_{i+1}^k) = \sum_{j=0}^{m-1} B_j^k \left(p_{i+j+1}^k - p_{i+j}^k\right),$$

where $B_j^k = \sum_{i=j+1}^{m} b_i^k,\ j \ge 1, B_0^k = \sum_{i=1}^{m} b_i^k - \frac{1}{2}$.

If

$$T_k^2 = \max_i \left\| p_{2i+1}^{k+1} - \frac{1}{2}(p_i^k + p_{i+1}^k) \right\|,$$

then

$$T_k^2 \le \sum_{j=0}^{m-1} \left|B_j^k\right| \max_i \left\| p_{i+1}^k - p_i^k \right\|,$$

This completes the proof.

Lemma 3.3. Given initial control polygon $p_i^0 = p_i,\ i \in Z$, let the values $p_i^k,\ k \ge 0$ be defined recursively by non-stationary subdivision scheme (2.1) together with necessary conditions (2.2). Suppose P^k be the piecewise linear interpolation to the values p_i^k, then

$$\left\| P^{k+1} - P^k \right\|_\infty \le \phi \max_i \left\| p_{i+1}^k - p_i^k \right\|, \tag{3.3}$$

where ϕ is defined in (2.4).

Proof: Let $\|.\|_\infty$ denote the uniform norm. Since the maximum difference between P^{k+1} and P^k is attained at a point on the $(k+1)th$ mesh, then

$$\left\| P^{k+1} - P^k \right\|_\infty \le \max\left\{T_k^1, T_k^2\right\}, \tag{3.4}$$

where

$$\begin{cases} T_k^1 = \max_i \left\| p_{2i}^{k+1} - p_i^k \right\|, \\ T_k^2 = \max_i \left\| p_{2i+1}^{k+1} - \frac{1}{2}(p_i^k + p_{i+1}^k) \right\|. \end{cases} \tag{3.5}$$

If

$$\phi = \max\left\{ \sum_{j=0}^{m-1} \left| A_j^k \right|, \sum_{j=0}^{m-1} \left| B_j^k \right| \right\},$$

then from (3.1), (3.2), (3.4) and (3.5) we have

$$\left\| P^{k+1} - P^k \right\|_\infty \le \phi \max_i \left\| p_{i+1}^k - p_i^k \right\|.$$

This completes the proof.

Lemma 3.4. Given initial control polygon $p_i^0 = p_i$, $i \in Z$, let the values p_i^k, $k \ge 0$ be defined recursively by non-stationary subdivision scheme (2.1) together with (2.2), if $\varphi < 1$, then

$$\max_i \left\| p_{i+1}^k - p_i^k \right\| \le (\varphi)^k \max_i \left\| p_{i+1}^0 - p_i^0 \right\|, \tag{3.6}$$

where φ is defined in (2.5).

Proof: We claim:

$$p_{2i+1}^k - p_{2i}^k = \sum_{j=0}^{m-1} C_j^{k-1} \left(p_{i+j+1}^{k-1} - p_{i+j}^{k-1} \right), \tag{3.7}$$

where $C_j^{k-1} = \sum_{i=0}^{j} (a_i^{k-1} - b_i^{k-1})$.

We can prove above claim by induction on m. Let $m = 1$ then from (2.1)

$$p_{2i+1}^k - p_{2i}^k = \left(b_0^{k-1} - a_0^{k-1} \right) p_i^{k-1} + \left(b_1^{k-1} - a_1^{k-1} \right) p_{i+1}^{k-1}.$$

From (2.2) we have $b_1^{k-1} - a_1^{k-1} = a_0^{k-1} - b_0^{k-1}$. This implies

$$p_{2i+1}^k - p_{2i}^k = \sum_{j=0}^{1-1} C_j^{k-1} \left(p_{i+j+1}^{k-1} - p_{i+j}^{k-1} \right),$$

where $C_j^{k-1} = \sum_{i=0}^{j} (a_i^{k-1} - b_i^{k-1})$. This proves our claim for $m = 1$.

Now we prove our claim for $m = 2$. From (2.1)

$$p_{2i+1}^k - p_{2i}^k = \left(b_0^{k-1} - a_0^{k-1} \right) p_i^{k-1} + \left(b_1^{k-1} - a_1^{k-1} \right) p_{i+1}^{k-1} + \left(b_2^{k-1} - a_2^{k-1} \right) p_{i+2}^{k-1}.$$

From (2.2) we have $b_2^{k-1} - a_2^{k-1} = a_0^{k-1} - b_0^{k-1} + a_1^{k-1} - b_1^{k-1}$. This implies

$$p_{2i+1}^k - p_{2i}^k = \left(a_0^{k-1} - b_0^{k-1} \right) \left(p_{i+1}^{k-1} + p_i^{k-1} \right) + \left(a_0^{k-1} - b_0^{k-1} + a_1^{k-1} - b_1^{k-1} \right) \left(p_{i+2}^{k-1} - p_{i+1}^{k-1} \right).$$

Thus we get

$$p_{2i+1}^k - p_{2i}^k = \sum_{j=0}^{2-1} C_j^{k-1} \left(p_{i+j+1}^{k-1} - p_{i+j}^{k-1} \right),$$

where $C_j^{k-1} = \sum_{i=0}^{j} (a_i^{k-1} - b_i^{k-1})$. This proves our claim for $m = 2$. Similarly we can find that claim is true for all m.

We also **claim:**

$$p_{2i+2}^{k}-p_{2i+1}^{k}=\sum_{j=0}^{m}D_j^{k-1}\left(p_{i+j+1}^{k-1}-p_{i+j}^{k-1}\right), \tag{3.8}$$

where $D_j^{k-1}=\sum_{i=0}^{j}(b_i^{k-1}-a_i^{k-1})+a_j^{k-1}=a_j^{k-1}-C_j^{k-1}$.

Similarly we can prove claim by induction on m. Let $m=1$ then from (2.1)

$$p_{2i+2}^{k}-p_{2i+1}^{k}=-b_0^{k-1}p_i^{k-1}+\left(a_0^{k-1}-b_1^{k-1}\right)p_{i+1}^{k-1}+a_1^{k-1}p_{i+2}^{k-1}.$$

Since from (2.2) $a_1^{k-1}=b_1^{k-1}-a_0^{k-1}+b_0^{k-1}$, then we have

$$p_{2i+2}^{k}-p_{2i+1}^{k}=\left\{\left(b_0^{k-1}-a_0^{k-1}\right)+a_0^{k-1}\right\}\left(p_{i+1}^{k-1}-p_i^{k-1}\right)+ \left\{\left(b_0^{k-1}-a_0^{k-1}\right)+\left(b_1^{k-1}-a_1^{k-1}\right)+a_1^{k-1}\right\}\left(p_{i+2}^{k-1}-p_{i+1}^{k-1}\right).$$

This implies

$$p_{2i+2}^{k}-p_{2i+1}^{k}=\sum_{j=0}^{1}D_j^{k-1}\left(p_{i+j+1}^{k-1}-p_{i+j}^{k-1}\right),$$

where $D_j^{k-1}=\sum_{i=0}^{j}(b_i^{k-1}-a_i^{k-1})+a_j^{k-1}=a_j^{k-1}-C_j^{k-1}$. This proves our claim for $m=1$. Similarly we can prove the claim for all m.

From (3.7) and (3.8) we get

$$\max_i\left\|p_{i+1}^{k}-p_i^{k}\right\|\le\left(\sum_{j=0}^{m-1}C_j^{k-1}\right)\max_i\left\|p_{i+1}^{k-1}-p_i^{k-1}\right\|,\quad \max_i\left\|p_{i+1}^{k}-p_i^{k}\right\|\le\left(\sum_{j=0}^{m}D_j^{k-1}\right)\max_i\left\|p_{i+1}^{k-1}-p_i^{k-1}\right\|.$$

Recursively we get

$$\max_i\left\|p_{i+1}^{k}-p_i^{k}\right\|\le\left(\hat{C}\right)^k\max_i\left\|p_{i+1}^{0}-p_i^{0}\right\|,\ \max_i\left\|p_{i+1}^{k}-p_i^{k}\right\|\le\left(\hat{D}\right)^k\max_i\left\|p_{i+1}^{0}-p_i^{0}\right\|,$$

where

$$\hat{C}=\max\left\{\sum_{j=0}^{m-1}\left|C_j^{k-1}\right|,\sum_{j=0}^{m-1}\left|C_j^{k-2}\right|,\ldots,\sum_{j=0}^{m-1}\left|C_j^{0}\right|\right\},$$

and

$$\hat{D}=\max\left\{\sum_{j=0}^{m}\left|D_j^{k-1}\right|,\sum_{j=0}^{m}\left|D_j^{k-2}\right|,\ldots,\sum_{j=0}^{m}\left|D_j^{0}\right|\right\}.$$

If $\varphi=\max\left\{\hat{C},\hat{D}\right\}$ then

$$\max_i\left\|p_{i+1}^{k}-p_i^{k}\right\|\le(\varphi)^k\max_i\left\|p_{i+1}^{0}-p_i^{0}\right\|.$$

This completes the proof.

Remarks 3.1. The main problem in Theorem 1 [3] is that its proof contains error on page 599 in equation (9). Its corrected form is given in Equation (3.7) of our Lemma 3.4.

Theorem 3.5. Given initial control polygon $p_i^0=p_i$, $i\in Z$, let the values p_i^k, $k\ge 0$ be defined recursively by non-stationary subdivision scheme (2.1) together with necessary conditions (2.2). Suppose P^k be the piecewise linear interpolation to the values p_i^k and P^∞ be the limit curve of the scheme (2.1). If

$$\varphi<1, \tag{3.9}$$

then error bounds between limit curve and its control polygon after k-fold subdivision are

$$\left\|P^k-P^\infty\right\|_\infty\le\phi\beta\left(\frac{(\varphi)^k}{1-\varphi}\right), \tag{3.10}$$

where $\beta=\max_i\left\|p_{i+1}^{0}-p_i^{0}\right\|$, ϕ and φ are defined in (2.4) and (2.5).

Proof: From (3.3) and (3.6) we have $\left\|P^{k+1}-P^{k}\right\|_{\infty} \leq \phi(\varphi)^{k} \max_{i}\left\|p_{i+1}^{0}-p_{i}^{0}\right\|$. If $\beta=\max_{i}\left\|p_{i+1}^{0}-p_{i}^{0}\right\|$ then $\left\|P^{k+1}-P^{k}\right\|_{\infty} \leq \phi\beta(\varphi)^{k}$. Using triangle inequality we have $\left\|P^{k}-P^{\infty}\right\|_{\infty} \leq \phi\beta\left(\frac{(\varphi)^{k}}{1-\varphi}\right)$. This completes the proof.

Remarks 3.2/ In non-stationary subdivision schemes the subdivision mask is updated during each subdivision level. In particular, if the subdivision mask does't change by increasing subdivision level then non-stationary schemes coincides with stationary schemes. In this particular case, Theorem 3.5 is exactly the same with Theorem 1(after correction) of Mustafa et al. [3].

Corollary 3.6. Given initial control polygon $p_i^0 = p_i$, $i \in Z$, let the values p_i^k, $k \geq 0$ be defined recursively by non-stationary subdivision scheme for curve interpolation [4]. Suppose P^k be the piecewise linear interpolation to the values p_i^k and P^∞ be the limit curve of the subdivision scheme. Then

$$\left\|P^{k}-P^{\infty}\right\|_{\infty} \leq \phi\beta\left(\frac{(\varphi)^{k}}{1-\varphi}\right),$$

where $\beta=\max_{i}\left\|p_{i+1}^{0}-p_{i}^{0}\right\|$, $\phi=\max\left\{1, \frac{1}{2}+\left|w^{0}\right|+\left|\frac{1}{2}+w^{0}\right|\right\}, \varphi=\frac{1}{2}+2\left|w^{0}\right|$ and $w^0 = w$ is initial weight parameter.

Proof: A non-stationary subdivision scheme for curve interpolation [4] have following subdivision mask

$$\left(a_0^k, a_1^k, a_2^k, a_3^k\right)=(0,1,0,0),\ \left(b_0^k, b_1^k, b_2^k, b_3^k\right)=\left(-w^k, \frac{1}{2}+w^k, \frac{1}{2}+w^k, -w^k\right).$$

Since above subdivision mask approximately satisfy (3.9) for $-0.2499 \leq w^k \leq 0.2499$, then by (3.10) we have the result.

4. Non-stationary subdivision surfaces

Given a set of control points $p_{i,j}^k \in R^N$, $i, j \in Z, N \geq 2$, where k is a nonnegative integer. A tensor product of non-stationary binary subdivision scheme (2.1) is defined by

$$\begin{cases} p_{2i,2j}^{k+1} = \sum_{r=0}^{m}\sum_{s=0}^{m} a_r^k a_s^k p_{i+r,j+s}^k \\ p_{2i,2j+1}^{k+1} = \sum_{r=0}^{m}\sum_{s=0}^{m} a_r^k b_s^k p_{i+r,j+s}^k \\ p_{2i+1,2j}^{k+1} = \sum_{r=0}^{m}\sum_{s=0}^{m} b_r^k a_s^k p_{i+r,j+s}^k \\ p_{2i+1,2j+1}^{k+1} = \sum_{r=0}^{m}\sum_{s=0}^{m} b_r^k b_s^k p_{i+r,j+s}^k \end{cases} \tag{4.1}$$

where m is greater than zero. The coefficients $\left\{a_j^k\right\}_{j=0}^m$ and $\left\{b_j^k\right\}_{j=0}^m$ are called the mask at the kth level of the subdivision scheme. If the mask is independent of k, then the scheme is called stationary, otherwise it is called non-stationary. A necessary condition for the convergence of the subdivision scheme (4.1) for arbitrary initial data, is that

$$\sum_{j=0}^{m} a_j^k \approx \sum_{j=0}^{m} b_j^k \approx 1. \tag{4.2}$$

4.1. Notations

We again gather here some new notations to set out our terminology needed in the coming Section.

$$\psi = \max\left\{\hat{A}\hat{C}, \hat{A}\hat{D}, \hat{B}\hat{C}\right\}, \tag{4.3}$$

where

$$\hat{A} = \max\left\{\sum_{r=0}^{m}\left|a_r^{j-1}\right|, j = 1,2,\ldots,k\right\}, \hat{B} = \max\left\{\sum_{r=0}^{m}\left|b_r^{j-1}\right|, j = 1,2,\ldots,k\right\},$$

$$\hat{C} = \max\left\{\sum_{r=0}^{m-1}\left|C_r^{j-1}\right|, j = 1,2,\ldots,k\right\}, \ \hat{D} = \max\left\{\sum_{r=0}^{m}\left|D_r^{j-1}\right|, j = 1,2,\ldots,k\right\},$$

$$C_r^k = \sum_{t=0}^{r}(a_t^k - b_t^k) \text{ and } D_r^k = a_r^k - C_r^k .$$

$$\begin{cases} \eta_1 = \left|a_0^k\right|\left(\sum_{t=1}^{m}\left|a_t^k\right| + \sum_{s=1}^{m-1}\left|A_s^k\right|\right), & \eta_2 = \left|a_0^k\right|\left(\sum_{t=1}^{m}\left|b_t^k\right| + \sum_{s=1}^{m-1}\left|B_s^k\right|\right) + \frac{1}{2}, \\ \eta_3 = \left|b_0^k\right|\left(\sum_{t=1}^{m}\left|a_t^k\right| + \sum_{s=1}^{m-1}\left|A_s^k\right|\right), & \eta_4 = \left|b_0^k\right|\left(\sum_{t=1}^{m}\left|b_t^k\right| + \sum_{s=1}^{m-1}\left|B_s^k\right|\right) + \frac{1}{4}, \end{cases} \tag{4.4}$$

where $A_s^k = \sum_{t=s+1}^{m} a_t^k$ and $B_s^k = \sum_{t=s+1}^{m} b_t^k$,

$$\begin{cases} \tau_1 = \sum_{t=1}^{m}\left|a_t^k\right| + \sum_{r=0}^{m}\left|a_r^k\right|\sum_{s=1}^{m-1}\left|A_s^k\right|, & \tau_2 = \sum_{t=1}^{m}\left|a_t^k\right| + \sum_{r=0}^{m}\left|b_r^k\right|\sum_{s=1}^{m-1}\left|A_s^k\right|, \\ \tau_3 = \sum_{t=1}^{m}\left|b_t^k\right| + \sum_{r=0}^{m}\left|a_r^k\right|\sum_{s=1}^{m-1}\left|B_s^k\right| + \frac{1}{2}, & \tau_4 = \sum_{t=1}^{m}\left|b_t^k\right| + \sum_{r=0}^{m}\left|b_r^k\right|\sum_{s=1}^{m-1}\left|B_s^k\right| + \frac{1}{2}, \end{cases} \tag{4.5}$$

and

$$\begin{cases} \xi_1 = \sum_{t=1}^{m}\left|a_t^k\right|\left(\sum_{t=1}^{m}\left|a_t^k\right| + \sum_{s=1}^{m-1}\left|A_s^k\right|\right), & \xi_2 = \sum_{t=1}^{m}\left|a_t^k\right|\left(\sum_{t=1}^{m}\left|b_t^k\right| + \sum_{s=1}^{m-1}\left|B_s^k\right|\right), \\ \xi_3 = \sum_{t=1}^{m}\left|b_t^k\right|\left(\sum_{t=1}^{m}\left|a_t^k\right| + \sum_{s=1}^{m-1}\left|A_s^k\right|\right), & \xi_4 = \sum_{t=1}^{m}\left|b_t^k\right|\left(\sum_{t=1}^{m}\left|b_t^k\right| + \sum_{s=1}^{m-1}\left|B_s^k\right|\right) + \frac{1}{4}. \end{cases} \tag{4.6}$$

$$\begin{cases} M_k^1 = \max_{i,j}\left\| p_{2i,2j}^{k+1} - p_{i,j}^k \right\|, \\ M_k^2 = \max_{i,j}\left\| p_{2i+1,2j}^{k+1} - \frac{1}{2}\left(p_{i,j}^k + p_{i+1,j}^k\right)\right\|, \\ M_k^3 = \max_{i,j}\left\| p_{2i,2j+1}^{k+1} - \frac{1}{2}\left(p_{i,j}^k + p_{i,j+1}^k\right)\right\|, \\ M_k^4 = \max_{i,j}\left\| p_{2i+1,2j+1}^{k+1} - \frac{1}{4}\left(p_{i,j}^k + p_{i+1,j}^k + p_{i,j+1}^k + p_{i+1,j+1}^k\right)\right\|. \end{cases} \tag{4.7}$$

$$\begin{cases} \Delta_{i,j,1}^k = p_{i+1,j}^k - p_{i,j}^k, \\ \Delta_{i,j,2}^k = p_{i,j+1}^k - p_{i,j}^k, \qquad \gamma_t^k = \max_{i,j}\left\|\Delta_{i,j,t}^k\right\|, t = 1,2,3. \\ \Delta_{i,j,3}^k = p_{i+1,j+1}^k - p_{i,j+1}^k, \end{cases} \tag{4.8}$$

5. The error bounds for non-stationary subdivision surfaces

The aim of this section is to estimate error bounds between limiting non-stationary subdivision surfaces and their control polygons. First we prove the following Lemmas needed for Theorem 5.7.

Lemma 5.1. Given initial control polygon $p_{i,j}^0 = p_{i,j}$, $i \in Z$, let the values $p_{i,j}^k$, $k \geq 0$ be defined recursively by non-stationary subdivision scheme (4.1) together with (4.2), then

$$M_k^1 \leq \eta_1\gamma_1^k + \tau_1\gamma_2^k + \xi_1\gamma_3^k , \tag{5.1}$$

where $\eta_1, \tau_1, \xi_1, M_k^1$ and $\gamma_t^k, t = 1,2,3,$ are defined in (4.4)-(4.8).

Proof: From (4.1) and (4.2) we get

$$p_{2i,2j}^{k+1} - p_{i,j}^k = \sum_{r=0}^{m} a_r^k \left(\sum_{s=0}^{m} a_s^k \left(p_{i+r,j+s}^k - p_{i,j}^k \right) \right). \tag{5.2}$$

Since

$$\begin{aligned}
&\sum_{s=0}^{m} a_s^k \left(p_{i+r,j+s}^k - p_{i,j}^k \right) = a_0^k \left(p_{i+r,j}^k - p_{i,j}^k \right) + a_1^k \left(p_{i+r,j+1}^k - p_{i,j}^k \right) + \\
&a_2^k \left(p_{i+r,j+2}^k - p_{i+r,j+1}^k + p_{i+r,j+1}^k - p_{i,j}^k \right) + \\
&a_3^k \left(p_{i+r,j+3}^k - p_{i+r,j+2}^k + p_{i+r,j+2}^k - p_{i+r,j+1}^k + p_{i+r,j+1}^k - p_{i,j}^k \right) + \ldots + \\
&a_m^k \left(p_{i+r,j+m}^k - p_{i+r,j+m-1}^k + \ldots + p_{i+r,j+3}^k - p_{i+r,j+2}^k + p_{i+r,j+2}^k - p_{i+r,j+1}^k + p_{i+r,j+1}^k - p_{i,j}^k \right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{s=0}^{m} a_s^k \left(p_{i+r,j+s}^k - p_{i,j}^k \right) = a_0^k \left(p_{i+r,j}^k - p_{i,j}^k \right) + \\
&\sum_{t=1}^{m} a_t^k \left(p_{i+r,j+1}^k - p_{i,j}^k \right) + \sum_{s=1}^{m-1} A_s^k \left(p_{i+r,j+s+1}^k - p_{i+r,j+s}^k \right),
\end{aligned}$$

where $A_s^k = \sum_{t=s+1}^{m} a_i^k$. Taking sum on both sides of above equation we get

$$\begin{aligned}
&\sum_{r=0}^{m} a_r^k \left(\sum_{s=0}^{m} a_s^k \left(p_{i+r,j+s}^k - p_{i,j}^k \right) \right) = a_0^k \sum_{r=0}^{m} a_r^k \left(p_{i+r,j}^k - p_{i,j}^k \right) + \\
&\sum_{t=1}^{m} a_t^k \left(\sum_{r=0}^{m} a_r^k \left(p_{i+r,j+1}^k - p_{i,j}^k \right) \right) + \sum_{r=0}^{m} a_r^k \left(\sum_{s=1}^{m-1} A_s^k \left(p_{i+r,j+s+1}^k - p_{i+r,j+s}^k \right) \right).
\end{aligned} \tag{5.3}$$

Since

$$\begin{aligned}
&\sum_{r=0}^{m} a_r^k \left(p_{i+r,j}^k - p_{i,j}^k \right) = a_1^k \left(p_{i+1,j}^k - p_{i,j}^k \right) + a_2^k \left(p_{i+2,j}^k - p_{i+1,j}^k + p_{i+1,j}^k - p_{i,j}^k \right) \\
&+ a_3^k \left(p_{i+3,j}^k - p_{i+2,j}^k + p_{i+2,j}^k - p_{i+1,j}^k + p_{i+1,j}^k - p_{i,j}^k \right) + \ldots + \\
&a_m^k \left(p_{i+m,j}^k - p_{i+m-1,j}^k + \ldots + p_{i+3,j}^k - p_{i+2,j}^k + p_{i+2,j}^k - p_{i+1,j}^k + p_{i+1,j}^k - p_{i,j}^k \right).
\end{aligned}$$

Hence

$$\sum_{r=0}^{m} a_r^k \left(p_{i+r,j}^k - p_{i,j}^k \right) = \sum_{t=1}^{m} a_t^k \left(p_{i+1,j}^k - p_{i,j}^k \right) + \sum_{s=1}^{m-1} A_s^k \left(p_{i+s+1,j}^k - p_{i+s,j}^k \right).$$

Similarly

$$\begin{aligned}
&\sum_{r=0}^{m} a_r^k \left(p_{i+r,j+1}^k - p_{i,j}^k \right) = a_0^k \left(p_{i,j+1}^k - p_{i,j}^k \right) + \\
&\sum_{t=1}^{m} a_t^k \left(p_{i+1,j+1}^k - p_{i,j}^k \right) + \sum_{s=1}^{m-1} A_s^k \left(p_{i+s+1,j+1}^k - p_{i+s,j+1}^k \right).
\end{aligned}$$

Substituting these sum into (5.3) and then from (5.2) we have

$$p_{2i,2j}^{k+1}-p_{i,j}^{k}=\left(a_0^k\sum_{t=1}^{m}a_t^k\right)\left(p_{i+1,j}^k-p_{i,j}^k\right)+\left(\sum_{t=1}^{m}a_t^k\right)^2\left(p_{i+1,j+1}^k-p_{i,j+1}^k\right)+$$
$$\left(a_0^k\sum_{t=1}^{m}a_t^k\right)\left(p_{i,j+1}^k-p_{i,j}^k\right)+a_0^k\sum_{s=1}^{m-1}A_s^k\left(p_{i+s+1,j}^k-p_{i+s,j}^k\right)+$$
$$\left(\sum_{t=1}^{m}a_t^k\right)\sum_{s=1}^{m-1}A_s^k\left(p_{i+s+1,j+1}^k-p_{i+s,j+1}^k\right)+\sum_{r=0}^{m}a_r^k\left(\sum_{s=1}^{m-1}A_s^k\left(p_{i+r,j+s+1}^k-p_{i+r,j+s}^k\right)\right).$$

This implies

$$M_k^1\le\left|a_0^k\right|\left(\sum_{t=1}^{m}\left|a_t^k\right|+\sum_{s=1}^{m-1}\left|A_s^k\right|\right)\max_{i,j}\left\|\Delta_{i,j,1}^k\right\|$$
$$+\left(\sum_{t=1}^{m}\left|a_t^k\right|+\sum_{r=0}^{m}\left|a_r^k\right|\sum_{s=1}^{m-1}\left|A_s^k\right|\right)\max_{i,j}\left\|\Delta_{i,j,2}^k\right\|$$
$$+\sum_{t=1}^{m}\left|a_t^k\right|\left(\sum_{t=1}^{m}\left|a_t^k\right|+\sum_{s=1}^{m-1}\left|A_s^k\right|\right)\max_{i,j}\left\|\Delta_{i,j,3}^k\right\|,$$

where $M_k^1=\max_{i,j}\left\|p_{2i,2j}^{k+1}-p_{i,j}^k\right\|$.

Using notations from (4.4)-(4.8) we can rewrite above inequality $M_k^1\le\eta_1\gamma_1^k+\tau_1\gamma_2^k+\xi_1\gamma_3^k$. This completes the proof.

By using the technique of Mustafa et al. [3] and the one used in Lemma 5.1 one can easily prove the following Lemmas.

Lemma 5.2. Given initial control polygon $p_{i,j}^0=p_{i,j},\ i\in Z$, let the values $p_{i,j}^k,\ k\ge 0$ be defined recursively by non-stationary subdivision scheme (4.1) together with (4.2), then

$$M_k^2\le\eta_2\gamma_1^k+\tau_2\gamma_2^k+\xi_2\gamma_3^k, \tag{5.4}$$

where $\eta_2,\tau_2,\xi_2,M_k^2$ and $\gamma_t^k, t=1,2,3,$ are defined in (4.4)-(4.8).

Lemma 5.3. Given initial control polygon $p_{i,j}^0=p_{i,j},\ i\in Z$, let the values $p_{i,j}^k,\ k\ge 0$ be defined recursively by non-stationary subdivision scheme (4.1) together with (4.2), then

$$M_k^3\le\eta_3\gamma_1^k+\tau_3\gamma_2^k+\xi_3\gamma_3^k, \tag{5.5}$$

where $\eta_3,\tau_3,\xi_3,M_k^3$ and $\gamma_t^k, t=1,2,3,$ are defined in (4.4)-(4.8).

Lemma 5.4. Given initial control polygon $p_{i,j}^0=p_{i,j},\ i\in Z$, let the values $p_{i,j}^k,\ k\ge 0$ be defined recursively by non-stationary subdivision scheme (4.1) together with (4.2), then

$$M_k^4\le\eta_4\gamma_1^k+\tau_4\gamma_2^k+\xi_4\gamma_3^k, \tag{5.6}$$

where $\eta_4,\tau_4,\xi_4,M_k^4$ and $\gamma_t^k, t=1,2,3,$ are defined in (4.4)-(4.8).

Lemma 5.5. Given initial control polygon $p_{i,j}^0=p_{i,j},\ i\in Z$, let the values $p_{i,j}^k,\ k\ge 0$ be defined recursively by non-stationary subdivision scheme (4.1) together with (4.2). Suppose P^k be the piecewise linear interpolation to the values $p_{i,j}^k$, then

$$\left\|P^{k+1}-P^k\right\|_\infty\le\eta\gamma_1^k+\tau\gamma_2^k+\xi\gamma_3^k, \tag{5.7}$$

where $\eta=\max\{\eta_1,\eta_2,\eta_3,\eta_4\}$, $\tau=\max\{\tau_1,\tau_2,\tau_3,\tau_4\}$ and $\xi=\max\{\xi_1,\xi_2,\xi_3,\xi_4\}$, where $\eta_t,\tau_t,\xi_t, t=1...4$ are defined in (4.4)-(4.6), $\gamma_t^k, t=1,2,3,$ are defined in (4.8).

Proof: Let $\|.\|_\infty$ denote the uniform norm. Since the maximum difference between P^{k+1} and P^k is attained at a point on the $(k+1)th$ mesh, then

$$\left\|P^{k+1}-P^{k}\right\|_\infty \le \max\left\{M_k^1, M_k^2, M_k^3, M_k^4\right\}, \tag{5.8}$$

where $M_k^1, M_k^2, M_k^3, M_k^4$ are defined in (4.7).

If $\eta=\max\{\eta_1,\eta_2,\eta_3,\eta_4\}$, $\tau=\max\{\tau_1,\tau_2,\tau_3,\tau_4\}$ and $\xi=\max\{\xi_1,\xi_2,\xi_3,\xi_4\}$, where if $\eta_t,\tau_t,\xi_t, t=1...4$ are defined in (4.4)-(4.6), then from (5.1), (5.4), (5.5), (5.6) and (5.8) we have

$$\left\|P^{k+1}-P^{k}\right\|_\infty \le \eta\gamma_1^k+\tau\gamma_2^k+\xi\gamma_3^k,$$

where $\gamma_t^k=\max\limits_{i,j}\left\|\Delta_{i,j,t}^k\right\|, t=1,2,3$.

Lemma 5.6. Given initial control polygon $p_{i,j}^0=p_{i,j},\ i\in Z,$ let the values $p_{i,j}^k,\ k\ge 0$ be defined recursively by non-stationary subdivision scheme (4.1) together with (4.2). If $\psi<1$, then

$$\gamma_t^k \le (\psi)^k\gamma_{0,}^k t=1,2,3, \tag{5.9}$$

where ψ and γ_t^k are defined in (4.3) and (4.8).

Proof: Using (4.1), (4.2) and by utilizing the approach used in the derivation of (3.7) and (3.8) we get

$$p_{2i+1,2j}^k-p_{2i,2j}^k=\sum_{s=0}^m a_s^{k-1}\left(\sum_{r=0}^m\left(b_r^{k-1}-a_r^{k-1}\right)p_{i+r,j+s}^{k-1}\right)=\sum_{s=0}^m a_s^{k-1}\left(\sum_{r=0}^{m-1}C_r^{k-1}\left(p_{i+r+1,j+s}^{k-1}-p_{i+r,j+s}^{k-1}\right)\right), \tag{5.10}$$

where $C_r^{k-1}=\sum\limits_{t=0}^{r}\left(a_t^{k-1}-b_t^{k-1}\right)$,

$$p_{2i+2,2j}^k-p_{2i+1,2j}^k=\sum_{s=0}^m a_s^{k-1}\left(\sum_{r=0}^{m}D_r^{k-1}\left(p_{i+r+1,j+s}^{k-1}-p_{i+r,j+s}^{k-1}\right)\right), \tag{5.11}$$

where $D_r^{k-1}=a_r^{k-1}-C_r^{k-1}$,

$$p_{2i+1,2j+1}^k-p_{2i,2j+1}^k=\sum_{s=0}^m b_s^{k-1}\left(\sum_{r=0}^{m-1}C_r^{k-1}\left(p_{i+r+1,j+s}^{k-1}-p_{i+r,j+s}^{k-1}\right)\right), \tag{5.12}$$

$$p_{2i+2,2j+1}^k-p_{2i+1,2j+1}^k=\sum_{s=0}^m b_s^{k-1}\left(\sum_{r=0}^{m}D_r^{k-1}\left(p_{i+r+1,j+s}^{k-1}-p_{i+r,j+s}^{k-1}\right)\right), \tag{5.13}$$

$$p_{2i+1,2j+2}^k-p_{2i,2j+2}^k=\sum_{s=0}^m a_s^{k-1}\left(\sum_{r=0}^{m-1}C_r^{k-1}\left(p_{i+r+1,j+s+1}^{k-1}-p_{i+r,j+s+1}^{k-1}\right)\right), \tag{5.14}$$

$$p_{2i+2,2j+2}^k-p_{2i+1,2j+2}^k=\sum_{s=0}^m a_s^{k-1}\left(\sum_{r=0}^{m}D_r^{k-1}\left(p_{i+r+1,j+s+1}^{k-1}-p_{i+r,j+s+1}^{k-1}\right)\right), \tag{5.15}$$

$$p_{2i,2j+1}^k-p_{2i,2j}^k=\sum_{r=0}^m a_r^{k-1}\left(\sum_{s=0}^{m-1}C_s^{k-1}\left(p_{i+r,j+s+1}^{k-1}-p_{i+r,j+s}^{k-1}\right)\right), \tag{5.16}$$

$$p_{2i,2j+2}^k-p_{2i,2j+1}^k=\sum_{r=0}^m a_r^{k-1}\left(\sum_{s=0}^{m}D_s^{k-1}\left(p_{i+r,j+s+1}^{k-1}-p_{i+r,j+s}^{k-1}\right)\right), \tag{5.17}$$

$$p_{2i+1,2j+1}^k-p_{2i+1,2j}^k=\sum_{r=0}^m b_r^{k-1}\left(\sum_{s=0}^{m-1}C_s^{k-1}\left(p_{i+r,j+s+1}^{k-1}-p_{i+r,j+s}^{k-1}\right)\right), \tag{5.18}$$

$$p^k_{2i+1,2j+2} - p^k_{2i+1,2j+1} = \sum_{r=0}^{m} b_r^{k-1}\left(\sum_{s=0}^{m-1} D_s^{k-1}\left(p^{k-1}_{i+r,j+s+1} - p^{k-1}_{i+r,j+s}\right)\right). \tag{5.19}$$

Using (5.10)-(5.13) we get

$$\max_{i,j}\left\|\Delta^k_{i,j,1}\right\| \le \left(\sum_{r=0}^{m}\left|a_r^{k-1}\right|\sum_{s=0}^{m-1}\left|C_s^{k-1}\right|\right)\max_{i,j}\left\|\Delta^{k-1}_{i,j,1}\right\|,$$

$$\max_{i,j}\left\|\Delta^k_{i,j,1}\right\| \le \left(\sum_{r=0}^{m}\left|a_r^{k-1}\right|\sum_{s=0}^{m}\left|D_s^{k-1}\right|\right)\max_{i,j}\left\|\Delta^{k-1}_{i,j,1}\right\|,$$

$$\max_{i,j}\left\|\Delta^k_{i,j,1}\right\| \le \left(\sum_{r=0}^{m}\left|b_r^{k-1}\right|\sum_{s=0}^{m-1}\left|C_s^{k-1}\right|\right)\max_{i,j}\left\|\Delta^{k-1}_{i,j,1}\right\|,$$

$$\max_{i,j}\left\|\Delta^k_{i,j,1}\right\| \le \left(\sum_{r=0}^{m}\left|b_r^{k-1}\right|\sum_{s=0}^{m}\left|D_s^{k-1}\right|\right)\max_{i,j}\left\|\Delta^{k-1}_{i,j,1}\right\|.$$

Recursively we get

$$\gamma_1^k \le \left(\sum_{r=0}^{m}\left|a_r^{k-1}\right|\sum_{s=0}^{m-1}\left|C_s^{k-1}\right|\sum_{r=0}^{m}\left|a_r^{k-2}\right|\sum_{s=0}^{m-1}\left|C_s^{k-2}\right|\ldots\sum_{r=0}^{m}\left|a_r^{0}\right|\sum_{s=0}^{m-1}\left|C_s^{0}\right|\right)\gamma_1^0,$$

$$\gamma_1^k \le \left(\sum_{r=0}^{m}\left|a_r^{k-1}\right|\sum_{s=0}^{m}\left|D_s^{k-1}\right|\sum_{r=0}^{m}\left|a_r^{k-2}\right|\sum_{s=0}^{m}\left|D_s^{k-2}\right|\ldots\sum_{r=0}^{m}\left|a_r^{0}\right|\sum_{s=0}^{m}\left|D_s^{0}\right|\right)\gamma_1^0,$$

$$\gamma_1^k \le \left(\sum_{r=0}^{m}\left|b_r^{k-1}\right|\sum_{s=0}^{m-1}\left|C_s^{k-1}\right|\sum_{r=0}^{m}\left|b_r^{k-2}\right|\sum_{s=0}^{m-1}\left|C_s^{k-2}\right|\ldots\sum_{r=0}^{m}\left|b_r^{0}\right|\sum_{s=0}^{m-1}\left|C_s^{0}\right|\right)\gamma_1^0,$$

$$\gamma_1^k \le \left(\sum_{r=0}^{m}\left|b_r^{k-1}\right|\sum_{s=0}^{m}\left|D_s^{k-1}\right|\sum_{r=0}^{m}\left|b_r^{k-2}\right|\sum_{s=0}^{m}\left|D_s^{k-2}\right|\ldots\sum_{r=0}^{m}\left|b_r^{0}\right|\sum_{s=0}^{m}\left|D_s^{0}\right|\right)\gamma_1^0,$$

where $\gamma_1^k = \max_{i,j}\left\|\Delta^k_{i,j,1}\right\|$.

From above inequalities and (4.3) we get $\gamma_1^k \le \left(\hat{A}\hat{C}\right)^k\gamma_1^0$, $\gamma_1^k \le \left(\hat{A}\hat{D}\right)^k\gamma_1^0$, $\gamma_1^k \le \left(\hat{B}\hat{C}\right)^k\gamma_1^0$, $\gamma_1^k \le \left(\hat{B}\hat{D}\right)^k\gamma_1^0$, where $\hat{A},\hat{B},\hat{C},\hat{D}$ are defined in (4.3).

If $\psi = \max\left\{\hat{A}\hat{C},\hat{A}\hat{D},\hat{B}\hat{C},\hat{B}\hat{D}\right\}$, then we get $\gamma_1^k \le (\psi)^k\gamma_1^0$. Using (5.14) and (5.15) recursively we get $\gamma_3^k \le (\psi)^k\gamma_3^0$. Similarly from (5.16)-(5.20) we get $\gamma_2^k \le (\psi)^k\gamma_2^0$. This completes the proof.

Remark 5.1. The main problem in Theorem 2 [3] is that its proof contains errors on page 609 in Equations (38), (40), (42), (44) and (46). Their corrected forms are given in Equations (5.10), (5.12), (5.14), (5.16) and (5.18) respectively of our Lemma 5.6.

Theorem 5.7. Given initial control polygon $p^0_{i,j} = p_{i,j}$, $i \in Z$, let the values $p^k_{i,j}$, $k \ge 0$ be defined recursively by non-stationary subdivision scheme (4.1) together with (4.2). Suppose P^k be the piecewise linear interpolation to the values $p^k_{i,j}$ and P^∞ be the limit surface of the subdivision scheme (4.1). If $\psi < 1$, then error bounds between limit surface and its control polygon after k-fold subdivision is

$$\left\|P^k - P^\infty\right\|_\infty \le \left\{\eta\gamma_1^0 + \tau\gamma_2^0 + \xi\gamma_3^0\right\}\left(\frac{(\psi)^k}{1-\psi}\right), \tag{5.20}$$

where $\eta = \max\{\eta_1,\eta_2,\eta_3,\eta_4\}$, $\tau = \max\{\tau_1,\tau_2,\tau_3,\tau_4\}$ and $\xi = \max\{\xi_1,\xi_2,\xi_3,\xi_4\}$, $\eta_t,\tau_t,\xi_t, t = 1\ldots4$ are defined in (4.4)-(4.6), $\gamma_t^0, t = 1,2,3,$ are defined in (4.8) and ψ is defined in (4.3).

Proof: From (5.7) we have $\left\|P^{k+1}-P^k\right\|_\infty \le \eta\gamma_1^k + \tau\gamma_2^k + \xi\gamma_3^k$. Using (5.9) we get

$$\left\|P^{k+1}-P^k\right\|_\infty \le \left(\eta\gamma_1^0 + \tau\gamma_2^0 + \xi\gamma_3^0\right)(\psi)^k .$$

By triangular inequality $\left\|P^k-P^\infty\right\|_\infty \le \left\{\eta\gamma_1^0 + \tau\gamma_2^0 + \xi\gamma_3^0\right\}\left(\frac{(\psi)^k}{1-\psi}\right)$. This completes the proof.

Remarks 5.2. Theorem 5.7 is a generalization of the corresponding Theorem 7 (after correction) of Mustafa et al. In particular, if the subdivision mask does't change by increasing subdivision level k then both results coincides.

By using Theorem 5.7 we can obtain the following computational formula of subdivision depth for non-stationary binary subdivision surfaces.

Theorem 5.8. Let k be subdivision depth and let d^k be the error bound between non-stationary binary subdivision surface and its k-level control mesh P^k. For arbitrary $\varepsilon > 0$, if $k \ge \log_{\psi^{-1}}\left(\frac{\eta\gamma_1^0 + \tau\gamma_2^0 + \xi\gamma_3^0}{\varepsilon(1-\psi)}\right)$ then $d^k \le \varepsilon$.

Proof: From (5.20), we have $d^k = \left\|P^k-P^\infty\right\|_\infty \le \left\{\eta\gamma_1^0 + \tau\gamma_2^0 + \xi\gamma_3^0\right\}\left(\frac{(\psi)^k}{1-\psi}\right)$, This implies, for arbitrary given $\varepsilon > 0$, when subdivision depth k satisfy the following inequality

$$k \ge \log_{\psi^{-1}}\left(\frac{\eta\gamma_1^0 + \tau\gamma_2^0 + \xi\gamma_3^0}{\varepsilon(1-\psi)}\right)$$

then $d^k \le \varepsilon$. This completes the proof.

6. References

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