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# Visualization of 2D Data By Rational Quadratic Functions

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**Abstract.** A local shape preserving interpolation scheme for 2D data is discussed by using piecewise rational quadratic function. To preserve the shape of the data in the view of curves, constraints are made on free parameters in the description of rational quadratic function.

**Keywords:** Visualization, Rational Quadratic Functions, Positive Curves, Monotonic Curves, Convex Curves.

## 1. Introduction

Visualization of scientific data in 2D and 3D is very vital aspect in CAGD. There are plenty of splines which can produce smooth curves but incapable to preserve the inherited shape of given data. Positivity, monotonicity and convexity are very important features of shape. There are many physical situations where entities only have meaning when their values appear in positive, monotonic or convex shape. Therefore it is very important to discuss shape preserving interpolation problems to provide a computationally economical and visually pleasing solution to the problems of different scientific phenomena.

Many people have worked in the area of shape preservation. For example, Fritsch and Carlson [4] and Fritsch and Butland [5] have discussed the piecewise cubic interpolation of monotonic data. Also, McAllister, Passow and Roulier [8] considered the piecewise polynomial interpolation of monotonic and convex data. Butt and Brodlie[3], Brodlie and Butt [2] have discussed the problem of shape preserving using piecewise cubic interpolation. The cubic interpolation method [2,3] requires the introduction of additional knots when used as shape preserving method. An algorithm for quadratic spline interpolation is given by McAllister and Roulier [9]. Schmidt and Hess [12] have used cubic polynomials and derived the necessary and sufficient conditions to make the interpolant positive. Sarfaraz et al [11] have used $C^1$ piecewise rational cubic functions.

This paper examines the problem of shape preservation of data that arose from some scientific phenomena or from some mathematical function. The rational quadratic function defined in Section 2 is used to preserve the shape of data by making the shape constraints on the free parameters in the description of rational quadratic function. The method under consideration in this paper has an important and advantageous feature that no additional points (knots) need to be supplied to preserve the shape of data.

## 2. Rational Quadratic Function

For given set of data points $(x_i, f_i), i = 1,2,...,n,$ where $x_1 < x_2 < ... < x_n$. The piecewise rational quadratic function $S_i(x)$ is defined over each subinterval $I_i = [x_i, x_{i+1}]$ as:

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)}, \tag{1}$$

with

$$p_i(\theta) = f_i(1-\theta)^2 + (r_i f_i + h_i d_i)(1-\theta)\theta + f_{i+1}\theta^2, \; q_i(\theta) = 1 + (r_i - 2)\theta(1-\theta),$$

where

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$$\theta = \frac{x - x_i}{h_i}, \quad 0 \le \theta \le 1, \qquad h_i = x_{i+1} - x_i,$$

$r_i$ is the shape parameter.

The rational quadratic function (1) has the following properties:

$$S(x_i) = f_i,\ S(x_{i+1}) = f_{i+1},\ S^{(1)}(x_i) = d_i,$$

where $S^{(1)}(x)$ denote the differentiation with respect to $x$ and $d_i$ denote derivative values (given or estimated by some method) at knots $x_i$.

## 2.1. Some Observations

- For $r_i = 2$, the rational quadratic (1) reduces to quadratic as:

$$S_i(x) = f_i(1-\theta)^2 + (2f_i + h_i d_i)(1-\theta)\theta + f_{i+1}\theta^2. \qquad (2)$$

- When $r_i \to 0$, equation (1) becomes:

$$S_i(x) = \frac{f_i(1-\theta)^2 + h_i d_i(1-\theta)\theta + f_{i+1}\theta^2}{1 - 2(1-\theta)\theta}.$$

In this case the curve gets loosened; it bulges outside the convex hull for negative values.

- When $r_i \to \infty, S(x) = f_i$.

Table 1

| x | 1.0 | 1.5 | 4.0 | 4.5 | 5.0 |
|---|---|---|---|---|---|
| f | 5.0 | 1.0 | 0.7 | 2.0 | 3.0 |

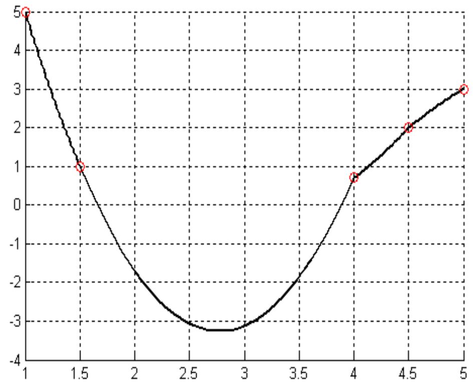


Figure 1: Curve to the data in Table 1, when $r_i = 2$

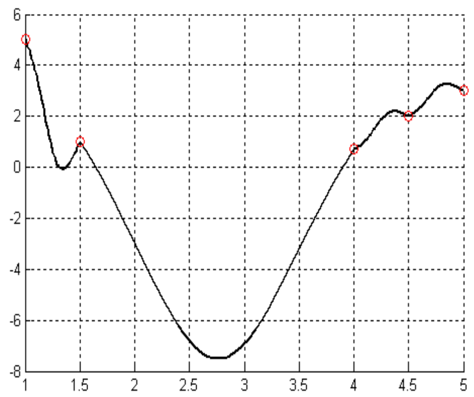


Figure 2: Curve to the data in Table 1, when $r_i = 0$

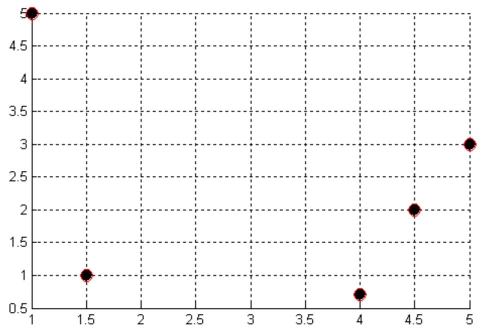


Figure 3: Curve to the data in Table 1, when $r_i \to \infty$

## 2.2. $C^1$ Rational Quadratic Function

The rational quadratic function preserves continuity of zeroth order; the order of continuity can be increased up to 1st order by applying following conditions on the rational quadratic function (1).

$$S^{(1)}(x_i) = d_i, \tag{3}$$

$$S^{(1)}(x_{i+1}) = d_{i+1}, \tag{4}$$

The equation (3) is satisfied by rational quadratic function. Thus, in order to achieve first order continuity, condition in (4) is also required.

Differentiating equation (1) with respect to *x*

$$S^{(1)}(x) = \frac{(r_i\Delta_i - d_i)\theta^2 + \Delta_i 2\theta(1-\theta) + d_i(1-\theta)^2}{[1+(r_i-2)\theta(1-\theta)]^2},$$

when $x = x_i$ *i.e.* $\theta = 0$

$$S^{(1)}(x) = d_i,$$

when $x = x_{i+1}$ *i.e.* $\theta = 1$

$$S^{(1)}(x) = (r_i\Delta_i - d_i),$$

$S(x)$ will preserve first order continuity if and if only if

$$S^{(1)}(x_{i+1}) = d_{i+1},$$

therefore

$$(r_i\Delta_i - d_i) = d_{i+1},$$

or

$$r_i = \frac{d_i + d_{i+1}}{\Delta_i}. \tag{5}$$

**Theorem 1.** The rational quadratic function (1) preserves continuity of 1st order if and only if equation (5) is satisfied.

## 3. Determination of derivatives

In most applications, derivatives parameters $d_i$ are not given and hence must be determined either from the data $(x_i, f_i)$ or by some appropriate methods. An obvious choice is mentioned here.

### 3.1. Arithmetic Mean Method

This is the three-point difference approximation with

$$d_i = 0, \text{ if } \Delta_{i-1} = 0 \text{ or } \Delta_i = 0,$$

otherwise

$$d_i = \frac{h_i\Delta_{i-1} + h_{i-1}\Delta_i}{h_i + h_{i+1}}, \ i = 2, 3, \ldots \text{n-1}.$$

and the end conditions are given as:

$$d_1 = 0, \text{ if } \Delta_1 = 0, \text{ or } \operatorname{sgn}(d_1^*) \neq \operatorname{sgn}(\Delta_1),$$

otherwise

$$d_1 = d_1^* = \Delta_1 + \frac{h_1(\Delta_1 - \Delta_2)}{h_1 + h_2},$$

$$d_n = 0, \text{ if } \Delta_{n-1} = 0 \text{ or } \operatorname{sgn}(d_n^*) \neq \operatorname{sgn}(\Delta_{n-1}),$$

otherwise

$$d_n = d_n^* = \Delta_{n-1} + \frac{h_{n-1}(\Delta_{n-1} - \Delta_{n-2})}{h_{n-1} + h_{n-2}}.$$

where

$$\Delta_i = \frac{f_{i+1} - f_i}{h_i}, i = 1, 2, \ldots, n.$$

## 4. Positive Curves

For given data points $(x_i, f_i)$, $i = 1, 2, \ldots, n$, where $f_1 > 0, f_2 > 0, \ldots, f_n > 0$. The curve $S(x)$ is positive on the whole interval if

$$S(x) > 0 \ \forall \ x_1 \leq x \leq x_n.$$

As $r_i > -2$ guarantees positive denominator of rational quadratic function (1) so the first condition on $r_i$ is

$$r_i > -2,$$

Now the positivity of $S(x)$ defined in (1) only is the positivity of numerator of (1)

*i.e.*

$$p_i(\theta) = f_i(1-\theta)^2 + (r_i f_i + h_i d_i)(1-\theta)\theta + f_{i+1}\theta^2,$$

and $p_i(\theta)$ is positive if

$$r_i > -\frac{h_i d_i}{f_i},$$

Hence $S(x) > 0$ if and only if

$$r_i > Max\left\{-2, -\frac{h_i d_i}{f_i}\right\} \quad (6)$$

**Theorem 2.** The rational quadratic function given in (1) preserves positivity in $[x_i, x_{i+1}]$, $i$=1,2,…,n if $r_i$ satisfies (6).

Equation (6) can be rearranged as:

$$r_i = l_i + Max\left\{-2, -\frac{h_i d_i}{f_i}\right\}, \text{ where } l_i > 0.$$

## 4.1. Demonstration

For the demonstration consider the positive data in Table 2. This data has come from the known volume of ***NAOH*** taken in a beaker and its conductivity was determined. ***HCL*** solution was added from the burette in steps drop by drop. After each addition volume of ***HCL*** (x) was stirred by gentle shaking and conductance (f) was determined as shown in Table 2. Application of the quadratic spline method produces the curve in Figure 4. This curve shows the negative value of conductance which is ridiculous. This flaw is recovered nicely in Figure 5 using positivity preserving rational quadratic scheme of Section 4.

Table 2

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| x | 2 | 3 | 7 | 8 | 9 | 13 | 14 |
| f | 10 | 2 | 3 | 7 | 2 | 3 | 10 |

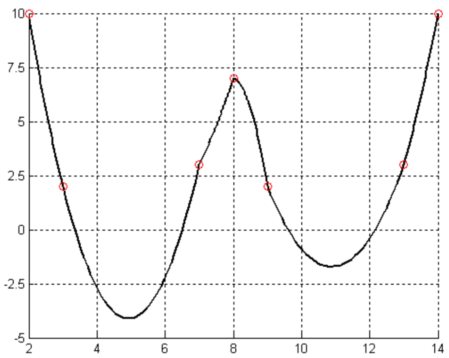


Figure 4: Quadratic curve to the data in Table 2

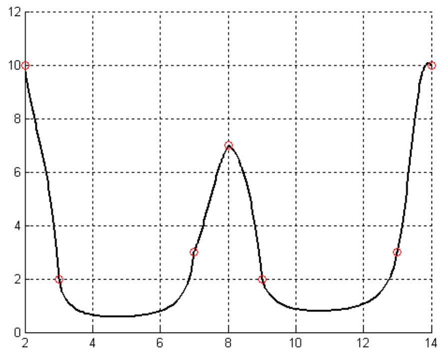


Figure 5: Shape preserving rational quadratic curve to data in Table 2

## 5. Monotonic curves

For given set of data points $(x_i, f_i),\ i = 1, 2, 3, ..., n$, where $x_1 < x_2 < ... < x_n$, let us assume monotonically increasing set of data such that

$$f_1 \le f_2 \le ... \le f_n,$$

or equivalently

$$\Delta_i \ge 0,\ i = 1, 2, \ldots, n-1.$$

and derivative parameter is chosen such that

$$d_i \ge 0,$$

Now the rational quadratic function (1) preserves monotonic curve through monotonic data if

$$S^{(1)}(x) > 0,$$

Where $S^{(1)}(x)$ can be obtained by differentiating (1) with respect to $x$ as

$$S^{(1)}(x) = \frac{\Sigma_{j=0}^{2} \binom{2}{j} A_{j,i} (1-\theta)^{2-j} \theta^{j}}{[1 + (r_i - 2)\theta(1-\theta)]^2},$$

Where

$$A_{0,i} = d_i,$$
$$A_{1,i} = \Delta_i,$$
$$A_{2,i} = r_i \Delta_i - d_i.$$

Now $S^{(1)}(x) > 0$ if and only if

$$A_{j,i} > 0,\ j = 0, 1, 2.$$

or

$$r_i > \frac{d_i}{\Delta_i}. \qquad (7)$$

**Theorem 3.** The rational quadratic function given in (1) preserves monotonicity in $[x_i, x_{i+1}]$, $i$=1,2,…,n if $r_i$ satisfies (7).

The equation (7) can be rearranged as:

$$r_i = m_i + \frac{d_i}{\Delta_i}, \quad \text{where } m_i > 0.$$

**Remark.** The case of monotonically decreasing data can be derived in a similar way.

## 5.1. Demonstration

We take a monotonic data as in Table 3. Figure 6 is produced by using quadratic spline method, which does not preserve monotonicity. Figure 7 shows the monotonic curve through monotonic data in Table 3 using monotonic rational quadratic scheme derived in Section 5.

Table 3

| x | 1.0 | 1.5 | 4.0 | 4.5 | 5.0 |
|---|---|---|---|---|---|
| f | 5.0 | 10.0 | 17.0 | 20.0 | 30.0 |

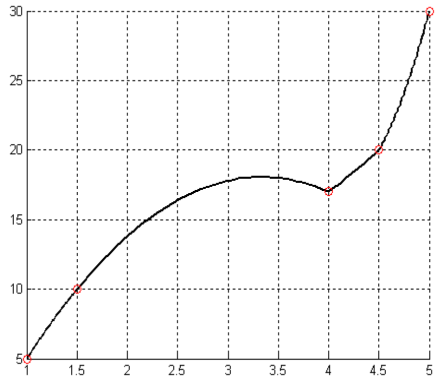


Figure 6: Quadratic curve to the data in Table 3

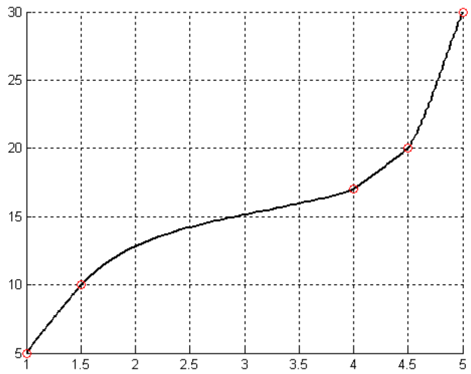


Figure 7: Shape preserving rational quadratic curve to data in Table 3

# 6. Convex curves

The data set $\{(x_i, f_i) : i = 1, 2, \ldots, n\}$ is said to be convex if $\Delta_1 \le \Delta_2 \le \ldots \le \Delta_{n-1} \le \Delta_n$ and it is strictly convex if $\Delta_1 < \Delta_2 < \ldots < \Delta_{n-1} < \Delta_n$. If derivative values at data points are also given then these values must satisfy $\Delta_i < d_i < \Delta_{i+1}$.

The rational quadratic function defined in (1) will preserve convex curve through convex data if in each subinterval

$$S^{(2)}(x) > 0,$$

Where $S^{(2)}(x)$ is given by

$$S^{(2)}(x) = \frac{\Sigma_{j=0}^{3} A_{j,i}(1-\theta)^{3-j}\theta^{j}}{h_i[1+(r_i-2)\theta(1-\theta)]^3},$$

Where

$$A_{0,i} = -2r_i d_i + 2\Delta_i + 2d_i,$$
$$A_{1,i} = -6d_i + 6\Delta_i,$$
$$A_{2,i} = 6r_i\Delta_i - 6d_i - 6\Delta_i,$$
$$A_{3,i} = 2r_i^2\Delta_i - 2r_i\Delta_i - 2r_i d_i + 2d_i - 2\Delta_i.$$

Clearly the denominator of $S^{(2)}(x)$ is positive if

$$r_i > -2,$$

and positivity of all the $A_{j,i}$ $j$ = 0,1,2,3. guarantees the positive numerator of $S^{(2)}(x)$ and $A_{j,i}$ >0, $j$ = 0,1,2,3 if

$$r_i > \frac{\Delta_i + d_i}{\Delta_i}.$$

Thus $S^{(2)}(x)$ is positive if

$$r_i > \max\left\{-2, \frac{\Delta_i + d_i}{\Delta_i}\right\}. \tag{8}$$

**Theorem 4.** The rational quadratic function given in (1) preserves convexity in $[x_i, x_{i+1}]$, $i$=1,2,…,n if $r_i$ satisfies (8).

The equation (8) can be rearranged as:

$$r_i = n_i + \max\left\{-2, \frac{\Delta_i + d_i}{\Delta_i}\right\}, \text{ where } n_i > 0.$$

## 6.1. Demonstration

An example of convex data is shown in Table 4. Application of the quadratic spline method produces the curve in Figure 8. This curve shows noise, which is misguiding. Figure 9 is produced by applying quadratic spline method on this convex data. One can see that the convexity nature of the data is preserved in a pleasing way.

Table 4

| X | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| F | 9 | 4 | 3 | 2.40 | 2.20 | 2.15 | 2.10 |

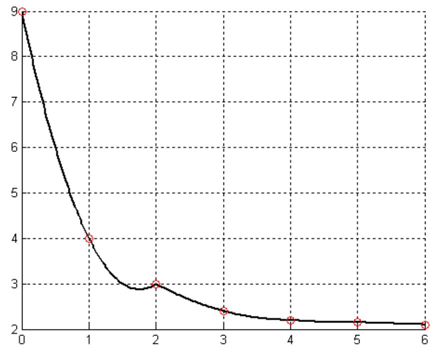


Figure 8: Quadratic curve to the data in Table 4

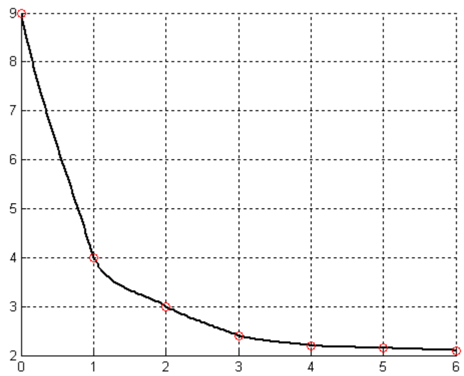


Figure 9: Shape preserving rational quadratic curve to data in Table 4

**Theorem 5.** The rational quadratic function preserves positivity, monotonicity and convexity through positive, monotone and convex data if the parameter $r_i$ satisfies the following condition

$$r_i > \max\left\{-2, -\frac{h_i d_i}{f_i}, 1+\frac{d_i}{\Delta_i}\right\},$$

in each subinterval $[x_i, x_{i+1}]$, $i$=1,2,…,n.

## 7. Conclusion

In this paper the problem of shape preserving curves is discussed and the constraints are made on free parameters in the description of rational quadratic function to preserve the shape of data.

In future this problem can be extended for the shape preserving rational biquadratic function.

## 8. References


[1] R. Asim. Visualization of Data Subject to Positive Constraint. Ph. D. thesis, School of Computer Studies. University of Leads, Leads, U. K. 2000

[2] K. W. Brodlie, and S. Butt. Preserving Convexity Using Piecewise Cubic Interpolation. *Computers and Graphics* 1991, **15**(1):15-23.

[3] S. Butt, and K. W. Brodlie. Preserving Positivity Using Piecewise Cubic Interpolation. *Computers and Graphics.* 1993, **17**(1):55-64.

[4] F. N. Fritsch, and R. E. Carlson. Monotone Piecewise Cubic Interpolation. *SIAM J. Numer. Anal.*, 1980, **17**(2): 238-246.

[5] F. N. Fritsch, and J. Butland. A Method for Constructing Local Monotone Piecewise Cubic Interpolants. *SIAM J. of Sci. Stat. Comput*. 1984, **5**:300-304.

[6] T. N. T.Goodman, and K. Unsworth. Shape Preserving Interpolation by Parametrically Defined Curves. *SIAM J.*

*Numer.Anal*., 1988, **25**:1-13.

[7] M. Z. Hussain, Shape Preserving Curves and Surfaces for Computer Graphics, Ph. D. thesis, University of the Punjab, Pakistan. 2002.

[8] D. F. McAllister, E. Passow, and J. A. Roulier. Algorithms for Compiting Shape Preserving Spline Interpolations to Data. *Math. Comp.*, 1977, **31**:717-725.

[9] D. F. McAllister, and Roulier, J. A. Roulier. An Algorithm for Computing a Shape Preserving Osculatory Quadratic Spline. *ACM Trans. Math. Software.* 1981, **7**:331-347.

[10] M. Sarfraz, M. Z. Hussain, and S. Butt. A Rational Spline for Visualizing Positive Data, *Proc. IEEE, International Conference on Information Visualization*. London, U. K, pp57-62, 2000.

[11] M. Sarfraz, S. Butt, and M. Z. Hussain. Visualization of Shaped Data by a Rational Cubic Spline Interpolation. *Computers and Graphics*. 2001, **25**(5):833-845.

[12] J. W.Schmidt, and W. Hess. Positivity of Cubic Polynomial on Intervals and Positive Spline Interpolation. *BIT* 1988, **28**:340-352.

[13] J. W. Schmidt. Positivity, Monotone and S-convex $C^1$ Interpolation on Rectangular Grids. *Computing*. 1992, **48**:363-371.