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Fuzzy Random Homogeneous Poisson Process and Compound Poisson Process

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Abstract. By dealing with interarrival times as exponentially distributed fuzzy random variables, a fuzzy random homogeneous Poisson process and a fuzzy random compound Poisson process are respectively defined. Several theorems on the two processes are provided, respectively.

Keywords Fuzzy Variables; Fuzzy Random Variables; Poisson Process; Fuzzy Random Homogeneous Poisson Process; Fuzzy Random Compound Poisson Process

1 Introduction

The imprecise data are very common in renewal processes, and fuzzy sets theory developed by Zadeh [19] is able to effectively deal with the imprecise information. Recently, fuzzy sets theory was used to renewal theory by a few authors. Based on the expected value operator of a fuzzy variable defined by Liu and Liu [11], Zhao and Liu [21] discussed a renewal process with fuzzy interarrival times and rewards, and established the fuzzy styles of elementary renewal theorem and renewal reward theorem. Li et al [6] defined a delayed renewal process with fuzzy interarrival times, and further discussed some properties on the average of the fuzzy renewal variable.

In practice, a hybrid uncertain process with randomness and fuzziness exists generally. Thus, randomness and fuzziness should be considered simultaneously. Fuzzy random theory is one of good tools to deal with such an uncertainty, where fuzzy random variables were introduced by Kwakernaak [4], Puri and Ralescu [17] to model a process which quantified "fuzzily" the outcomes of a random experiment. Hwang [3] investigated a renewal process in which the interarrival times were expressed as independent and identically distributed (iid) fuzzy random variables, and provided a theorem for the fuzzy rate of the fuzzy random renewal process. Dozzi et al [2] defined a fuzzy-set-indexed renewal counting process, and gave the elementary renewal theorem. Popova and Wu [16] considered a renewal reward process with random interarrival times and fuzzy rewards, and presented a theorem on the asymptotic average fuzzy reward per unit time.

Homogeneous Poisson process is a special one of Poisson processes, where interarrival times and the rate of the process are two important quantities. Conventionally, interarrival times are expressed as iid exponentially distributed random variables and the rate of the process is assumed to be a constant. Mixed Poisson process, a special case of doubly stochastic Poisson process introduced by Cox [1] is an extension of homogeneous Poisson process, where the rate related to the process is assumed as a random variable. Some conclusions on mixed Poisson process can be found in Mcfadden [14]. However, an interval estimate or a point estimate of the rate is provided by the experiment data often with quite small sample sizes. Sometimes, it is more realistic and appropriate to characterize the rate as a fuzzy variable rather than a crisp number or a random variable. In this paper, by adopting the definition of fuzzy random variables defined by Liu and Liu [9] for a homogeneous

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Poisson process, the concept of a fuzzy random homogeneous Poisson process is defined, where interarrival times are assumed as iid exponentially distributed fuzzy random variables and the rate of the process is depicted as a fuzzy variable. As an extension of the compound Poisson process, fuzzy random compound Poisson process is further defined.

2 Fuzzy Variables

Let Θ be a universe, $P(\Theta)$ the power set of Θ, Pos a possibility measure (see Zadeh [20]), and $(\Theta, P(\Theta), \mathrm{Pos})$ a possibility space.

Fuzzy variable is an important concept in fuzzy sets theory. Kaufmann [5] first used it as a generalization of the concept of a Boolean variable, and Nahmias [15] defined it as a function from a pattern space to the set of real numbers.

Definition 2.1 *[8] A fuzzy variable ξ is*

(1) continuous if $\mathrm{Pos}\{\xi = r\}$ *is a continuous function of* r;

(2) positive if $\mathrm{Pos}\{\xi \le 0\} = 0$;

(3) nonnegative if $\mathrm{Pos}\{\xi < 0\} = 0$;

(4) discrete if there exists a countable sequence $\{x_1, x_2, \cdots\}$ *such that* $\mathrm{Pos}\{\xi \ne x_1, \xi \ne x_2, \cdots\} = 0$.

Definition 2.2 *[12] Let ξ be a fuzzy variable on the possibility space* $(\Theta, P(\Theta), \mathrm{Pos})$*, and* $\alpha \in (0, 1]$. *Then*

$$\xi_\alpha^L = \inf\left\{r \;\middle|\; \mathrm{Pos}\{\xi \le r\} \ge \alpha\right\} \quad \text{and} \quad \xi_\alpha^U = \sup\left\{r \;\middle|\; \mathrm{Pos}\{\xi \ge r\} \ge \alpha\right\}$$

are called the α-pessimistic value and the α-optimistic value of ξ, respectively.

Definition 2.3 *[8, 15] The fuzzy variables $\xi_1, \xi_2, \cdots$ are independent if and only if*

$$\mathrm{Pos}\{\xi_i \in B_i,\ i = 1, 2, \cdots\} = \min_{i \ge 1} \mathrm{Pos}\{\xi_i \in B_i\}$$

for any sets $B_1, B_2, \cdots$ of $\Re$.

Definition 2.4 *[8] The fuzzy variables ξ and η are identically distributed if and only if*

$$\mathrm{Pos}\{\xi \in B\} = \mathrm{Pos}\{\eta \in B\}$$

for any set B of $\Re$.

Proposition 2.1 *Let ξ and η be two independent fuzzy variables. Then for any $\alpha \in (0, 1]$,*

$$(\xi + \eta)_\alpha^L = \xi_\alpha^L + \eta_\alpha^L, \qquad (\xi + \eta)_\alpha^U = \xi_\alpha^U + \eta_\alpha^U.$$

Proposition 2.2 *If ξ and η are two independently nonnegative fuzzy variables, for any $\alpha \in (0, 1]$,*

$$(\xi \cdot \eta)_\alpha^L = \xi_\alpha^L \cdot \eta_\alpha^L, \qquad (\xi \cdot \eta)_\alpha^U = \xi_\alpha^U \cdot \eta_\alpha^U.$$

Proposition 2.3 *If ξ is a nonnegative fuzzy variable, for any $\alpha \in (0, 1]$ and positive integer k,*

$$\left(\xi^k\right)_\alpha^L = \left(\xi_\alpha^L\right)^k, \quad \left(\xi^k\right)_\alpha^U = \left(\xi_\alpha^U\right)^k.$$

In order to define the expected value of a fuzzy variable ξ, Liu and Liu [11] introduced a set function Cr of a fuzzy event $\{\xi \geq r\}$ as

$$\mathrm{Cr}\{\xi \geq r\} = \frac{1}{2}\left(\mathrm{Pos}\{\xi \geq r\} + 1 - \mathrm{Pos}\{\xi < r\}\right).$$

Based on the credibility measure, the expected value of a fuzzy variable ξ is defined as follows,

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r. \tag{1}$$

When the right side of (1) is of form $\infty - \infty$, the expected value operator is not defined. Especially, if ξ is a nonnegative fuzzy variable, its expected value is defined as $E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r$.

Proposition 2.4 *[12] Let ξ be a fuzzy variable with finite expected value $E[\xi]$, then we have*

$$E[\xi] = \frac{1}{2}\int_0^1 \left(\xi_\alpha^L + \xi_\alpha^U\right)\mathrm{d}\alpha.$$

3 Fuzzy Random Variables

Let $(\Omega,\ A,\ \mathrm{Pr})$ be a probability space and F a collection of fuzzy variables defined on the possibility space $(\Theta, P(\Theta), \mathrm{Pos})$.

Definition 3.1 *[9] A fuzzy random variable ξ is a function $\xi : \Omega \to F$ such that for any Borel set B of $\Re$, $\mathrm{Pos}\{\xi(\omega) \in B\}$ is a measurable function of ω.*

Definition 3.2 (Exponentially Distributed Fuzzy Random Variable) *For each $\omega \in \Omega$, let $\xi_\alpha^L(\omega)$ and $\xi_\alpha^U(\omega)$ be the α-pessimistic value and the α-optimistic value of $\xi(\omega)$, respectively. A fuzzy random variable ξ defined on the probability space $(\Omega,\ A,\ \mathrm{Pr})$ is said to be exponential if $\xi_\alpha^L(\omega)$ and $\xi_\alpha^U(\omega)$ are exponentially distributed random variables whose density functions are defined as*

$$f_{\xi_\alpha^L(\omega)}(x) = \begin{cases} \lambda_\alpha^U \cdot e^{-\lambda_\alpha^U \cdot x}, & x \geq 0 \\ 0, & x < 0, \end{cases} \quad \textit{and} \quad f_{\xi_\alpha^U(\omega)}(x) = \begin{cases} \lambda_\alpha^L \cdot e^{-\lambda_\alpha^L \cdot x}, & x \geq 0 \\ 0, & x < 0, \end{cases} \tag{2}$$

where λ is a fuzzy variable defined on the possibility space $(\Theta, P(\Theta), \mathrm{Pos})$ with

$$\mathrm{Pos}\left\{\lambda = \lambda_\alpha^L\right\} = \min\left\{\mathrm{Pos}\left\{\xi(\omega) = \xi_\alpha^U(\omega)\right\}, \omega \in \Omega\right\},$$

$$\mathrm{Pos}\left\{\lambda = \lambda_\alpha^U\right\} = \min\left\{\mathrm{Pos}\left\{\xi(\omega) = \xi_\alpha^L(\omega)\right\}, \omega \in \Omega\right\}.$$

An exponentially distributed fuzzy random variable is denoted by $\xi \sim EXP(\lambda)$, and the fuzziness of fuzzy random variable ξ is said to be characterized by the fuzzy variable λ.

Definition 3.3 *[8] A fuzzy random variable ξ is nonnegative if and only if $\mathrm{Pos}\{\xi(\omega) < 0\} = 0$ for any $\omega \in \Omega$.*

Definition 3.4 *[8] (Fuzzy Random Arithmetic On Different Spaces) Let $f : \Re^n \to \Re$ be a Borel measurable function, and $\xi_1, \xi_2, \cdots, \xi_n$ fuzzy random variables on the probability spaces $(\Omega_i,\ A_i,\ \mathrm{Pr_i})$ respectively. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is a fuzzy random variable defined on the product probability space $(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n, A_1 \times A_2 \times \cdots A_n, \mathrm{Pr}_1 \times \mathrm{Pr}_2 \times \cdots \times \mathrm{Pr}_n)$ as*

$$\xi(\omega_1, \omega_2, \cdots, \omega_n) = f(\xi_1(\omega_1), \xi_2(\omega_2), \cdots, \xi_n(\omega_n))$$

for any $(\omega_1, \omega_2, \cdots, \omega_n) \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$.

Definition 3.5 *[10] Let ξ be a fuzzy random variable defined on $(\Omega,\ A,\ \Pr)$, then the average chance, denoted by Ch, of a fuzzy random event $\{\xi \le r\}$ is defined as*

$$\mathrm{Ch}\{\xi \le r\} = \int_0^1 \Pr\left\{\omega \in \Omega \;\middle|\; \mathrm{Cr}\{\xi(\omega) \le r\} \ge \alpha\right\} \mathrm{d}\alpha. \tag{3}$$

Remark 3.1 *[22] The equation (3) is equivalent to the following form*

$$\mathrm{Ch}\{\xi \le r\} = \frac{1}{2}\int_0^1 \left(\Pr\left\{\omega \in \Omega \;\middle|\; \xi_\alpha^L(\omega) \le r\right\} + \Pr\left\{\omega \in \Omega \;\middle|\; \xi_\alpha^U(\omega) \le r\right\}\right) \mathrm{d}\alpha.$$

Proposition 3.1 *[10] The average chance Ch of a fuzzy random event $\{\xi \le r\}$ is self dual, i.e.,*

$$\mathrm{Ch}\{\xi \ge r\} = 1 - \mathrm{Ch}\{\xi < r\}.$$

Definition 3.6 *[9] Let ξ be a fuzzy random variable defined on $(\Omega, A, \Pr)$. Then its expected value is defined by*

$$E[\xi] = \int_\Omega \left(\int_0^{+\infty} \mathrm{Cr}\{\xi(\omega) \ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi(\omega) \le r\}\mathrm{d}r\right) \Pr(\mathrm{d}\omega) \tag{4}$$

provided that at least one of the two integrals is finite. Especially, if ξ is a nonnegative fuzzy random variable, then $E[\xi] = \int_\Omega \int_0^{+\infty} \mathrm{Cr}\{\xi(\omega) \ge r\}\mathrm{d}r \Pr(\mathrm{d}\omega)$.

Proposition 3.2 *Based on Fubini theorem and Proposition 2.4, the equation (4) can be written as*

$$E[\xi] = \frac{1}{2}\int_0^1 \left(E\left[\xi_\alpha^L(\omega)\right] + E\left[\xi_\alpha^U(\omega)\right]\right) \mathrm{d}\alpha. \tag{5}$$

Definition 3.7 *[11] The fuzzy random variables $\xi_1, \xi_2, \cdots, \xi_n$ are iid if and only if*

$$(\mathrm{Pos}\{\xi_i(\omega) \in B_1\}, \mathrm{Pos}\{\xi_i(\omega) \in B_2\}, \cdots, \mathrm{Pos}\{\xi_i(\omega) \in B_m\}), \qquad i = 1, 2, \cdots, n$$

are iid random vectors for any positive integer m and any Borel sets B_i of $\Re$.

4 Fuzzy Random Homogeneous Poisson Process with Fuzzy Rates

Let ξ_i, known as interarrival times, be iid nonnegative fuzzy random variables defined on $(\Omega_i, A_i, \mathrm{Pr_i})$ with $\xi_i \sim EXP(\lambda_i)$. Furthermore, let $N(t)$ denote the total number of events that have occurred by time t. That is,

$$N(t) = \max_{n \ge 0} \left\{n \;\middle|\; 0 \le \xi_1 + \xi_2 + \cdots + \xi_n \le t\right\}. \tag{6}$$

The counting process $\{N(t), t \ge 0\}$ is called a *fuzzy random homogeneous Poisson process with fuzzy rates λ_i* and $N(t)$ is called a *Poisson fuzzy random variable with fuzzy rates λ_i.*

Theorem 4.1 *If $\{N(t), t \ge 0\}$ is a fuzzy random homogeneous Poisson process with fuzzy rates λ_i, and $\xi_i, i = 1, 2, \cdots$ are iid nonnegative exponentially distributed fuzzy random interarrival times, then we have*

$$E[\xi_1] = E\left[\frac{1}{\lambda_1}\right].$$

Proof. Since $\xi_1 \sim EXP(\lambda_1)$, by Definition 3.2, we have

$$E\left[\xi_{1,\alpha}^L(\omega)\right] = \frac{1}{\lambda_{1,\alpha}^U}, \quad E\left[\xi_{1,\alpha}^U(\omega)\right] = \frac{1}{\lambda_{1,\alpha}^L}. \tag{7}$$

It follows from Proposition 3.2 that

$$\begin{aligned} E[\xi_1] &= \frac{1}{2}\int_0^1 \left(E\left[\xi_{1,\alpha}^L(\omega)\right] + E\left[\xi_{1,\alpha}^U(\omega)\right]\right) \mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \left(\frac{1}{\lambda_{1,\alpha}^U} + \frac{1}{\lambda_{1,\alpha}^L}\right) \mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \left(\left(\frac{1}{\lambda_1}\right)_\alpha^L + \left(\frac{1}{\lambda_1}\right)_\alpha^U\right) \mathrm{d}\alpha \quad \text{(by Proposition 2.4)} \\ &= E\left[\frac{1}{\lambda_1}\right]. \end{aligned}$$

The proof is finished.

Theorem 4.2 *If $\{N(t), t \geq 0\}$ is a fuzzy random homogeneous Poisson process with fuzzy rates λ_i, and $\xi_i, i = 1, 2, \cdots$ are iid nonnegative exponentially distributed fuzzy random interarrival times, then we have*

$$E[N(t)] = t \cdot E[\lambda_1].$$

Proof. For any ω, fuzzy variables $\xi_i(\omega)$ are defined on the possibility spaces $(\Theta_i, P(\Theta_i), \mathrm{Pos}_i)$. Assume that $\Theta_1 = \Theta_2 = \cdots$ For any given $\alpha \in (0, 1]$, there exist points $\theta_i', \theta_i'' \in \Theta_i$ with $\mathrm{Pos}\left\{\theta_i'\right\} \geq \alpha, \mathrm{Pos}\left\{\theta_i''\right\} \geq \alpha$ such that

$$\xi_{i,\alpha}^L(\omega) = \xi_i(\omega)\left(\theta_i'\right), \quad \xi_{i,\alpha}^U(\omega) = \xi_i(\omega)\left(\theta_i''\right).$$

Taking $\theta_1' = \theta_2' = \cdots, \theta_1'' = \theta_2'' = \cdots$, we obtain

$$N(t)(\omega)\left(\theta''\right) = \max_{n\geq 0}\left\{n \mid 0 \leq \xi_{1,\alpha}^U(\omega) + \xi_{2,\alpha}^U(\omega) + \cdots + \xi_{n,\alpha}^U(\omega) \leq t\right\}, \tag{8}$$

$$N(t)(\omega)\left(\theta'\right) = \max_{n\geq 0}\left\{n \mid 0 \leq \xi_{1,\alpha}^L(\omega) + \xi_{2,\alpha}^L(\omega) + \cdots + \xi_{n,\alpha}^L(\omega) \leq t\right\}, \tag{9}$$

where $\theta' = \left(\theta_1', \theta_2', \cdots\right), \theta'' = \left(\theta_1'', \theta_2'', \cdots\right)$.

For any $\theta_i \in \Theta_i$ with $\mathrm{Pos}\{\theta_i\} \geq \alpha$, by Definition 2.2, we have

$$\xi_{i,\alpha}^L(\omega) \leq \xi_i(\omega)(\theta_i) \leq \xi_{i,\alpha}^U(\omega). \tag{10}$$

Consequently,

$$N(t)(\omega)\left(\theta''\right) \leq N(t)(\omega)(\theta) \leq N(t)(\omega)\left(\theta'\right), \tag{11}$$

where $\theta = (\theta_1, \theta_2, \cdots)$. Let $N(t)_\alpha^L(\omega)$ and $N(t)_\alpha^U(\omega)$ be the α-pessimistic value and the α-optimistic value of $N(t)(\omega)$ for any ω. Again by Definition 2.2, we have

$$N(t)_\alpha^L(\omega) = N(t)(\omega)\left(\theta''\right), \quad N(t)_\alpha^U(\omega) = N(t)(\omega)\left(\theta'\right).$$

Thus, equations (8) and (9) can be written as

$$N(t)_\alpha^L(\omega) = \max_{n\ge 0}\left\{n \mid 0 \le \xi_{1,\alpha}^U(\omega) + \xi_{2,\alpha}^U(\omega) + \cdots + \xi_{n,\alpha}^U(\omega) \le t\right\}, \tag{12}$$

$$N(t)_\alpha^U(\omega) = \max_{n\ge 0}\left\{n \mid 0 \le \xi_{1,\alpha}^L(\omega) + \xi_{2,\alpha}^L(\omega) + \cdots + \xi_{n,\alpha}^L(\omega) \le t\right\}. \tag{13}$$

Since $\xi_{i,\alpha}^L(\omega)$ and $\xi_{i,\alpha}^U(\omega)$ are iid random variables, respectively, then $\left\{N(t)_\alpha^L(\omega), t \ge 0\right\}$ defined by (12) is a homogeneous Poisson process with constant rate $\lambda_{1,\alpha}^L$ and $\left\{N(t)_\alpha^U(\omega), t \ge 0\right\}$ defined by (13) is a homogeneous Poisson process with constant rate $\lambda_{1,\alpha}^U$. By the results of a homogeneous Poisson process (see Ross [18]),

$$E\left[N(t)_\alpha^L(\omega)\right] = t\cdot\lambda_{1,\alpha}^L, \quad E\left[N(t)_\alpha^U(\omega)\right] = t\cdot\lambda_{1,\alpha}^U. \tag{14}$$

It follows from Proposition 3.2 that

$$\begin{aligned} E[N(t)] &= \frac{1}{2}\int_0^1 \left(E\left[N(t)_\alpha^L(\omega)\right] + E\left[N(t)_\alpha^U(\omega)\right]\right) \mathrm{d}\alpha \\ &= \frac{t}{2}\int_0^1 \left(\lambda_{1,\alpha}^L + \lambda_{1,\alpha}^U\right) \mathrm{d}\alpha \\ &= t\cdot E[\lambda_1]. \end{aligned}$$

The proof is finished.

Theorem 4.3 *If $\{N(t), t\ge 0\}$ is a fuzzy random homogeneous Poisson process with fuzzy rates λ_i, and $\xi_i, i = 1, 2, \cdots$ are iid nonnegative exponentially distributed fuzzy random interarrival times, then we have*

$$E\left[N^2(t)\right] = t\cdot E[\lambda_1] + t^2\cdot E\left[\lambda_1^2\right].$$

Proof. Note that, for any ω, the fuzzy variable $N(t)(\omega)$ is defined on $(\Theta, P(\Theta), \mathrm{Pos})$, where

$$\Theta = \Theta_1\times\Theta_2\times\cdots, \quad \mathrm{Pos}\{\theta\} = \mathrm{Pos}\{\theta_1\}\wedge\mathrm{Pos}\{\theta_2\}\wedge\cdots$$

for any $\theta = (\theta_1, \theta_2, \cdots) \in \Theta$.

For any $\theta\in\Theta$ with $\mathrm{Pos}\{\theta\}\ge\alpha$, by Definition 2.2, we have

$$N(t)_\alpha^L(\omega) \le N(t)(\omega)(\theta) \le N(t)_\alpha^U(\omega).$$

Consequently,

$$\left(N(t)_\alpha^L(\omega)\right)^2 \le N^2(t)(\omega)(\theta) \le \left(N(t)_\alpha^U(\omega)\right)^2.$$

Again by Definition 2.2,

$$N^2(t)_\alpha^L(\omega) = \left(N(t)_\alpha^L(\omega)\right)^2, \quad N^2(t)_\alpha^U(\omega) = \left(N(t)_\alpha^U(\omega)\right)^2,$$

where $N^2(t)_\alpha^L(\omega)$ and $N^2(t)_\alpha^U(\omega)$ are the α-pessimistic value and the α-optimistic value of $N^2(t)(\omega)$ for each ω. For the homogeneous Poisson processes defined by (12) and (13), by the results of a homogeneous Poisson process (see Ross [18]), we obtain

$$E\left[N^2(t)_\alpha^L(\omega)\right] = E\left[\left(N(t)_\alpha^L(\omega)\right)^2\right] = t\cdot\lambda_{1,\alpha}^L + t^2\cdot\left(\lambda_{1,\alpha}^L\right)^2 = t\cdot\lambda_{1,\alpha}^L + t^2\cdot\left(\lambda_1^2\right)_\alpha^L,$$

$$E\left[N^2(t)_\alpha^U(\omega)\right] = E\left[\left(N(t)_\alpha^U(\omega)\right)^2\right] = t\cdot\lambda_{1,\alpha}^U + t^2\cdot\left(\lambda_{1,\alpha}^U\right)^2 = t\cdot\lambda_{1,\alpha}^U + t^2\cdot\left(\lambda_1^2\right)_\alpha^U.$$

From Proposition 3.2, we have

$$\begin{aligned}
E\left[N^2(t)\right] &= \frac{1}{2}\int_0^1 \left(E\left[N^2(t)_\alpha^L(\omega)\right] + E\left[N^2(t)_\alpha^U(\omega)\right]\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(t\cdot\lambda_{1,\alpha}^L + t^2\cdot\left(\lambda_1^2\right)_\alpha^L + t\cdot\lambda_{1,\alpha}^U + t^2\cdot\left(\lambda_1^2\right)_\alpha^U\right) \mathrm{d}\alpha \\
&= \frac{t}{2}\int_0^1 \left(\lambda_{1,\alpha}^L + \lambda_{1,\alpha}^U\right) \mathrm{d}\alpha + \frac{t^2}{2}\int_0^1 \left(\left(\lambda_1^2\right)_\alpha^L + \left(\lambda_1^2\right)_\alpha^U\right) \mathrm{d}\alpha \\
&= t\cdot E[\lambda_1] + t^2\cdot E\left[\lambda_1^2\right].
\end{aligned}$$

The proof is finished.

Theorem 4.4 *If $\{N(t), t \geq 0\}$ is a fuzzy random homogeneous Poisson process with fuzzy rates λ_i, and $\xi_i, i = 1, 2, \cdots$ are iid nonnegative exponentially distributed fuzzy random interarrival times, then we have*

$$E[N(t)\cdot N(t+s)] = t\cdot E[\lambda_1] + t\cdot(t+s)\cdot E\left[\lambda_1^2\right].$$

Proof. For any $\theta_1, \theta_2 \in \Theta$ with $\mathrm{Pos}\{\theta_1\} \geq \alpha, \mathrm{Pos}\{\theta_2\} \geq \alpha$, we have

$$N(t)_\alpha^L(\omega) \leq N(t)(\omega)(\theta_1) \leq N(t)_\alpha^U(\omega), \tag{15}$$

$$N(t+s)_\alpha^L(\omega) \leq N(t+s)(\omega)(\theta_2) \leq N(t+s)_\alpha^U(\omega), \tag{16}$$

and consequently,

$$N(t)_\alpha^L(\omega)\cdot N(t+s)_\alpha^L(\omega) \leq N(t)(\omega)(\theta_1)\cdot N(t+s)(\omega)(\theta_2) \leq N(t)_\alpha^U(\omega)\cdot N(t+s)_\alpha^U(\omega).$$

By Definition 2.2, we have

$$(N(t)\cdot N(t+s))_\alpha^L(\omega) = N(t)_\alpha^L(\omega)\cdot N(t+s)_\alpha^L(\omega),$$
$$(N(t)\cdot N(t+s))_\alpha^U(\omega) = N(t)_\alpha^U(\omega)\cdot N(t+s)_\alpha^U(\omega),$$

where $(N(t)\cdot N(t+s))_\alpha^L(\omega)$ and $(N(t)\cdot N(t+s))_\alpha^U(\omega)$ are the α-pessimistic value and the α-optimistic value of $(N(t)\cdot N(t+s))(\omega)$ for each ω. For the homogeneous Poisson processes defined by (12) and (13), by the results of a homogeneous Poisson process (see Ross [18]), we have

$$E\left[(N(t)\cdot N(t+s))_\alpha^L(\omega)\right] = t\cdot\lambda_{1,\alpha}^L + t\cdot(t+s)\cdot\left(\lambda_{1,\alpha}^L\right)^2 = t\cdot\lambda_{1,\alpha}^L + t\cdot(t+s)\cdot\left(\lambda_1^2\right)_\alpha^L,$$
$$E\left[(N(t)\cdot N(t+s))_\alpha^U(\omega)\right] = t\cdot\lambda_{1,\alpha}^U + t\cdot(t+s)\cdot\left(\lambda_{1,\alpha}^U\right)^2 = t\cdot\lambda_{1,\alpha}^U + t\cdot(t+s)\cdot\left(\lambda_1^2\right)_\alpha^U.$$

From Proposition 3.2, we obtain

$$\begin{aligned}
E\left[N(t)\cdot(t+s)\right] &= \frac{1}{2}\int_0^1 \left(E\left[(N(t)\cdot N(t+s))_\alpha^L(\omega)\right] + E\left[(N(t)\cdot N(t+s))_\alpha^U(\omega)\right]\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(t\cdot\lambda_{1,\alpha}^L + t\cdot(t+s)\cdot\left(\lambda_1^2\right)_\alpha^L + t\cdot\lambda_{1,\alpha}^U + t\cdot(t+s)\cdot\left(\lambda_1^2\right)_\alpha^U\right) \mathrm{d}\alpha \\
&= \frac{t}{2}\int_0^1 \left(\lambda_{1,\alpha}^L + \lambda_{1,\alpha}^U\right) \mathrm{d}\alpha + \frac{t\cdot(t+s)}{2}\int_0^1 \left(\left(\lambda_1^2\right)_\alpha^L + \left(\lambda_1^2\right)_\alpha^U\right) \mathrm{d}\alpha \\
&= t\cdot E[\lambda_1] + t\cdot(t+s)\cdot E\left[\lambda_1^2\right].
\end{aligned}$$

The proof is finished.

Theorem 4.5 *If $\{N(t), t \geq 0\}$ is a fuzzy random homogeneous Poisson process with fuzzy rates λ_i, and $\xi_i, i = 1, 2, \cdots$ are iid nonnegative exponentially distributed fuzzy random interarrival times, then for any $a \geq 0$, we have*

$$E[N(t+a) - N(t)] = E[N(t+a)] - E[N(t)] = a \cdot E[\lambda_1].$$

Proof. It follows immediately from Theorem 4.2 that $E[N(t+a)] - E[N(t)] = a \cdot E[\lambda_1]$. In the following, we prove

$$E[N(t+a) - N(t)] = E[N(t+a)] - E[N(t)]. \tag{17}$$

For any $\theta_1, \theta_2 \in \Theta$ with $\text{Pos}\{\theta_1\} \geq \alpha$, $\text{Pos}\{\theta_2\} \geq \alpha$, from (15), (16) and using (10),

$$N(t+a)_\alpha^L(\omega) - N(t)_\alpha^L(\omega) \leq N(t+a)(\omega)(\theta_2) - N(t)(\omega)(\theta_1) \leq N(t+a)_\alpha^U(\omega) - N(t)_\alpha^U(\omega)$$

By Definition 2.2,

$$(N(t+a) - N(t))_\alpha^L(\omega) = N(t+a)_\alpha^L(\omega) - N(t)_\alpha^L(\omega), \tag{18}$$

$$(N(t+a) - N(t))_\alpha^U(\omega) = N(t+a)_\alpha^U(\omega) - N(t)_\alpha^U(\omega), \tag{19}$$

by taking expectations,

$$E\left[(N(t+a) - N(t))_\alpha^L(\omega)\right] = E\left[N(t+a)_\alpha^L(\omega) - N(t)_\alpha^L(\omega)\right] = E\left[N(t+a)_\alpha^L(\omega)\right] - E\left[N(t)_\alpha^L(\omega)\right],$$

$$E\left[(N(t+a) - N(t))_\alpha^U(\omega)\right] = E\left[N(t+a)_\alpha^U(\omega) - N(t)_\alpha^U(\omega)\right] = E\left[N(t+a)_\alpha^U(\omega)\right] - E\left[N(t)_\alpha^U(\omega)\right].$$

From Proposition 3.2, we obtain

$$\begin{aligned}
&E[N(t+a) - N(t)] \\
&= \frac{1}{2}\int_0^1 \left(E\left[(N(t+a) - N(t))_\alpha^L(\omega)\right] + E\left[(N(t+a) - N(t))_\alpha^U(\omega)\right]\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(E\left[N(t+a)_\alpha^L(\omega)\right] - E\left[N(t)_\alpha^L(\omega)\right] + E\left[N(t+a)_\alpha^U(\omega)\right] - E\left[N(t)_\alpha^U(\omega)\right]\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(E\left[N(t+a)_\alpha^L(\omega)\right] + E\left[N(t)_\alpha^U(\omega)\right]\right) \mathrm{d}\alpha - \frac{1}{2}\int_0^1 \left(E\left[N(t)_\alpha^L(\omega)\right] + E\left[N(t)_\alpha^U(\omega)\right]\right) \mathrm{d}\alpha \\
&= E[N(t+a)] - E[N(t)].
\end{aligned}$$

The proof is finished.

Theorem 4.6 *If $\{N(t), t \geq 0\}$ is a fuzzy random homogeneous Poisson process with fuzzy rates λ_i, and $\xi_i, i = 1, 2, \cdots$ are iid nonnegative exponentially distributed fuzzy random interarrival times, then for any $a \geq 0$, we have*

$$E[N(t+a) - N(t)] = E[N(t+a)] - E[N(t)] = E[N(a)].$$

Proof. It follows immediately from Theorem 4.2.

Theorem 4.7 *If $\{N(t), t \geq 0\}$ is a fuzzy random homogeneous Poisson process with fuzzy rates λ_i, and $\xi_i, i = 1, 2, \cdots$ are iid nonnegative exponentially distributed fuzzy random interarrival times, then we have*

$$\text{Ch}\{N(t) = 0\} = E\left[e^{-t \cdot \lambda_1}\right]. \tag{20}$$

Proof. Note that $\{N(t)=0\} \Longleftrightarrow \{\xi_1 > t\}$. Since $\xi_1 \sim EXP(\lambda_1)$, by Definition 3.2,

$$\Pr\left\{\xi_{1,\alpha}^L(\omega) > t\right\} = e^{-t\cdot\lambda_{1,\alpha}^U}, \quad \Pr\left\{\xi_{1,\alpha}^U(\omega) > t\right\} = e^{-t\cdot\lambda_{1,\alpha}^L}. \tag{21}$$

Since $e^{-x\cdot t}$ is a decreasing function of x for any fixed $t > 0$, then

$$\left(e^{-\lambda_1\cdot t}\right)_\alpha^L = e^{-t\cdot\lambda_{1,\alpha}^U}, \quad \left(e^{-\lambda_1\cdot t}\right)_\alpha^U = e^{-t\cdot\lambda_{1,\alpha}^L}.$$

In addition,

$$E\left[e^{-t\cdot\lambda_1}\right] = \frac{1}{2}\int_0^1 \left(\left(e^{-\lambda_1\cdot t}\right)_\alpha^L + \left(e^{-\lambda_1\cdot t}\right)_\alpha^U\right) \mathrm{d}\alpha = \frac{1}{2}\int_0^1 \left(e^{-t\cdot\lambda_{1,\alpha}^L} + e^{-t\cdot\lambda_{1,\alpha}^U}\right) \mathrm{d}\alpha.$$

Hence, by Remark 3.1 and (21),

$$\begin{aligned}
\mathrm{Ch}\{N(t)=0\} &= \mathrm{Ch}\{\xi_1 > t\} \\
&= \frac{1}{2}\int_0^1 \left(\Pr\left\{\xi_{1,\alpha}^L(\omega) > t\right\} + \Pr\left\{\xi_{1,\alpha}^U(\omega) > t\right\}\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(e^{-t\cdot\lambda_{1,\alpha}^L} + e^{-t\cdot\lambda_{1,\alpha}^U}\right) \mathrm{d}\alpha \\
&= E\left[e^{-t\cdot\lambda_1}\right].
\end{aligned}$$

The proof is finished.

Theorem 4.8 *If $\{N(t), t \geq 0\}$ is a fuzzy random homogeneous Poisson process with fuzzy rates λ_i, and $\xi_i, i = 1, 2, \cdots$ are iid nonnegative exponentially distributed fuzzy random interarrival times, then for any positive integer k, we have*

$$\mathrm{Ch}\{N(t+s) - N(t) \leq k\} = \mathrm{Ch}\{N(s) \leq k\} = \frac{1}{2}\sum_{j=0}^{k}\int_0^1 \left(e^{-\lambda_{1,\alpha}^L\cdot s}\frac{\left(\lambda_{1,\alpha}^L\cdot s\right)^j}{j!} + e^{-\lambda_{1,\alpha}^U\cdot s}\frac{\left(\lambda_{1,\alpha}^U\cdot s\right)^j}{j!}\right) \mathrm{d}\alpha.$$

Proof. For the homogeneous Poisson processes defined by (12) and (13), by the results of a homogeneous Poisson process (see Ross [18]), we have

$$\Pr\left\{N(s)_\alpha^L(\omega) \leq k\right\} = \sum_{j=0}^{k} e^{-\lambda_{1,\alpha}^L\cdot s}\frac{\left(\lambda_{1,\alpha}^L\cdot s\right)^j}{j!}, \quad \Pr\left\{N(s)_\alpha^U(\omega) \leq k\right\} = \sum_{j=0}^{k} e^{-\lambda_{1,\alpha}^U\cdot s}\frac{\left(\lambda_{1,\alpha}^U\cdot s\right)^j}{j!}.$$

By Remark 3.1, we obtain

$$\begin{aligned}
\mathrm{Ch}\{N(s) \leq k\} &= \frac{1}{2}\int_0^1 \left(\Pr\left\{N(s)_\alpha^L(\omega) \leq k\right\} + \Pr\left\{N(s)_\alpha^U(\omega) \leq k\right\}\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(\sum_{j=0}^{k} e^{-\lambda_{1,\alpha}^L\cdot s}\frac{\left(\lambda_{1,\alpha}^L\cdot s\right)^j}{j!} + \sum_{j=0}^{k} e^{-\lambda_{1,\alpha}^U\cdot s}\frac{\left(\lambda_{1,\alpha}^U\cdot s\right)^j}{j!}\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\sum_{j=0}^{k}\int_0^1 \left(e^{-\lambda_{1,\alpha}^L\cdot s}\frac{\left(\lambda_{1,\alpha}^L\cdot s\right)^j}{j!} + e^{-\lambda_{1,\alpha}^U\cdot s}\frac{\left(\lambda_{1,\alpha}^U\cdot s\right)^j}{j!}\right) \mathrm{d}\alpha.
\end{aligned}$$

For any $\theta_1, \theta_2 \in \Theta$ with $\text{Pos}\{\theta_1\} \geq \alpha$ and $\text{Pos}\{\theta_2\} \geq \alpha$, using (18), (19) and by the results of a homogeneous Poisson process (see Ross [18]),

$$(N(t+s) - N(t))_\alpha^L(\omega) = N(s)_\alpha^L(\omega), \quad (N(t+s) - N(t))_\alpha^U(\omega) = N(s)_\alpha^U(\omega).$$

Consequently,

$$\Pr\left\{(N(t+s) - N(t))_\alpha^L(\omega) \leq k\right\} = \Pr\left\{N(s)_\alpha^L(\omega) \leq k\right\},$$
$$\Pr\left\{(N(t+s) - N(t))_\alpha^U(\omega) \leq k\right\} = \Pr\left\{N(s)_\alpha^U(\omega) \leq k\right\}.$$

It follows from Remark 3.1 that

$$\text{Ch}\{N(t+s) - N(t) \leq k\} = \text{Ch}\{N(s) \leq k\}. \tag{22}$$

The proof is finished.

Theorem 4.9 *If $\{N(t), t \geq 0\}$ is a fuzzy random homogeneous Poisson process with fuzzy rates λ_i, and $\xi_i, i = 1, 2, \cdots$ are iid nonnegative exponentially distributed fuzzy random interarrival times, then for any positive integer k, we have*

$$\text{Ch}\{N(t+s) - N(t) \geq k\} = 1 - \frac{1}{2}\sum_{j=0}^{k-1}\int_0^1 \left(e^{-\lambda_{1,\alpha}^L \cdot s}\frac{\left(\lambda_{1,\alpha}^L \cdot s\right)^j}{j!} + e^{-\lambda_{1,\alpha}^U \cdot s}\frac{\left(\lambda_{1,\alpha}^U \cdot s\right)^j}{j!}\right) \mathrm{d}\alpha.$$

Proof. It immediately follows from Proposition 3.1 and Theorem 4.8. The proof is finished.

Theorem 4.10 *Let $\{N(t), t \geq 0\}$ be a fuzzy random homogeneous Poisson process with fuzzy rates λ_i, and $\xi_i, i = 1, 2, \cdots$ iid nonnegative exponentially distributed fuzzy random interarrival times. If $E[\lambda_1] < +\infty$, then we have*

$$\text{Ch}\{N(t+h) - N(t) \geq 1\} = \text{Ch}\{N(h) \geq 1\} = h \cdot E[\lambda_1] + o(h), \tag{23}$$

where $\lim\limits_{h\to 0} \frac{o(h)}{h} = 0$.

Proof. For the homogeneous Poisson processes defined by (12) and (13), by the results of a homogeneous Poisson process (see Ross [18]), we have

$$\Pr\left\{N(h)_\alpha^L(\omega) \geq 1\right\} = \lambda_{1,\alpha}^L + o(h), \quad \Pr\left\{N(h)_\alpha^U(\omega) \geq 1\right\} = \lambda_{1,\alpha}^U + o(h).$$

That is,

$$\lim_{h\to 0}\frac{\Pr\left\{N(h)_\alpha^L(\omega) \geq 1\right\} - \lambda_{1,\alpha}^L}{h} = 0, \quad \lim_{h\to 0}\frac{\Pr\left\{N(h)_\alpha^U(\omega) \geq 1\right\} - \lambda_{1,\alpha}^U}{h} = 0.$$

From Remark 3.1 and the dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{h\to 0}\frac{\text{Ch}\{N(h) \geq 1\} - h \cdot E[\lambda_1]}{h} \\
&= \lim_{h\to 0}\frac{1}{2}\int_0^1 \left(\frac{\Pr\left\{N(h)_\alpha^L(\omega) \geq 1\right\}}{h} + \frac{\Pr\left\{N(h)_\alpha^U(\omega) \geq 1\right\}}{h}\right)\mathrm{d}\alpha - E[\lambda_1] \\
&= \frac{1}{2}\int_0^1 \lim_{h\to 0}\left(\frac{\Pr\left\{N(h)_\alpha^L(\omega) \geq 1\right\}}{h} + \frac{\Pr\left\{N(h)_\alpha^U(\omega) \geq 1\right\}}{h}\right)\mathrm{d}\alpha - E[\lambda_1] \\
&= \frac{1}{2}\int_0^1 \left(\lambda_{1,\alpha}^L + \lambda_{1,\alpha}^U\right)\mathrm{d}\alpha - E[\lambda_1] \\
&= 0,
\end{aligned}$$

which implies

$$\mathrm{Ch}\{N(h) \geq 1\} = h \cdot E[\lambda_1] + o(h).$$

Furthermore, it follows from (22) that the proof is finished.

Theorem 4.11 *If $\{N(t), t \geq 0\}$ is a fuzzy random homogeneous Poisson process with fuzzy rates λ_i, and $\xi_i, i = 1, 2, \cdots$ are iid nonnegative exponentially distributed fuzzy random interarrival times, then we have*

$$\mathrm{Ch}\{N(t+h) - N(t) \geq 2\} = \mathrm{Ch}\{N(h) \geq 2\} = o(h).$$

Proof. For the homogeneous Poisson processes defined by (12) and (13), from the results of a homogeneous Poisson process (see Ross [18]), we have

$$\Pr\left\{N(h)^L_\alpha(\omega) \geq 2\right\} = o(h), \quad \Pr\left\{N(h)^U_\alpha(\omega) \geq 2\right\} = o(h).$$

That is,

$$\lim_{h\to 0} \frac{\Pr\left\{N(h)^L_\alpha(\omega) \geq 2\right\}}{h} = 0, \quad \lim_{h\to 0} \frac{\Pr\left\{N(h)^U_\alpha(\omega) \geq 2\right\}}{h} = 0.$$

By Remark 3.1, we obtain

$$\begin{aligned}\lim_{h\to 0} \frac{\mathrm{Ch}\{N(h) \geq 2\}}{h} &= \lim_{h\to 0} \frac{1}{2}\int_0^1 \left(\frac{\Pr\left\{N(h)^L_\alpha(\omega) \geq 2\right\}}{h} + \frac{\Pr\left\{N(h)^U_\alpha(\omega) \geq 2\right\}}{h} \right) \mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \left(\lim_{h\to 0}\frac{\Pr\left\{N(h)^L_\alpha(\omega) \geq 2\right\}}{h} + \lim_{h\to 0}\frac{\Pr\left\{N(h)^U_\alpha(\omega) \geq 2\right\}}{h} \right) \mathrm{d}\alpha \\ &= 0.\end{aligned}$$

Furthermore, it follows from (22) that the result holds.

5 Fuzzy Random Compound Poisson Process

Let $\eta_1, \eta_2, \cdots$ be iid nonnegative fuzzy random variables defined on the probability spaces $(\Omega_i, A_i, \Pr_i)$, independent of $N(t)$, a Poisson fuzzy random variable with fuzzy rates λ_i. Let

$$X(t) = \sum_{i=1}^{N(t)} \eta_i. \tag{24}$$

Then $\{X(t), t \geq 0\}$ is called *a fuzzy random compound Poisson process with fuzzy rates* λ_i and $X(t)$ is called *a compound Poisson fuzzy random variable with fuzzy rates* λ_i.

Theorem 5.1 *If $\{X(t), t \geq 0\}$ is a fuzzy random compound Poisson process with fuzzy rates λ_i, then*

$$E[X(t)] = t \cdot E[\lambda_1 \cdot \eta_1].$$

Proof. For any given $\omega \in \Omega$, fuzzy variables $\eta_i(\omega)$ are defined on $(\Gamma_i, P(\Gamma_i), \mathrm{Pos}_i)$. Assume that $\Gamma_1 = \Gamma_2 = \cdots$ For any $\theta \in \Theta, \gamma_i \in \Gamma_i$ with $\mathrm{Pos}\{\theta\} \geq \alpha, \mathrm{Pos}\{\gamma_i\} \geq \alpha$, since

$$N(t)^L_\alpha(\omega) \leq N(t)(\omega)(\theta) \leq N(t)^U_\alpha(\omega), \quad \eta^L_{i,\alpha}(\omega) \leq \eta_i(\omega)(\gamma_i) \leq \eta^U_{i,\alpha}(\omega),$$

then

$$\sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega) \le \sum_{i=1}^{N(t)(\omega)(\theta)} \eta_i(\omega)(\gamma_i) \le \sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega).$$

That is,

$$\sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega) \le X(t)(\omega)(\theta,\gamma_i) \le \sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega). \tag{25}$$

Using Definition 2.2, we have

$$X(t)_\alpha^L(\omega) = \sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega), \quad X(t)_\alpha^U(\omega) = \sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega). \tag{26}$$

Since $\eta_{i,\alpha}^L(\omega)$ and $\eta_{i,\alpha}^U(\omega), i=1,2,\cdots$ are iid random variables, respectively, $\left\{X(t)_\alpha^L(\omega), t\ge 0\right\}$ and $\left\{X(t)_\alpha^U(\omega), t\ge 0\right\}$ are two compound Poisson processes. Applying Wald's Equation to (26),

$$E\left[X(t)_\alpha^L(\omega)\right] = E\left[\sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega)\right] = E\left[N(t)_\alpha^L(\omega)\right]\cdot E\left[\eta_{1,\alpha}^L(\omega)\right],$$

$$E\left[X(t)_\alpha^U(\omega)\right] = E\left[\sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega)\right] = E\left[N(t)_\alpha^U(\omega)\right]\cdot E\left[\eta_{1,\alpha}^U(\omega)\right].$$

In addition, by (14),

$$E\left[X(t)_\alpha^L(\omega)\right] = t\cdot\lambda_{1,\alpha}^L\cdot E\left[\eta_{1,\alpha}^L(\omega)\right], \quad E\left[X(t)_\alpha^U(\omega)\right] = t\cdot\lambda_{1,\alpha}^U\cdot E\left[\eta_{1,\alpha}^U(\omega)\right].$$

Since, for any $\theta_1\in\Theta_1$, $\gamma_1\in\Gamma_1$ with $\mathrm{Pos}\{\theta_1\}\ge\alpha$ and $\mathrm{Pos}\{\gamma_1\}\ge\alpha$,

$$\lambda_{1,\alpha}^L\cdot\eta_{1,\alpha}^L(\omega) \le \lambda_1(\theta_1)\cdot\eta_1(\omega)(\gamma_1) \le \lambda_{1,\alpha}^U\cdot\eta_{1,\alpha}^U(\omega).$$

It follows from Definition 2.2 that

$$(\lambda_1\cdot\eta_1)_\alpha^L(\omega) = \lambda_{1,\alpha}^L\cdot\eta_{1,\alpha}^L(\omega), \quad (\lambda_1\cdot\eta_1)_\alpha^U(\omega) = \lambda_{1,\alpha}^U\cdot\eta_{1,\alpha}^U(\omega).$$

Consequently, by taking expectations,

$$E\left[(\lambda_1\cdot\eta_1)_\alpha^L(\omega)\right] = \lambda_{1,\alpha}^L\cdot E\left[\eta_{1,\alpha}^L(\omega)\right], \quad E\left[(\lambda_1\cdot\eta_1)_\alpha^U(\omega)\right] = \lambda_{1,\alpha}^U\cdot E\left[\eta_{1,\alpha}^U(\omega)\right],$$

and then,

$$\begin{aligned} E[\lambda_1\cdot\eta_1] &= \frac{1}{2}\int_0^1 \left(E\left[(\lambda\cdot\eta_1)_\alpha^L(\omega)\right] + E\left[(\lambda\cdot\eta_1)_\alpha^U(\omega)\right]\right)\mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \left(\lambda_{1,\alpha}^L\cdot E\left[\eta_{1,\alpha}^L(\omega)\right] + \lambda_{1,\alpha}^U\cdot E\left[\eta_{1,\alpha}^U(\omega)\right]\right)\mathrm{d}\alpha. \end{aligned}$$

Hence, by Proposition 3.2,

$$\begin{aligned} E[X(t)] &= \frac{1}{2}\int_0^1 \left(E\left[X(t)_\alpha^L(\omega)\right] + E\left[X(t)_\alpha^U(\omega)\right]\right)\mathrm{d}\alpha \\ &= \frac{t}{2}\int_0^1 \left(\lambda_{1,\alpha}^L\cdot E\left[\eta_{1,\alpha}^L(\omega)\right] + \lambda_{1,\alpha}^U\cdot E\left[\eta_{1,\alpha}^U(\omega)\right]\right)\mathrm{d}\alpha \\ &= t\cdot E[\lambda_1\cdot\eta_1]. \end{aligned}$$

The proof is finished.

Remark 5.1 *If $N(t)$ degenerates to a Poisson random variable with constant rate λ, then the result in Theorem 5.1 degenerates the following form*

$$E[X(t)] = \lambda \cdot t \cdot E[\eta_1].$$

Remark 5.2 *If η_i degenerate to iid random variables, then the result in Theorem 5.1 degenerates the following form*

$$E[X(t)] = t \cdot E[\lambda_1] \cdot E[\eta_1].$$

Theorem 5.2 *If $\{X(t), t \geq 0\}$ is a fuzzy random compound Poisson process with fuzzy rates λ_i, then*

$$E\left[X^2(t)\right] = t \cdot E\left[\lambda_1 \cdot \eta_1^2\right] + \frac{t^2}{2}\int_0^1 \left(\left(\lambda_{1,\alpha}^L \cdot E\left[\eta_{1,\alpha}^L(\omega)\right]\right)^2 + \left(\lambda_{1,\alpha}^U \cdot E\left[\eta_{1,\alpha}^U(\omega)\right]\right)^2\right) \mathrm{d}\alpha.$$

Proof. Using (25),

$$\left(\sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega)\right)^2 \leq \left(\sum_{i=1}^{N(t)(\omega)(\theta)} \eta_i(\omega)(\gamma_i)\right)^2 \leq \left(\sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega)\right)^2.$$

That is,

$$\left(\sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega)\right)^2 \leq X^2(t)(\omega, \gamma_i) \leq \left(\sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega)\right)^2.$$

From Definition 2.2,

$$X^2(t)_\alpha^L(\omega) = \left(\sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega)\right)^2, \quad X^2(t)_\alpha^U(\omega) = \left(\sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega)\right)^2.$$

By the results of a compound Poisson process (see Ross [18]),

$$E\left[X^2(t)_\alpha^L(\omega)\right] = t \cdot \lambda_{1,\alpha}^L \cdot E\left[\left(\eta_1^2\right)_\alpha^L(\omega)\right] + t^2 \cdot \left(\lambda_{1,\alpha}^L\right)^2 \cdot E^2\left[\eta_{1,\alpha}^L(\omega)\right]$$

$$E\left[X^2(t)_\alpha^U(\omega)\right] = t \cdot \lambda_{1,\alpha}^U \cdot E\left[\left(\eta_1^2\right)_\alpha^U(\omega)\right] + t^2 \cdot \left(\lambda_{1,\alpha}^U\right)^2 \cdot E^2\left[\eta_{1,\alpha}^U(\omega)\right].$$

It is easy to prove that

$$\left(\lambda_1 \cdot \eta_1^2\right)_\alpha^L(\omega) = \lambda_{1,\alpha}^L \cdot E\left[\left(\eta_1^2\right)_\alpha^L(\omega)\right], \quad \left(\lambda_1 \cdot \eta_1^2\right)_\alpha^U(\omega) = \lambda_{1,\alpha}^U \cdot E\left[\left(\eta_1^2\right)_\alpha^U(\omega)\right]$$

and then

$$E\left[\lambda_1 \cdot \eta_1^2\right] = \frac{1}{2}\int_0^1 \left(\lambda_{1,\alpha}^L \cdot E\left[\left(\eta_1^2\right)_\alpha^L(\omega)\right] + \lambda_{1,\alpha}^U \cdot E\left[\left(\eta_1^2\right)_\alpha^U(\omega)\right]\right) \mathrm{d}\alpha.$$

Using Proposition 3.2,

$$\begin{aligned} E\left[X^2(t)\right] &= \frac{1}{2}\int_0^1 \left(E\left[X^2(t)_\alpha^L(\omega)\right] + E\left[X^2(t)_\alpha^U(\omega)\right]\right) \mathrm{d}\alpha \\ &= \frac{t}{2}\int_0^1 \left(\lambda_{1,\alpha}^L \cdot E\left[\left(\eta_1^2\right)_\alpha^L(\omega)\right] + \lambda_{1,\alpha}^U \cdot E\left[\left(\eta_1^2\right)_\alpha^U(\omega)\right]\right) \mathrm{d}\alpha \\ &\quad + \frac{1}{2}\int_0^1 \left(\left(t \cdot \lambda_{1,\alpha}^L \cdot E\left[\eta_{1,\alpha}^L(\omega)\right]\right)^2 + \left(t \cdot \lambda_{1,\alpha}^U \cdot E\left[\eta_{1,\alpha}^U(\omega)\right]\right)^2\right) \mathrm{d}\alpha \\ &= t \cdot E\left[\lambda_1 \cdot \eta_1^2\right] + \frac{t^2}{2}\int_0^1 \left(\left(\lambda_{1,\alpha}^L \cdot E\left[\eta_{1,\alpha}^L(\omega)\right]\right)^2 + \left(\lambda_{1,\alpha}^U \cdot E\left[\eta_{1,\alpha}^U(\omega)\right]\right)^2\right) \mathrm{d}\alpha. \end{aligned}$$

The proof is finished.

Remark 5.3 *If $N(t)$ degenerates to a Poisson random variable with constant rate λ, then the result in Theorem 5.2 degenerates the following form*

$$E\left[X^2(t)\right]=t\cdot\lambda\cdot E\left[\eta_1^2\right]+\frac{t^2\cdot\lambda^2}{2}\int_0^1\left(E^2\left[\eta_{1,\alpha}^L(\omega)\right]+E^2\left[\eta_{1,\alpha}^U(\omega)\right]\right)\mathrm{d}\alpha.$$

Remark 5.4 *If η_i degenerate to iid random variables, then the result in Theorem 5.2 degenerates the following form*

$$E\left[X^2(t)\right]=t\cdot E[\lambda_1]\cdot E\left[\eta_1^2\right]+t^2\cdot E\left[\lambda_1^2\right]\cdot E^2[\eta_1].$$

Theorem 5.3 *If $\{X(t),t\geq 0\}$ is a fuzzy random compound Poisson process with fuzzy rates λ_i, then we have*

$$\begin{aligned}&E\left[X^3(t)\right]=t\cdot E\left[\lambda_1\cdot\eta_1^3\right]\\&+\frac{3t^2}{2}\int_0^1\left(\left(\lambda_{1,\alpha}^L\right)^2\cdot E\left[\eta_{1,\alpha}^L(\omega)\right]\cdot E\left[\left(\eta_1^2\right)_\alpha^L(\omega)\right]+\left(\lambda_\alpha^U\right)^2\cdot E\left[\eta_{1,\alpha}^U(\omega)\right]\cdot E\left[\left(\eta_1^2\right)_\alpha^U(\omega)\right]\right)\mathrm{d}\alpha\\&+\frac{t^3}{2}\int_0^1\left(\left(\lambda_{1,\alpha}^L\right)^3\cdot E^3\left[\eta_{1,\alpha}^L(\omega)\right]+\left(\lambda_\alpha^U\right)^3\cdot E^3\left[\eta_{1,\alpha}^U(\omega)\right]\right)\mathrm{d}\alpha.\end{aligned}$$

Proof. Similarly, we have

$$X^3(t)_\alpha^L(\omega)=\left(\sum_{i=1}^{N(t)_\alpha^L(\omega)}\eta_{i,\alpha}^L(\omega)\right)^3,\quad X^3(t)_\alpha^U(\omega)=\left(\sum_{i=1}^{N(t)_\alpha^U(\omega)}\eta_{i,\alpha}^U(\omega)\right)^3.$$

By the results of a compound Poisson process (see Ross[18]),

$$\begin{aligned}&E\left[X^3(t)_\alpha^L(\omega)\right]\\&=t\cdot\lambda_{1,\alpha}^L\cdot E\left[\left(\eta_1^3\right)_\alpha^L(\omega)\right]+3\cdot\left(\lambda_{1,\alpha}^L\right)^2\cdot E\left[\eta_{1,\alpha}^L(\omega)\right]\cdot E\left[\left(\eta_1^2\right)_\alpha^L(\omega)\right]+t^3\cdot\left(\lambda_{1,\alpha}^L\right)^3\cdot E^3\left[\eta_{1,\alpha}^L(\omega)\right],\end{aligned}$$

$$\begin{aligned}&E\left[X^3(t)_\alpha^U(\omega)\right]\\&=t\cdot\lambda_{1,\alpha}^U\cdot E\left[\left(\eta_1^3\right)_\alpha^U(\omega)\right]+3\cdot\left(\lambda_{1,\alpha}^U\right)^2\cdot E\left[\eta_{1,\alpha}^U(\omega)\right]\cdot E\left[\left(\eta_1^2\right)_\alpha^U(\omega)\right]+t^3\cdot\left(\lambda_{1,\alpha}^U\right)^3\cdot E^3\left[\eta_{1,\alpha}^U(\omega)\right].\end{aligned}$$

It is easy to prove that

$$\left(\lambda_1\cdot\eta_1^3\right)_\alpha^L(\omega)=\lambda_{1,\alpha}^L\cdot E\left[\left(\eta_1^3\right)_\alpha^L(\omega)\right],\quad\left(\lambda_1\cdot\eta_1^3\right)_\alpha^U(\omega)=\lambda_{1,\alpha}^U\cdot E\left[\left(\eta_1^3\right)_\alpha^U(\omega)\right]$$

and then

$$E\left[\lambda_1\cdot\eta_1^3\right]=\frac{1}{2}\int_0^1\left(\lambda_{1,\alpha}^L\cdot E\left[\left(\eta_1^3\right)_\alpha^L(\omega)\right]+\lambda_{1,\alpha}^U\cdot E\left[\left(\eta_1^3\right)_\alpha^U(\omega)\right]\right)\mathrm{d}\alpha.$$

Using Proposition 3.2,

$$\begin{aligned}
E\left[X^3(t)\right] &= \frac{1}{2}\int_0^1 \left(E\left[X^3(t)_\alpha^L(\omega)\right] + E\left[X^3(t)_\alpha^U(\omega)\right]\right)\mathrm{d}\alpha \\
&= \frac{t}{2}\int_0^1 \left(\lambda_{1,\alpha}^L \cdot E\left[\left(\eta_1^3\right)_\alpha^L(\omega)\right] + \lambda_{1,\alpha}^U \cdot E\left[\left(\eta_1^3\right)_\alpha^U(\omega)\right]\right)\mathrm{d}\alpha \\
&\quad + \frac{1}{2}\int_0^1 \left(3\cdot\left(\lambda_{1,\alpha}^L\right)^2 \cdot E\left[\eta_{1,\alpha}^L(\omega)\right]\cdot E\left[\left(\eta_1^2\right)_\alpha^L(\omega)\right] + 3\cdot\left(\lambda_{1,\alpha}^U\right)^2 \cdot E\left[\eta_{1,\alpha}^U(\omega)\right]\cdot E\left[\left(\eta_1^2\right)_\alpha^U(\omega)\right]\right)\mathrm{d}\alpha \\
&\quad + \frac{1}{2}\int_0^1 \left(t^3\cdot\left(\lambda_{1,\alpha}^U\right)^3 \cdot E^3\left[\eta_{1,\alpha}^U(\omega)\right] + t^3\cdot\left(\lambda_{1,\alpha}^U\right)^3\cdot E^3\left[\eta_{1,\alpha}^U(\omega)\right]\right)\mathrm{d}\alpha \\
&= t\cdot E\left[\lambda_1\cdot\eta_1^3\right] \\
&\quad + \frac{3t^2}{2}\int_0^1 \left(\left(\lambda_{1,\alpha}^L\right)^2 \cdot E\left[\eta_{1,\alpha}^L(\omega)\right]\cdot E\left[\left(\eta_1^2\right)_\alpha^L(\omega)\right] + \left(\lambda_\alpha^U\right)^2 \cdot E\left[\eta_{1,\alpha}^U(\omega)\right]\cdot E\left[\left(\eta_1^2\right)_\alpha^U(\omega)\right]\right)\mathrm{d}\alpha \\
&\quad + \frac{t^3}{2}\int_0^1 \left(\left(\lambda_{1,\alpha}^L\right)^3 \cdot E^3\left[\eta_{1,\alpha}^L(\omega)\right] + \left(\lambda_\alpha^U\right)^3\cdot E^3\left[\eta_{1,\alpha}^U(\omega)\right]\right)\mathrm{d}\alpha.
\end{aligned}$$

The proof is finished.

Remark 5.5 *If $N(t)$ degenerates to a Poisson random variable with constant rate λ, then the result in Theorem 5.3 degenerates the following form*

$$\begin{aligned}
E\left[X^3(t)\right] &= t\cdot\lambda\cdot E\left[\eta_1^3\right] \\
&\quad + \frac{3t^2\cdot\lambda^2}{2}\int_0^1 \left(E\left[\eta_{1,\alpha}^L(\omega)\right]\cdot E\left[\left(\eta_1^2\right)_\alpha^L(\omega)\right] + E\left[\eta_{1,\alpha}^U(\omega)\right]\cdot E\left[\left(\eta_1^2\right)_\alpha^U(\omega)\right]\right)\mathrm{d}\alpha \\
&\quad + \frac{t^3\cdot\lambda^3}{2}\int_0^1 \left(E^3\left[\eta_{1,\alpha}^L(\gamma)\right] + E^3\left[\eta_{1,\alpha}^U(\gamma)\right]\right)\mathrm{d}\alpha.
\end{aligned}$$

Remark 5.6 *If η_i degenerate to iid random variables, then the result in Theorem 5.3 degenerates the following form*

$$E\left[X^3(t)\right] = t\cdot E[\lambda_1]\cdot E\left[\eta_1^3\right] + 3t^2\cdot E\left[\lambda_1^2\right]\cdot E[\eta_1]\cdot E\left[\eta_1^2\right] + t^3\cdot E\left[\lambda_1^3\right]\cdot E^3[\eta_1].$$

Theorem 5.4 *Let $\{X(t), t\geq 0\}$ be a fuzzy random compound Poisson process with fuzzy rates λ_i. If h is a strictly increasing nonnegative function, then*

$$E[X(t)\cdot h(X(t))] = E[\lambda_1\cdot\eta_1\cdot h(X(t)+\eta_1)].$$

Proof. Since h is a strictly increasing nonnegative function, using (25),

$$h\left(\sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega)\right) \leq h\left(X(t)(\omega)(\theta,\gamma_i)\right) \leq h\left(\sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega)\right),$$

$$\left(\sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega)\right)\cdot h\left(\sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega)\right) \leq X(t)(\omega)(\theta,\gamma_i)\cdot h(X(t)(\omega)(\theta,\gamma_i)),$$

$$X(t)(\omega)(\theta, \gamma_i) \cdot h(X(t)(\omega)(\theta, \gamma_i)) \leq \left(\sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega) \right) \cdot h \left(\sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega) \right).$$

From Definition 2.2,

$$(X(t) \cdot h(X(t))_\alpha^L(\omega) = \left(\sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega) \right) \cdot h \left(\sum_{i=1}^{N(t)_\alpha^L(\omega)} \eta_{i,\alpha}^L(\omega) \right),$$

$$(X(t) \cdot h(X(t))_\alpha^L(\omega) = \left(\sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega) \right) \cdot h \left(\sum_{i=1}^{N(t)_\alpha^U(\omega)} \eta_{i,\alpha}^U(\omega) \right).$$

By the results of a compound Poisson process (see Ross [18]),

$$(X(t) \cdot h(X(t))_\alpha^L(\omega) = \lambda_{1,\alpha}^L \cdot \eta_{1,\alpha}^L(\omega) \cdot h \left(X(t)_\alpha^L(\omega) + \eta_{1,\alpha}^L(\omega) \right),$$

$$(X(t) \cdot h(X(t))_\alpha^U(\omega) = \lambda_{1,\alpha}^U \cdot \eta_{1,\alpha}^U(\omega) \cdot h \left(X(t)_\alpha^U(\omega) + \eta_{1,\alpha}^U(\omega) \right).$$

It is easy to prove that

$$E\left[(\lambda_1 \cdot \eta_1 \cdot h(X(t) + \eta_1)_\alpha^L(\omega) \right] = \lambda_{1,\alpha}^L \cdot E\left[\eta_{1,\alpha}^L(\omega) \cdot h\left(X(t)_\alpha^L(\omega) + \eta_{1,\alpha}^L(\omega) \right) \right],$$

$$E\left[(\lambda_1 \cdot \eta_1 \cdot h(X(t) + \eta_1)_\alpha^U(\omega) \right] = \lambda_{1,\alpha}^U \cdot E\left[\eta_{1,\alpha}^U(\omega) \cdot h\left(X(t)_\alpha^U(\omega) + \eta_{1,\alpha}^U(\omega) \right) \right],$$

and consequently,

$$\begin{aligned} & E[\lambda_1 \cdot \eta_1 \cdot h(X(t) + \eta_1)] \\ & = \frac{1}{2} \int_0^1 \left(\lambda_{1,\alpha}^L \cdot E\left[\eta_{1,\alpha}^L(\omega) \cdot h\left(X(t)_\alpha^L(\omega) + \eta_{1,\alpha}^L(\omega) \right) \right] \right. \\ & \quad \left. + \lambda_{1,\alpha}^U \cdot E\left[\eta_{1,\alpha}^U(\omega) \cdot h\left(X(t)_\alpha^U(\omega) + \eta_{1,\alpha}^U(\omega) \right) \right] \right) \mathrm{d}\alpha. \end{aligned}$$

It follows from Proposition 3.2 that

$$\begin{aligned} & E[X(t) \cdot h(X(t))] = \frac{1}{2} \int_0^1 \left(E\left[(X(t) \cdot h(X(t))_\alpha^L(\omega) \right] + E\left[(X(t) \cdot h(X(t))_\alpha^U(\omega) \right] \right) \mathrm{d}\alpha \\ & = \frac{1}{2} \int_0^1 \left(\lambda_{1,\alpha}^L \cdot E\left[\eta_{1,\alpha}^L \cdot h\left(X(t)_\alpha^L(\omega) + \eta_{1,\alpha}^L(\omega) \right) \right] + \lambda_{1,\alpha}^U \cdot E\left[\eta_{1,\alpha}^U \cdot h\left(X(t)_\alpha^U(\omega) + \eta_{1,\alpha}^U(\omega) \right) \right] \right) \mathrm{d}\alpha \\ & = E[\lambda_1 \cdot \eta_1 \cdot h(X(t) + \eta_1)]. \end{aligned}$$

The proof is finished.

Remark 5.7 *If $N(t)$ degenerates to a Poisson random variable with rate constant λ, then the result in Theorem 5.4 degenerates the following form*

$$E[X(t) \cdot h(X(t))] = \lambda \cdot E[\eta_1 \cdot h(X(t) + \eta_1)].$$

Remark 5.8 *If η_i degenerate to random variables, then the result in Theorem 5.4 degenerates the following form*

$$E[X(t) \cdot h(X(t))] = E[\lambda_1] \cdot E[\eta_1 \cdot h(X(t) + \eta_1)].$$

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