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Numerical Methods for Finding Multiple Solutions of a Superlinear Problem

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Abstract. Using two numerical methods, we will obtain numerical positive solutions of the equation $-\Delta u = \lambda f(u)$ with Dirichlet boundary condition in a bounded domain Ω, where $\lambda > 0$ and $f(u)$ is a superlinear function of u. We study the behavior of the branches of numerical positive solutions for varying λ.

Keywords: Superlinear equation; Multiple positive solutions; Mountain pass Lemma; sub and super-solutions

AMS Subject Classification: 35J60, 35B30

1. Introduction

We are interested in the positive solutions of the problem

$$\begin{gathered} -\Delta u(x) = \lambda f(x, u(x)), x \in \Omega \\ u(x) = 0, x \in \partial\Omega \end{gathered} \tag{1.1}$$

where Ω is a bounded and smooth domain in R^N with boundary $\partial\Omega$, $\lambda > 0$ is a real parameter, and

$$f(u) = au - bu^2 + cu^3 (a, b, c > 0).$$

In this paper we study numerical solutions of the equation (1.1) that arises in wide fields of physics, and so it has been studied by several authors. Among others it describes problems of thermal self-ignition, diffusion phenomena induced by nonlinear sources or a ball of isothermal gas in gravitational. In this paper we concentrate on the numerical positive solutions of temperature distribution in an object heated by the application of a uniform electric current suggested in [4]. In fact we show that the first eigenvalue of the problem

$$\begin{aligned} -\Delta u(x) &= \lambda a u(x), x \in \Omega \\ u(x) &= 0, x \in \partial\Omega \end{aligned} . \tag{1.2}$$

This is a bifurcation point of the branch of numerical solutions and so there is another branch of solutions that for any positive λ admits a numerical solution.

Let H be the Sobolev space $H_0^{1,2}(\Omega)$ with inner product (see [1]) $\langle u, v \rangle = \int_\Omega u.v dx$.

We define $J : H \to \Re$ by

$$J(u) = \int_\Omega (\frac{1}{2} |\nabla u|^2 - F(u)) dx \tag{1.3}$$

where $F(u) = \int_0^u f(t) dt$. And we have

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$$J'(u)v = \langle \nabla J(u), v \rangle = \int_{\Omega} \{\nabla u.\nabla v - f(u)v\} dx \tag{1.4}$$

Define

$$\gamma(u) = J'(u)u = \langle \nabla J(u), u \rangle = \int_{\Omega} \{|\nabla u|^2 - uf(u)\} dx \tag{1.5}$$

$$J''(u)(v,w) = \int_{\Omega} \{\nabla v.\nabla w - f'(u)vw\} dx \tag{1.6}$$

We can do integration by parts on (1.4) to get

$$J'(u)v = -\int_{\Omega} \{\Delta u + f(u)\} v dx$$

Therefore a classical solution to the PDE (1) is a critical point of J. By definition, critical points of J are weak solution to (1). By regularity theory for elliptic boundary value problems, u is a classical solution to our problem if and only if u is a weak solution to (1). In other words, critical points of the action functional J precisely the classical solutions of the PDE. We consider in this paper

$$S = \{u \in H - \{0\} : \langle \nabla J(u,u) \rangle = 0\},$$

where we note that nontrivial solutions to (1) are in S, and S is a closed subset of H. S is known as the Nehari manifold (see [5]) and it is clear that all critical points of J(u) must lie in S. It can be shown that all of the condition of Mountain Pass Lemma (see [2]) are satisfied for functional J, so there exists a critical point $\hat{u}$ of J that we call it mountain pass type solution and without loss of generality we may assume $\hat{u}$ is positive.

In the next section we present useful numerical methods and introduce the framework of the procedure to findnumerical solutions.

2. Numerical Results

In this section we present our numerical results are based on ”mountain pass lemma” and ”sub and supersolutions”.

Mountain pass algorithm: At first we present a numerical algorithm for finding mountain pass type solution. Consider the problem

$$-\Delta u(x) = \lambda f(x, u(x)), x \in \Omega$$
$$u(x) = 0, x \in \partial\Omega$$

where $f(u)$ is superlinear. Given a nonzero element $u \in H$ and a piece-wise smooth region $\Omega \subset R^N$, we will use the notation **u** to represent an array of real numbers agreeing with u on a grid $\Omega \subset \overline{\Omega}$.

We will take the grid to be regular.

At each step of the iterative process, we are required to project nonzero elements of H onto the submanifold S.

Projection of $\nabla J(u)$ on to the ray $\{\lambda u : \lambda > 0\}$ is given by

$$p_u(J(u)) = \frac{< \nabla u, u >}{< u, u >} u = \frac{\gamma(u)}{\| u \|^2} u .$$

Let u be a nonzero element of H, represented by u over the grid Ω. Let $s_1 = 0.5$ or another perhaps optimally determined small positive constant. Define $u_0 = u$ and

$$u_{k+1} = u_k + s_1 \frac{\gamma(u_k)}{\| u_k \|^2} u_k, k \geq 0$$

We will use notation $p_1(u) = \lim_{k \to \infty} u_k$, provided that limit exists, to represent unique positive multiple of u lying on S.

We use following algorithm to find the solution:

1. Initialize u_0 with appropriate initial guess.

2. Project u on to S. (ray projection in ascent direction element u onto S, that we explain it above).

The standard L^2 gradient is not the gradient we are considering,

$$
\begin{aligned}
&< \nabla J(u), v > \\
&= J'(u)v = \int_\Omega (\nabla u.\nabla v - f(u)v)dx \\
&= \int_\Omega (\nabla u.\nabla v - (-\Delta)(-\Delta)^{-1} f(u)v)dx \\
&= \int \{(\nabla u.\nabla v + \nabla(-\Delta)^{-1} f(u).\nabla u)\} \\
&= < u - (-\Delta)^{-1} f(u), v > .
\end{aligned}
$$

3. Begin loop with k = 0.

3.1. Solve linear system $-\Delta grad = f(u_k)$ for grad allows one to explicitly construct the array $\nabla J(u_k) \equiv u_k - grad$, representing $\nabla J(u)$.

3.2. Take gradient descent

$$u_k = u_k - s_2 \nabla J(u_k).$$

3.3. Reproject u_k on to S.

3.4. Increment k and repeat step (3.1), (3.2), (3.3) until convergence criteria are met:

$$\| \nabla J(u_k) \|_2^2 \approx 0, \| \Delta u_k + f(u_k) \|_2^2 \approx 0.$$

The obtained results shows there is an array of solutions that has the norm above the horizontal asymptote 0.383 when we define

$$\| u \| = \| u \|_\infty = \sup_{x \in [0,1]} u(x).$$

For brevity we express just some of those numerical results:

Approximation of u for $\lambda = eps$

$x \setminus y$	0.1	0.3	0.5	0.7	0.9
01	0.143×10^{36}	0.416×10^{36}	0.547×10^{36}	0.416×10^{36}	0.143×10^{36}
0.3	0.416×10^{36}	1.290×10^{36}	1.748×10^{36}	1.290×10^{36}	0.416×10^{36}
0.5	0.547×10^{36}	1.748×10^{36}	2.402×10^{36}	1.748×10^{36}	0.547×10^{36}
0.7	0.416×10^{36}	1.290×10^{36}	1.748×10^{36}	1.290×10^{36}	0.416×10^{36}
0.9	0.143×10^{36}	0.416×10^{36}	0.547×10^{36}	0.416×10^{36}	0.143×10^{36}

$$\| u \|_\infty = 2.402 \times 10^{36}$$

We executed our code for $\lambda = eps$, that $eps = 2.22 \times 10^{-16}$ in MATLAB.

Approximation of u for $\lambda = 10$

$x \setminus y$	0.1	0.3	0.5	0.7	0.9
01	0.073×10^3	0.211×10^3	0.276×10^3	0.211×10^3	0.073×10^3
0.3	0.211×10^3	0.647×10^3	0.873×10^3	0.647×10^3	0.211×10^3
0.5	0.276×10^3	0.873×10^3	1.195×10^3	0.873×10^3	0.276×10^3
0.7	0.211×10^3	0.647×10^3	0.873×10^3	0.647×10^3	0.211×10^3
0.9	0.073×10^3	0.211×10^3	0.276×10^3	0.211×10^3	0.073×10^3

$$\| u \|_\infty = 1.195\times10^3$$

Approximation of u for $\lambda = 100$

$x \setminus y$	0.1	0.3	0.5	0.7	0.9
0.1	0.989	2.700	3.433	2.700	0.989
0.3	2.700	7.676	10.009	7.676	2.700
0.5	3.433	10.009	13.234	10.009	3.433
0.7	2.700	7.676	10.009	7.676	2.700
0.9	0.989	2.700	3.433	2.700	0.989

$$\| u \|_\infty = 13.234$$

Approximation of u for $\lambda = 10000$

$x \setminus y$	0.1	0.3	0.5	0.7	0.9
0.1	0.037	0.097	0.119	0.097	0.037
0.3	0.097	0.250	0.306	0.250	0.097
0.5	0.119	0.306	0.376	0.306	0.119
0.7	0.097	0.250	0.306	0.250	0.097
0.9	0.037	0.097	0.119	0.097	0.037

$$\| u \|_\infty = 0.376$$

According above tables $\| u \|_\infty \to 0.383$ as $\lambda \to \infty$.

It is well-known that there must always exists a solution for problems such as (2) between a sub-solution $\underline{v}$ and a super-solution $\overline{u}$ such that $\underline{v} \le \overline{u}$ for all $x \in \Omega$ (see [3]).

Consider the boundary value problem

$$\begin{cases} \Delta u(x) + f(x,u(x)) = 0 & on \ \ \Omega \\ u(x) = 0 & on \ \ \partial\Omega. \end{cases} \tag{2.1}$$

Let $\overline{u}$, $\underline{u} \in C^2(\overline{\Omega})$ satisfy $\overline{u} \ge \underline{u}$ as well as

$$\Delta\overline{u}(x) + f(x,\overline{u}(x)) \le 0 \quad on\Omega \quad \overline{u} \ge 0 \tag{2.2}$$

$$\Delta\underline{v}(x) + f(x,\underline{v}(x)) \ge 0 \quad on\Omega \quad \overline{u} \le 0 \tag{2.3}$$

Choose a number $c > 0$ such that

$$c + \tfrac{\partial f(x,u)}{\partial u} > 0 \quad \forall (x,u) \in \overline{\Omega} \times [\underline{v}, \overline{u}]$$

and such that the operator $(\Delta - c)$ with Dirichlet boundary condition has its spectrum strictly contained in the open left-half complex plane. Then the mapping

$$T : \phi \to w, \quad w = T\phi, \quad \phi \in C^2(\overline{\Omega}), \quad \phi(x) \in [\underline{v}, \overline{u}], \quad \forall x \in \overline{\Omega} \quad (3.1)$$

where $w(x)$ is the unique solution of the BVP

$$\begin{cases} \Delta w(x) - cw(x) = -[c\phi(x) + f(x,\phi(x))] & on \;\; \Omega \\ w(x) = 0 & on \;\; \partial\Omega, \end{cases} \tag{2.4}$$

is monotone, i.e. for any ϕ_1, ϕ_2 satisfying (3.1) and $\phi_1 \le \phi_2$, we have $T\phi_1, T\phi_2$ satisfies (3.1), and $T\phi_1 \le T\phi_2$ on Ω.

Consequently, by letting $f_c(x,u) = cu + f(x,u),$ the iterations

$$\begin{cases} u_0(x) = \overline{u}(x) & \\ (\Delta - c)u_{n+1}(x) = -f_c(x, u_n(x)) & on \;\; \Omega, \\ & n = 0,1,2,\ldots \\ u_{n+1}(x) = 0 & on \;\; \partial\Omega, \end{cases} \tag{2.5}$$

and

$$\begin{cases} v_0(x) = \underline{v}(x) & \\ (\Delta - c)v_{n+1}(x) = -f_c(x, v_n(x)) & on \;\; \Omega, \\ & n = 0,1,2,\ldots \\ v_{n+1}(x) = 0 & on \;\; \partial\Omega, \end{cases} \tag{2.6}$$

yield iteration u_n and v_n satisfying

$$\underline{v} = v_0 \le v_1 \le \cdots \le v_n \le \cdots \le u_n \le \cdots \le u_1 \le u_0 = \overline{u},$$

so that the limits

$$u_\infty(x) = \lim_{n\to\infty} u_n(x), \qquad v_\infty(x) = \lim_{n\to\infty} v_n(x)$$

exists in $C^2(\overline{\Omega})$. We have

(i) $v_\infty(x) \le u_\infty(x)$ on $\overline{\Omega}$

(ii) u_∞ and v_∞ are, respectively, stable from above and below;

(iii) if $u_\infty \not\equiv v_\infty$ and both u_∞ and v_∞ are asymptotically stable, then there exists an unstable solution $\phi \in C^2(\overline{\Omega})$ such that $v_\infty \le \phi \le u_\infty$.

We use following algorithm

Sub- and super-solution algorithm

1. Find a subsolution v_0 and a supersolution u_0. Choose a number $c > 0$;

2. Solve the boundary value problem

$$\begin{cases} -\Delta w_{n+1}(x) - cw_{n+1}(x) = -f_c(x, w_n(x)) & on\Omega \\ u(x) = 0 & on\partial\Omega. \end{cases} \tag{2.7}$$

for $w_n = v_n$ and $w_n = u_n$, respectively;

. If $\| w_{n+1} - w_n \| < \varepsilon$, output and stop. Else go to step 2.

We will use the notation $\mathbf{u}$ to represent an array of real numbers agreeing with u on a grid $\Omega \subset \overline{\Omega}$. We will take the grid to be regular.

We consider the problem $-\Delta u(x) = \lambda f(x, u(x))$ with $\Omega = [0,1]\times[0,1]$ and $f(u) = au - bu^2 + cu^3 (a,b,c > 0)$.

Let $a = c = 1, b = 3$, it is clear $f(u) \geq 0$ where $0 \leq u < 0.383$. So we consider $\underline{u} = 0$ as subsolution and $\bar{v} = 0.39$ as supersolution, it can be easily shown that our sub and supsolutions satisfy in (2.2),(2.3).

The obtained results shows there is an array of solution that before λ_1^+ (that we obtain λ_1^+ is around 20 since before it

Our code do not converges to positive solution) is identically zero and after it has the norm less than the horizontal asymptote 0.385 when we define

$$\| u \| = \| u \|_\infty = \sup_{x \in [0,1]} u(x)$$

(see the following tables). We repeated monotone iteration until our sub and super solution coincide.

For brevity we express just some of those numerical results.

Approximation of u for $\lambda = 21$

$x \backslash y$	0.1	0.3	0.5	0.7	0.9
0.1	0.0031	0.0080	0.0099	0.0080	0.0031
0.3	0.0080	0.0208	0.0255	0.0208	0.0080
0.5	0.0208	0.0255	0.0314	0.0255	0.0208
0.7	0.0080	0.0208	0.0255	0.0208	0.0080
0.9	0.0031	0.0080	0.0099	0.0080	0.0031

$$\| u \|_\infty = 0.014$$

Approximation of u for $\lambda = 100$

$x \backslash y$	0.1	0.3	0.5	0.7	0.9
0.1	0.0863	0.1680	0.1836	0.1680	0.0863
0.3	0.1680	0.3084	0.3222	0.3084	0.1680
0.5	0.3084	0.3222	0.3566	0.3222	0.3084
0.7	0.1680	0.3084	0.3222	0.3084	0.1680
0.9	0.0863	0.1680	0.1836	0.1680	0.0863

$$\| u \|_\infty = 0.3566$$

Approximation of u for $\lambda = 1000$

$x \backslash y$	0.1	0.3	0.5	0.7	0.9
0.1	0.3049	0.3422	0.3426	0.3422	0.3049
0.3	0.3422	0.3819	0.3816	0.3819	0.3422
0.5	0.3426	0.3816	0.3819	0.3816	0.3426
0.7	0.3422	0.3819	0.3816	0.3819	0.3422
0.9	0.3049	0.3422	0.3426	0.3422	0.3049

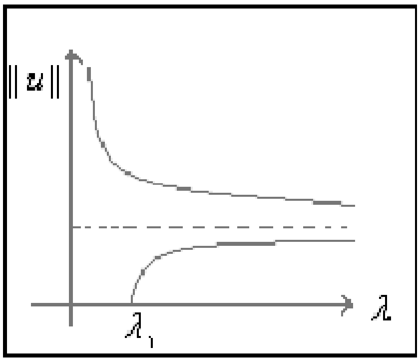
$$\| u \|_\infty = 0.3819$$

Approximation of u for $\lambda = 10000$

$x \backslash y$	0.1	0.3	0.5	0.7	0.9
0.1	0.3731	0.3775	0.3775	0.3775	0.3731
0.3	0.3775	0.3819	0.3820	0.3819	0.3775
0.5	0.3775	0.3820	0.3820	0.3820	0.3775
0.7	0.3775	0.3819	0.3820	0.3819	0.3775
0.9	0.3731	0.3775	0.3775	0.3775	0.3731

$$\| u \|_{\infty} = 0.3820$$

So by using the results tables we can draw the bifurcation diagram in the plane $(\lambda, \| u \|)$.



Bifurcation diagram

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