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# The Characteristics of Some Kinds of Special Quasirings *

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**Abstract.** By using quasi-group, we introduce some kinds of special quasi-rings and discuss their characteristics. According to the concept of monogenic semigroup we introduce definition of monogenic semiring and discuss the numbers and structures of monogenic semiring .Then we display their relations as following:

$$\{division\quad ring\} \begin{array}{l} \text{Ø}\,\{\text{int}\,egral\quad domain\}(\notin \{strong\quad biquasiring\}) \\ \text{Ø}\,\{strong\quad biquasiring\}(\notin \{\text{int}\,egral\quad domain\}) \end{array}$$

$$\left.\begin{array}{l} \text{Ø}\,\{ring\} \\ \text{Ø}\,\{biquasiring\} \\ \text{Ø}\,\{monogenic\quad semiring\} \end{array}\right\} \text{Ø}\,\{quasiring\}\,\text{Ø}\,\{semiring\}$$

**Keywords:** Semiring, Quasiring, Biquasiring, Strong Biquasiring, Monogenic Semiring

**AMS:** Subject Classification (2000): 20M07, 20M10

## 1. Introduction

A semiring[1] $S = (S,+,\bullet)$ is an algebra where both the additive reduct $(S,+)$ and the multiplicative reduct $(S,\bullet)$ are semigroups and such that the following distributive laws hold:

$$x(y+z) = xy + xz, \quad (x+y)z = xz + yz\,.$$

A semiring $(S,+,\bullet)$ is called a nilsemiring if there are additive idempotent 0 and for any $a \in S$ ,there are $n \in N$ , such that $na = 0$ .

An ideal (prime ideal) of a semiring $(S,+,\bullet)$ is the ideal (prime ideal) of the $(S,\bullet)$.

An ideal I of a semiring S is called a K-ideal[1], if

$$\forall a,b \in S, a \in I \text{ and } (a+b \in I \text{ or } b+a \in I) \Rightarrow b \in I\,.$$

Let $(S,+,\bullet)$ be a semiring. If I is a K-ideal, then I must be an ideal of $(S,\bullet)$ , not be an ideal of $(S,+)$ .

A semiring $(S,+,\bullet)$ is called a distributive semiring[1] if it satisfies the dual distributive laws:

$$x + yz = (x+y)(x+z), \quad yz + x = (y+x)(z+x)$$

In this paper, $\text{Reg}^{+}$, $\text{Reg}^{\bullet}$ are regular sets of $(S,+)$, $(S,\cdot)$ respectively. $E^{+}$ , $E^{\bullet}$ are idempotent sets of $(S,+)$, $(S,\bullet)$ respectively.

**Theorem 1.1.** Let $(S,+,\bullet)$ be a semiring. Then $(\text{Re}\,g^{+},\bullet)$ is an ideal of $(S,\bullet)$ .

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**Proof:**

$$\forall a \in \mathrm{Re}\,g^{+}, y \in S$$
$$\Rightarrow \exists x \in S[a + x + a = a]$$
$$\Rightarrow ay = (a + x + a)y = ay + xy + ay, xy \in S$$
$$\Rightarrow ay \in \mathrm{Re}\,g^{+}.$$

Similarly, $ya \in \mathrm{Re}\,g^{+}$.

$(\mathrm{Re}\,g^{+},+), (\mathrm{Re}\,g^{\bullet},+)$ is not always an ideal of $(S,+)$; $(\mathrm{Re}\,g^{\bullet},\bullet)$ is not always an ideal of $(S,\bullet)$.

A semigroup S is called a quasi-regular semigroup ($\Pi$-regular semigroup), if for any $a \in S$, there exist $n \in N$ and $x \in S$ such that $a^n x a^n = a^n$.

A semigroup S is called a quasigroup[2] if S is quasi-regular semigroup and S contains a unique idempotent.

A semiring $(S,+,\bullet)$ is called a quasi-ring[7],if $(S,+)$ is a quasi-Able-group.

**Lemma 1.2.** [2] Let S be a quasigroup.

(1) Reg(S) is a group;

(2) Reg(S) is an ideal of S.

**Proof:** (1) Obvious.

(2) Let e be the unique idempotent of S. For any $a \in \mathrm{Re}\,g(S)$ and $c \in S$, there exist x, $y \in S$ and $n \in N$ such that $ax = e$ and $c^n y = e$ so that $(ac)(c^{n-1}yx)(ac) = e(ac) = (ea)c = ac$. Hence $ac \in \mathrm{Re}\,g(S)$. Similarly, $ca \in \mathrm{Re}\,g(S)$. Thus, $\mathrm{Re}\,g(S)$ is an ideal of $S$.

**Theorem 1.3.** Let $(S,+,\bullet)$ be a distributive semiring. Then $(\mathrm{Re}\,g^{\bullet},+)$ is an ideal of $(S,+)$.

**Proof:**

$$\forall a \in \mathrm{Re}\,g^{\bullet}, \forall y \in S$$
$$\Rightarrow \exists x \in S(axa = a)$$
$$\Rightarrow a + y = axa + y = (a + y)(x + y)(a + y)$$
$$\Rightarrow a + y \in \mathrm{Re}\,g^{\bullet}$$

Similarly, $y + a \in \mathrm{Re}\,g^{+}$.

## 2. Biquasirings and strong biquasiring

**Definition 2.1.** The quasiring $(S,+,\bullet)$ is called a biquasiring, if $(S \setminus \{0\},\bullet)$ is a quasigroup, where 0 is the unique idempotent of $(S,+)$.

It is easy to see $(Z,+,\bullet)$ is a ring for integer's additive and multiplication operation, therefore a quasiring. However, it is not a biquasiring, the class of quasirings also properly contains the class of biquasirings.

The following example is displayed the class of biquasirings.

Example. Let $S = \{0,1,2,3\}$ satisfying

| + | 0 | 1 | 2 | 3 | $\bullet$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 0 | 0 | 2 | 1 | 1 | 1 | 1 |
| 3 | 0 | 0 | 0 | 0 | 3 | 1 | 1 | 1 | 1 |

It is clear to verify that (S,+) and (S\0,•) are null semigroups, and obviously quasi-groups, so (S,+,•) is biquasiring, not a ring.

**Theorem 2.2.** Let $(S,+,\bullet)$ be a biquasiring. Then

(1) $(\mathrm{Re}g^{\bullet},+)$ is also a subsemigroup of;

(2) $\forall a\in S\left[\exists n,m\in N\left(na^{m}\in \mathrm{Re}g^{\bullet}\cap \mathrm{Re}g^{+}\right)\right]$;

(3) $(\mathrm{Re}g^{\bullet}\cap \mathrm{Re}g^{+},\bullet)$ is an ideal of $(S,\bullet)$.

**Proof:** (1) A. Assume 0 is the unique additive idempotent, that is, 0+0=0, so $0\cdot 0\cdot 0=0\cdot(0+0)\cdot 0=0\cdot 0\cdot 0+0\cdot 0\cdot 0$, therefore $0\cdot 0\cdot 0$ is an idempotent, hence $0\cdot 0\cdot 0=0$, that is $0\in \mathrm{Re}g^{\bullet}$.

B. According to that $S\setminus\{0\}$ is a quasigroup, for any $a,b\in \mathrm{Re}g^{\bullet}$, there exist $a^{-1},b^{-1}\in \mathrm{Re}g^{\bullet}$, such that $aa^{-1}=a^{-1}a=bb^{-1}=b^{-1}b=e, e\in E^{\bullet}\setminus\{0\}$, since $(b^{-1}+a^{-1})b\in S$, $(\mathrm{Re}g^{\bullet},\bullet)$ is an ideal of $(S,\bullet)$:

a. If $b^{-1}+a^{-1}\neq 0$, then $a(b^{-1}+a^{-1})b=ae+eb=a+b\in \mathrm{Re}g^{\bullet}$.

b. If $b^{-1}+a^{-1}=0$, then $b(b^{-1}+a^{-1})a=b\cdot 0a=0$

$\Rightarrow ea+be=0\Rightarrow a+b=0\in \mathrm{Re}g^{\bullet}$

Hence, $(\mathrm{Re}g^{\bullet},+)$ is a subsemigroup of $(S,+)$.

(2) For any $a\in S$, there are $m\in N$ such that $a^{m}\in \mathrm{Re}g^{\bullet}$, according to Theorem 1.3, $(\mathrm{Re}g^{\bullet},+)$ is a subsemigroup , so there are $n\in N$, $na^{m}\in \mathrm{Re}g^{\bullet}$, therefore $na^{m}\in \mathrm{Re}g^{\bullet}\cap \mathrm{Re}g^{+}$.

(3) According to Theorem 1.1, $(\mathrm{Re}g^{+},\bullet)$ is an ideal of $(S,\bullet)$, so it is a subsemigroup . For any $a\in \mathrm{Re}g^{\bullet}\cap \mathrm{Re}g^{+}$, that is, $a\in \mathrm{Re}g^{+}$, there are $x\in S$, such that $a+x+a=a$, so for any $b\in S$, there are $ab=(a+x+a)b=ab+xb+ab$, so $ab\in \mathrm{Re}g^{+}$. According to Lemma 1.2, $(\mathrm{Re}g^{\bullet},\bullet)$ is an ideal of $(S,\bullet)$, so $ab\in \mathrm{Re}g^{\bullet}$. Therefore $ab\in \mathrm{Re}g^{\bullet}\cap \mathrm{Re}g^{+}$, that is, $(\mathrm{Re}g^{\bullet}\cap \mathrm{Re}g^{+},\bullet)$ is an ideal of $(S,\bullet)$.

**Theorem2.3.** Let $(S,+,\bullet)$ be a distributive biquaisring. Then

(1) $\mathrm{Re}g^{\bullet}\cap \mathrm{Re}g^{+}=\{0\}$;

(2) If $\mathrm{Re}g^{\bullet}\neq\{0\}$, then $(\mathrm{Re}g^{+},\bullet)$ is a nilsemigroup;

(3) $(\mathrm{Re}g^{\bullet},+)$ is a nilsemigroup.

**Proof:** Since $\mathrm{Re}g^{\bullet}\setminus\{0\}$ is a quasigroup and $0\in \mathrm{Re}g^{\bullet}$, if $\mathrm{Re}g^{\bullet}\neq\{0\}$, then there are $e\in E^{\bullet}\setminus\{0\}$ and $0\in \mathrm{Re}g^{+}$, so $e+0\in \mathrm{Re}g^{+}$.

(1)

$$\begin{aligned}
&\forall x\in(\ \mathrm{Re}g^{\bullet}\cap \mathrm{Re}g^{+}\ )\setminus\{0\}\\
&\Rightarrow \exists x'\in S(e=xx'\in E^{\cdot}\setminus\{0\})\\
&\Rightarrow e+0x=(e+0)(e+x)=e+ex+0\\
&\Rightarrow ex+e+0=e+0\\
&\Rightarrow ex+0=0
\end{aligned}$$

(Since $e+0\in \mathrm{Re}g^{+}$, there are inverse element $-(e+0)\in \mathrm{Re}g^{+}$)

According to Theorem 1.1 and $x\in \mathrm{Re}g^{+}$, there are $ex\in \mathrm{Re}g^{+}$, so $ex+0=ex=0$, according to $x\in \mathrm{Re}g^{\bullet}$, hence $ex=x=0$.

(2) According to (1), for any $x\in \mathrm{Re}g^{+}, e\in E^{\bullet}\setminus\{0\}$, there are $e+0x=(e+0)(e+x)=ex+e+0$, so

$ex = 0$ .Since any $x \in S$ ,there are n, such that $x^n \in \mathrm{Re}\,g^{\bullet}$ , so $e = x^n x'$ and $x^n x' x^n = x^n$ ,therefore $x^n x' x = 0$, hence $x^n x' x^n = 0$ ,that is, $x^n = 0$ ,so x is a multiplicative nilpotent element. Hence $(\mathrm{Re}\,g^{+}, \bullet)$ is a nilsemigroup.

(3) For any $x \in \mathrm{Re}\,g^{\bullet}$ there are

$$ex = x \Rightarrow e + 0x = (e+0)(e+x) = ex + e + 0\,.$$

that is, $ex + 0 = 0$ ,hence $x + 0 = 0$ . For any $x \in S$ there are n such that $nx \in \mathrm{Re}\,g^{+}$ , so $nx + n \cdot 0 = n \cdot x = 0$ ,hence $nx = 0$ ,that is, $x$ is a additive nilpotent element. Hence $(\mathrm{Re}\,g^{\bullet}, +)$ is a nilsemigroup.

**Definition 2.4.** A semiring $(S, +, \bullet)$ is called a strong biquasiring, if

(1) $(S, +)$ is an abelian group;

(2) $(S \setminus \{0\}, \cdot)$ is a quasigroup.

**Remark 2.5.** (1) The strong biquasiring is equal to the quasiring of $[3,4,5]$ .

(2) A strong biquasiring is a ring and the following relations are holds:

$$\{division \quad ring\} \begin{array}{l} \text{Ø} \{\mathrm{int}\,egral \quad domain\}(\notin \{strong \quad biquasiring\}) \\ \text{Ø} \{strong \quad biquasiring\}(\notin \{\mathrm{int}\,egral \quad domain\}) \end{array}$$

$$\left.\left\{\begin{array}{l} \text{Ø} \{ring\} \\ \text{Ø} \{biquasiring\} \\ \text{Ø} \{monogenic \quad semiring\} \end{array}\right.\right\} \text{Ø} \{quasiring\} \text{ Ø} \{semiring\}$$

(3) If the (2) of Definition 2.4 is changed to that $(S, \bullet)$ is a quasigroup , then $(S, \bullet)$ is a nilsemigroup and $E^{\bullet} = \mathrm{Re}\,g^{\bullet} = \{0\}$ ,so $(S, +, \bullet)$ is a nilring.

(4) According to $(S \setminus \{0\}, \bullet)$ is a quasigroup, if $\mathrm{Re}\,g^{\bullet} = E^{\bullet}$, then $(S \setminus \{0\}, \bullet)$ is also a nilsemigroup, but the idempotent of $S \setminus \{0\}$ is e and $e \neq 0$ .

(5) The subring $S_1$ of a strong biquasiring S is not always strong biquasiring. For example:

Let $(Q, +, \bullet)$ be a division ring, $(Z, +, \bullet)$ be its subring, where Q,Z is the rational numbers set, integers set respectively , "+" and "·" is the usual addition and multiplication respectively. But $(Z, +, \bullet)$ is not a biquasiring.

**Theorem 2.6.** Let $(S, +, \bullet)$ be a strong biquasiring. Then $(\mathrm{Re}\,g^{\bullet}, +, \bullet)$ is the greatest division subring and $S / \mathrm{Re}\,g^{\bullet}$ is a nilring.

**Proof:** Since $(S \setminus \{0\}, \bullet)$ is a quasigroup, $(\mathrm{Re}\,g^{\bullet} \setminus \{0\}, \bullet)$ is the greatest subgroup and an ideal of $(S \setminus \{0\}, \bullet)$ by Lemma 1.2.

In the following, we only prove that $(\mathrm{Re}g^{\bullet}, +)$ is a subgroup of $(S, +)$ and $S / \mathrm{Re}\,g^{\bullet}$ is a nilring.

(1) Since $0 \cdot 0 \cdot 0 = 0, 0 \in \mathrm{Re}\,g^{\bullet}$

(2)

$$\begin{array}{l} \forall a \in \mathrm{Re}\,g^{\bullet} \setminus \{0\} \\ \Rightarrow \exists a^{-1} \in \mathrm{Re}\,g^{\bullet} \setminus \{0\} (aa^{-1} = a^{-1}a = e, aa^{-1}a = a) \\ \Rightarrow -a(-a^{-1}) = a \cdot a^{-1} = e \quad and \quad -a(-a^{-1})(-a) = -a, \\ (\sin ce \quad there \quad are \quad -a, -a^{-1} \in S) \\ \Rightarrow -a \in \mathrm{Re}\,g^{\bullet} \setminus \{0\} \end{array}$$

(3) $\forall a,b\in \mathrm{Re}\,g^{\bullet}$ ,if $a=0$ or $b=0$ , then $a+b\in \mathrm{Re}\,g^{\bullet}$ , if $a\neq 0,b\neq 0$ , then there are $a^{-1},b^{-1}\in \mathrm{Re}\,g^{\bullet}\setminus\{0\}$, such that $aa^{-1}=a^{-1}a=bb^{-1}=b^{-1}b=e$, since $(b^{-1}+a^{-1})b\in S$ , according to lemma 1.2 $\mathrm{Re}\,g^{\bullet}\setminus\{0\}$ is an ideal of $S\setminus\{0\}$ if $b^{-1}+a^{-1}\neq 0$ , then $a(b^{-1}+a^{-1})b=ae+eb=a+b\in \mathrm{Re}\,g^{\bullet}$ . If $b^{-1}+a^{-1}=0$ , then $a^{-1}=-b^{-1}$ , so $b=bb^{-1}b=aa^{-1}b=-ab^{-1}b=-a$, therefore $a+b=0\in \mathrm{Re}\,g^{\bullet}$.

(4) The distributive laws is hold clearly.

It is clear that $(\mathrm{Re}\,g^{\bullet},+,\bullet)$ is a division ring, since $(\mathrm{Re}\,g^{\bullet},\bullet)$ is the set of greatest regular elements, $(\mathrm{Re}\,g^{\bullet},+,\bullet)$ is the greatest division subring.

(5) According to Lemma 1.2, $(\mathrm{Re}\,g^{\bullet}\setminus\{0\},\bullet)$ is an ideal of $(S\setminus\{0\},\cdot)$, so $(\mathrm{Re}\,g^{\bullet},\bullet)$ is an ideal of $(S,\bullet)$, that is, $(\mathrm{Re}\,g^{\bullet},+,\bullet)$ is an ideal subring, so $S/\mathrm{Re}\,g^{\bullet}$ is a corgruence ring.

now,

$$\begin{aligned}&\forall \overline{x}\in S/\mathrm{Re}\,g^{\bullet}\Rightarrow x\in S\\&\Rightarrow \exists n\in N(x^{n}\in \mathrm{Re}\,g^{\bullet})\\&\Rightarrow \overline{x}^{n}=\overline{x^{n}}=\overline{0}\end{aligned}$$

that is, $S/\mathrm{Re}\,g^{\bullet}$ is a nilring.

**Corollary 2.7.**[3] A strong biquasiring $(S,+,\cdot)$ is the nil-extension of the greatest division subring $(\mathrm{Re}\,g^{\bullet},+,\bullet)$.

**Theorem 2.8.** Let $(S,+,\cdot)$ be a distributive strong biquasiring. Then $S=\{0\}$.

**Proof:** Since $(S,+,\bullet)$ is a strong biquasiring, $\mathrm{Re}\,g^{+}=S$, $\mathrm{Re}\,g^{\bullet}\cap \mathrm{Re}\,g^{+}=\mathrm{Re}\,g^{\bullet}$ .According to Corollary 2.7 and S is a distributive biquasiring, $\mathrm{Re}\,g^{\bullet}\cap \mathrm{Re}\,g^{+}=\{0\}$ .So $\mathrm{Re}\,g^{\bullet}=\{0\}$, if $S\neq\{0\}$, since $(S\setminus\{0\},\bullet)$ is a quasigroup , so for any $x\in S\setminus\{0\}$, there are $n\in N$ ,such that $x^{n}\in \mathrm{Re}\,g^{\bullet}\setminus\{0\}$ . But $\mathrm{Re}\,g^{\bullet}=\{0\}$. It is a contradiction,so $S=\{0\}$.

**Remark 2.9.** (1). If I is a K-ideal, then I must be an ideal of $(S,\bullet)$, not be an ideal of $(S,+)$.

(2) Let S be a quasiring. If $S\neq \mathrm{Re}\,g^{+}$, then $(\mathrm{Re}\,g^{+},\bullet)$ not be a K-ideal.

According to Theorem 1.1, $(\mathrm{Re}\,g^{+},\bullet)$ is an ideal of $(S,\bullet)$, since $0\in \mathrm{Re}\,g^{+}$, $(\mathrm{Re}\,g^{+},+)$ is an ideal, for any $b\in S\setminus \mathrm{Re}\,g^{+}$ ,there are $0+b\in \mathrm{Re}\,g^{+}$. It is contradiction.

(3) Let S be a distributive semiring. Then $(\mathrm{Re}\,g^{\bullet},+)$ is an ideal of $(S,+)$, not a K-ideal.

According to Theorem 1.3, $(\mathrm{Re}\,g^{\bullet},+)$ is an ideal of $(S,+)$ . For any $b\in S\setminus \mathrm{Re}\,g^{\bullet}$ , Since $0\in \mathrm{Re}\,g^{\bullet}$,there are $0\cdot b=0\in \mathrm{Re}\,g^{\bullet}$,so it is contradiction.

**Theorem 2.10.** Let $(S,+,\bullet)$ be a strong biquasiring. Then $(\mathrm{Re}\,g^{\bullet},\bullet)$ is a K-ideal.

**Proof:** $\forall a\in \mathrm{Re}\,g^{\bullet}$ , there are $x\in S$ , such that $axa=a$ ,since $(S,+)$ is an abelian group, so $-a\in S,-x\in S$ ,therefore, $-a(-x)(-a)=-a$ ,that is $-a\in \mathrm{Re}\,g^{\bullet}$.

If $a+b\in \mathrm{Re}\,g^{\bullet}$, since $\mathrm{Re}\,g^{\bullet}$ is the greatest division subring, so it is closed about "+", therefore

$$b+0=b+a-a=a+b-a\in \mathrm{Re}\,g^{\bullet},$$

that is, there are $y\in S$ such that $(b+0)y(b+0)=b+0$ ,hence $byb+0=b+0$ ,since $(S,+)$ is an abelian group , so $byb+0=byb, b+0=b$, hence $byb=b$ ,that is $b\in \mathrm{Re}\,g^{\bullet}$,so $\mathrm{Re}\,g^{\bullet}$ is a K-ideal.

## 3. Monogenic semiring

A semigroup is called a monogenic semigroup[6], if S=<a>, we refer to <a> a semigroup generated by the element a.

**Definition 3.1.** A semiring $(S,+,\bullet)$ is called a monogenic semiring, if

(1) $(S,+)$ is a monogenic semigroup;

(2) $(S,\cdot)$ is a semigroup.

It is easy to see a finite monogenic semigroup is a kind of special quasiring.

Let $S=\{a,2a,...,na\}$ is a finite monogenic semiring. Then S is a special quasiring. The number of monogenic semiring is discussed as following:

A. If $|S|=1$, that is, $S=\{a\}$. The number of monogenic semiring is one by Definition 3.1.

B. If $|S|=2$, that is, $S=\{a,2a\}$.It is easy to see that $(S,+)$ have two operation structures[3]such as ($B_{21}$ ), ($B_{22}$).

| + | a | 2a |
|---|---|---|
| a | 2a | a |
| 2a | a | 2a |

(B1)

| + | a | 2a |
|---|---|---|
| a | 2a | 2a |
| 2a | 2a | 2a |

(B2)

| • | a | 2a |
|---|---|---|
| a | a | 2a |
| 2a | 2a | 2a |

(B11)

| • | a | 2a |
|---|---|---|
| a | 2a | 2a |
| 2a | 2a | 2a |

(B12)

| • | a | 2a |
|---|---|---|
| a | a | 2a |
| 2a | 2a | 2a |

(B21)

| • | a | 2a |
|---|---|---|
| a | 2a | 2a |
| 2a | 2a | 2a |

(B22)

By using $(S,\bullet)$ is a semigroup and $(S,+,\bullet)$ satisfies the distributive law, we have the following result especially.

a. If $(S,+)$ have (B1) type structure, we can obtain the structure:

$a_1$. If $a\cdot a=a$, then

$$a\cdot 2a=a\cdot(a+a)=a\cdot a+a\cdot a=a+a=2a$$

$$2a\cdot a=(a+a)\cdot a=a\cdot a+a\cdot a=a+a=2a\ ;$$

$$2a\cdot 2a=(a+a)\cdot 2a=a\cdot 2a+a\cdot 2a=2a+2a=2a\ ;$$

$a_2$. If $a\cdot a=2a$, then

$$a\cdot 2a=a\cdot(a+a)=a\cdot a+a\cdot a=2a+2a=2a\ ;$$

$$2a\cdot a=(a+a)\cdot a=a\cdot a+a\cdot a=2a+2a=2a\ ;$$

$$2a\cdot 2a=(a+a)\cdot 2a=a\cdot 2a+a\cdot 2a=2a+2a=2a\ .$$

That is (B11), (B12).

b. If $(S,+)$ have ($B_2$) type structure, we can obtain the structures similarly:(B21,B22).

Therefore, when $|S|=2$, the numbers of the monogenic semigroup are four.

c. f ｜S｜=3, that is , S={a,2a,3a },then we can obtain that S have three opertion structures by [3]:

| + | a | 2a | 3a |
|---|---|---|---|
| a | 2a | 3a | a |
| 2a | 3a | a | 2a |
| 3a | a | 2a | 3a |

(C1)

| + | a | 2a | 3a |
|---|---|---|---|
| a | 2a | 3a | a |
| 2a | 3a | 2a | 3a |
| 3a | 2a | 3a | 2a |

(C2)

| + | a | 2a | 3a |
|---|---|---|---|
| a | 2a | 3a | 3a |
| 2a | 3a | 3a | 3a |
| 3a | 3a | 3a | 3a |

(C3)

| ● | a | 2a | 3a |
|---|---|---|---|
| a | 2a | a | 3a |
| 2a | a | 2a | 3a |
| 3a | 3a | 3a | 3a |

(C11)

| ● | a | 2a | 3a |
|---|---|---|---|
| a | a | 2a | 3a |
| 2a | 2a | a | 3a |
| 3a | 3a | 3a | 3a |

(C12)

| ● | a | 2a | 3a |
|---|---|---|---|
| a | 3a | 3a | 3a |
| 2a | 3a | 3a | 3a |
| 3a | 3a | 3a | 3a |

(C13)

a. If $(S,+)$ have (C1) type structure, we can obtain the structure:

$a_1$. If $a \cdot a = a$, then $a \cdot 2a = a \cdot (a+a) = a + a = 2a$;

$$a \cdot 3a = a \cdot (a+2a) = a + a \cdot 2a = a + 2a = 3a;$$

$$2a \cdot 2a = (a+a) \cdot 2a = a \cdot 2a + a \cdot 2a = 2a + 2a = a;$$

$$2a \cdot 3a = (a+a) \cdot 3a = a \cdot 3a + a \cdot 3a = 3a + 3a = 3a;$$

$$3a \cdot 3a = (a+2a) \cdot 3a = a \cdot 3a + 2a \cdot 3a = 3a + 3a = 3a;$$

that is (C11), similarly, we can obtain (C12),(C13).

b. If $(S,+)$ have (C2) type structure, we can obtain the structures similarly (C21), (C22), (C23).

| ● | a | 2a | 3a |
|---|---|---|---|
| a | a | 2a | 3a |
| 2a | 2a | 2a | 2a |
| 3a | 3a | 2a | 2a |

(C21)

| ● | a | 2a | 3a |
|---|---|---|---|
| a | 2a | 2a | 2a |
| 2a | 2a | 2a | 2a |
| 3a | 2a | 2a | 2a |

(C22)

| ● | a | 2a | 3a |
|---|---|---|---|
| a | 3a | 2a | 3a |
| 2a | 2a | 2a | 2a |
| 3a | 3a | 2a | 3a |

(C23)

| ● | A | 2a | 3a |
|---|---|---|---|
| a | A | 2a | 3a |
| 2a | 2a | 3a | 3a |
| 3a | 3a | 3a | 3a |

(C31)

| ● | a | 2a | 3a |
|---|---|---|---|
| a | 2a | 3a | 3a |
| 2a | 3a | 3a | 3a |
| 3a | 3a | 3a | 3a |

(C32)

| ● | a | 2a | 3a |
|---|---|---|---|
| a | 3a | 3a | 3a |
| 2a | 3a | 3a | 3a |
| 3a | 3a | 3a | 3a |

(C33)

c. If $(S,+)$ have (C3) type structure, we can obtain the structures similarly (C31) (C32) (C33).

Therefore, when $|S| = 3$, the numbers of the monogenic semigroup are nine.

Problem: If｜S｜=n, are the numbers of the monogenic semigroup $n^2$?

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