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Visualization of Data Subject to Positive Constraints

Malik Zawwar Hussain [1 +] and Maria Hussain [2]

[1, 2] Department of Mathematics, University of the Punjab, Lahore- Pakistan.

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Abstract. In this paper, first free parameters are constrained in the description of rational cubic function [6] to preserve the shape of data that lies above the straight line. Then, rational cubic function is extended to rational bicubic partially blended function (Coons patches). A local positivity preserving scheme is developed for positive data by making constraints on free parameters in the description of rational bicubic partially blended patches. We also develop at the end the constraints for visualizing a data that lie above the plane.

Keywords: Visualization, Rational Function, Interpolation, Positive Surfaces, Free Parameters.

1. Introduction

Scientific visualization, is the representation of data graphically for gaining understanding and insight into the data. Sometimes it is also referred to visual data analysis. Visualization involves research in computer graphics, image processing, high performance computing, meteorological monitoring, maps, data plots, drawing and many other areas. It enriches the process of scientific discovery and fosters profound and unexpected insights but its recent emphasized is on computer graphics and visualization in scientific computing.

In Computer Graphics environment, a user is usually in need of an interpolating scheme which possesses certain characteristics like shape preservation, shape control to visualize the data in a pleasant way. The properties those are used to quantify shape are positivity, monotonicity and convexity. Problem of positivity is described as: if entries in the sample data are positive, then interpolating curve and surface is positive. Preserving positivity is particularly important in visualizing entities that cannot be negative e.g. amount of rainfall, population, volume, area, density and concentration of sugar in blood.

Some interest has been shown in this area in [1-14] and references there in. Brodlie, Mashwama and Butt in [2] preserved the positivity of 3D positive data by the rearrangement of data. They inserted one or more knots where required to preserve the shape of data. Piah, Goodman and Unsworth in [10] have discussed the problem of positivity preserving scattered data interpolation. Nadler in [9] also discussed the problem of non-negative interpolation. They have considered non-negative data arranged over a triangular mesh and have interpolated each triangular patch using a bivariate quadratic function. Zawwar in [5] preserved the shape of 3D data by rational bicubic tensor product surface.

In section 2, Rational cubic function [6] used in this paper is described. In section 3, a scheme is developed to preserve the shape of data that is lying above the straight line. In section 4, rational cubic function [6] is extended to rational bicubic partially blended patches (Coons Patches). The advantageous feature of these partially blended patches is that they inherit all the properties of the network of boundary curves [4]. In section 5 derivative approximation scheme is introduced. In section 6, a scheme is developed to preserve the shape of positive data by making constraints on free parameters in the description of rational bicubic partially blended patches. In section 7, constraints for visualizing a data that lie above the plane are developed. The section 8 discusses and demonstrates the schemes developed in section 3, 6 and 7. Section 9 concludes the paper.

2. Rational Cubic Function

In this section, we introduce the piecewise rational cubic function [6] used in this paper. Let (x_i, f_i),

[+] *E-mail address*: malikzawwar@math.pu.edu.pk

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$i = 0,1,2,...,n$ be given set of data points where $x_0 < x_1 < \ldots < x_n$. Piecewise rational cubic function is defined over each interval $I_i = [x_i, x_{i+1}]$ as:

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)}, \tag{1}$$

where

$$p_i(\theta) = v_i f_i (1-\theta)^3 + [(2u_i v_i + v_i) f_i + v_i h_i d_i](1-\theta)^2 \theta + [(2u_i v_i + u_i) f_{i+1} - u_i h_i d_{i+1}](1-\theta)\theta^2 + u_i f_{i+1}\theta^3,$$

$$q_i(\theta) = v_i (1-\theta)^2 + 2u_i v_i (1-\theta)\theta + u_i \theta^2,$$

$$h_i = x_{i+1} - x_i, \quad \theta = \frac{(x - x_i)}{h_i}.$$

The rational cubic function (1) has the following properties:

$$S(x_i) = f_i, \quad S(x_{i+1}) = f_{i+1}, \; S^{(1)}(x_i) = d_i, \quad S^{(1)}(x_{i+1}) = d_{i+1}.$$

$S^{(1)}(x)$ denotes the derivative with respect to x and d_i denotes derivative values (given or estimated by some method) at knot x_i. $S(x) \in C^{(1)}[x_0, x_n]$ has u_i and v_i as free parameters in the interval $[x_i, x_{i+1}]$. We note that in each interval I_i, when we take $u_i = 1$ and $v_i = 1$, the piecewise rational cubic function reduces to standard Cubic Hermite. Zawwar and Jamaludin in $[6]$ have developed the following result:

Theorem 2.1. The piecewise rational cubic function (1) preserves positivity if free parameters u_i and v_i satisfy the following condition in each interval $[x_i, x_{i+1}]$

$$u_i > Max\left\{0, -\frac{h_i d_i}{2 f_i} + 1\right\}. \quad v_i > Max\left\{0, \frac{h_i d_{i+1}}{2 f_{i+1}} - 1\right\}.$$

3. Visualization of 2D Constrained Data

In this section we consider data lying above a straight line and constraints are developed on free parameters for visualizing this data.

Let $(x_i, f_i), i = 0,1,2,...,n$ be given data points lying above the straight line $y = mx + c$ i.e.

$$f_i \geq mx_i + c, \quad \forall \quad i = 0,1,2,...,n.$$

The curve will lie above the straight line if rational cubic function (1) satisfies the condition:

$$S(x) > mx + c, \quad \forall \; x \in [x_0, x_n]. \tag{2}$$

For each subinterval $[x_i, x_{i+1}]$, above relation can be expressed as:

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)} > mx_i + c. \tag{3}$$

The equation of straight line in parameter θ is defined as: $a_i(1-\theta) + b_i\theta$, where

$$a_i = mx_i + c \; and \; b_i = mx_{i+1} + c.$$

The parametric form of equation (3) in terms of parameter θ is:

$$\frac{p_i(\theta)}{q_i(\theta)} > a_i(1-\theta) + b_i\theta, \quad \theta \in [0\,1], \quad i = 0,1,2,...,n. \tag{4}$$

$$q_i(\theta) > 0 \text{ if } u_i > 0 \; and \; v_i > 0.$$

Multiply both sides of (4) by $q_i(\theta)$ and rearranging we obtain:

$$U_i(\theta) > 0,$$

$$U_i(\theta) = \sum_{i=0}^{3} (1-\theta)^{3-i}\theta^i A_i, \tag{5}$$

with

$$A_0 = v_i f_i - v_i a_i,\ A_1 = (2u_i v_i + v_i) f_i + v_i h_i d_i - 2u_i v_i a_i - v_i b_i,$$

$$A_2 = (2u_i v_i + u_i) f_{i+1} - u_i h_i d_{i+1} - 2u_i v_i b_i - u_i a_i,\quad A_3 = u_i f_{i+1} - u_i b_i.$$

$$U_i(\theta) > 0 \text{ if } A_i > 0, i = 0,1,2,3.$$

$$A_0 > 0 \text{ if } v_i > 0.$$

$$A_1 > 0 \text{ if } u_i > \frac{-f_i - h_i d_i + b_i}{2(f_i - a_i)}.$$

$$A_2 > 0 \text{ if } v_i > \frac{-f_{i+1} + h_i d_{i+1} + a_i}{2(f_{i+1} - b_i)}.$$

$$A_3 > 0 \text{ if } u_i > 0.$$

All this discussion is summarized in the following theorem:

Theorem 3.1. The rational cubic function (1) preserves the shape of data that lies above the straight line, if in each subinterval $[x_i, x_{i+1}]$, free parameters u_i and v_i satisfy the following conditions:

$$u_i = l_i + Max\left\{0, \frac{-f_i - h_i d_i + b_i}{2(f_i - a_i)}\right\},\quad l_i > 0.\quad v_i = m_i + Max\left\{0, \frac{-f_{i+1} + h_i d_{i+1} + a_i}{2(f_{i+1} - b_i)}\right\},\quad m_i > 0.$$

4. Bicubic Partially Blended Rational Function

The piecewise rational cubic function (1) is extended to bicubic partially blended rational function $S(x, y)$ over rectangular domain $D = [x_0, x_m] \times [y_0, y_n]$. Let $\pi : a = x_0 < x_1 < \ldots < x_m = b$ be partition of $[a\ \ b]$ and $\tilde{\pi} : c = y_0 < y_1 < \ldots < y_n = d$ be partition of $[c\ \ d]$. Rational bicubic function is defined over each rectangular patch $[x_i\ \ x_{i+1}] \times [y_j\ \ y_{j+1}]$ where $i = 0,1,2,...,m-1;\ j = 0,1,2,...,n-1$ as:

$$S(x, y) = -AFB^T, \tag{6}$$

where

$$F = \begin{pmatrix} 0 & S(x, y_j) & S(x, y_{j+1}) \\ S(x_i, y) & S(x_i, y_j) & S(x_i, y_{j+1}) \\ S(x_{i+1}, y) & S(x_{i+1}, y_j) & S(x_{i+1}, y_{j+1}) \end{pmatrix},$$

$$A = [-1\quad \phi_{0,3}^i(x)\quad \phi_{3,3}^i(x)],\ B = [-1\quad \phi_{0,3}^j(y)\quad \phi_{3,3}^j(y)].$$

$\phi_{0,3}^i(x), \phi_{3,3}^i(x), \phi_{0,3}^j(y)$ and $\phi_{3,3}^j(y)$ are the Cubic Hermite blending functions. $S(x, y_j)$, $S(x, y_{j+1})$, $S(x_i, y)$ and $S(x_{i+1}, y)$ are rational cubic functions defined on the sides of $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ using (1) with

$S(x, y_j)$	$F_{i,j}$	$F_{i,j}^x$	$F_{i+1,j}^x$	$F_{i+1,j}$	$u_{i,j}$	$v_{i,j}$
$S(x, y_{j+1})$	$F_{i,j+1}$	$F_{i,j+1}^x$	$F_{i+1,j+1}^x$	$F_{i+1,j+1}$	$u_{i,j+1}$	$v_{i,j+1}$
$S(x_i, y)$	$F_{i,j}$	$F_{i,j}^y$	$F_{i,j+1}^y$	$F_{i,j+1}$	$\hat{u}_{i,j}$	$\hat{v}_{i,j}$
$S(x_{i+1}, y)$	$F_{i+1,j}$	$F_{i+1,j}^y$	$F_{i+1,j+1}^y$	$F_{i+1,j+1}$	$\hat{u}_{i+1,j}$	$\hat{v}_{i+1,j}$

$$S(x, y_j) = \frac{\sum_{i=0}^{3} (1-\theta)^{3-i} \theta^i A_i}{q_1(\theta)}, \tag{7}$$

with

$$A_0 = v_{i,j} F_{i,j}, \;\; A_1 = (2u_{i,j} v_{i,j} + v_{i,j}) F_{i,j} + v_{i,j} h_i F_{i,j}^x, \;\; A_2 = (2u_{i,j} v_{i,j} + u_{i,j}) F_{i+1,j} - u_{i,j} h_i F_{i+1,j}^x,$$

$$A_3 = u_{i,j} F_{i+1,j}, \;\; q_1(\theta) = (1-\theta)^2 v_{i,j} + 2u_{i,j} v_{i,j} (1-\theta)\theta + \theta^2 u_{i,j}.$$

$$S(x, y_{j+1}) = \frac{\sum_{i=0}^{3} (1-\theta)^{3-i} \theta^i B_i}{q_2(\theta)}, \tag{8}$$

with

$$B_0 = v_{i,j+1} F_{i,j+1}, \;\; B_1 = (2u_{i,j+1} v_{i,j+1} + v_{i,j+1}) F_{i,j+1} + v_{i,j+1} h_i F_{i,j+1}^x,$$

$$B_2 = (2u_{i,j+1} v_{i,j+1} + u_{i,j+1}) F_{i+1,j+1} - u_{i,j+1} h_i F_{i+1,j+1}^x, \;\; B_3 = u_{i,j+1} F_{i+1,j+1},$$

$$q_2(\theta) = (1-\theta)^2 v_{i,j+1} + 2u_{i,j+1} v_{i,j+1} (1-\theta)\theta + \theta^2 u_{i,j+1}.$$

$$S(x_i, y) = \frac{\sum_{i=0}^{3} (1-\phi)^{3-i} \phi^i C_i}{q_3(\phi)}, \tag{9}$$

with

$$C_0 = \hat{v}_{i,j} F_{i,j}, \;\; C_1 = (2\hat{u}_{i,j} \hat{v}_{i,j} + \hat{v}_{i,j}) F_{i,j} + \hat{v}_{i,j} \hat{h}_j F_{i,j}^y, \;\; C_2 = (2\hat{u}_{i,j} \hat{v}_{i,j} + \hat{u}_{i,j}) F_{i,j+1} - \hat{u}_{i,j} \hat{h}_j F_{i,j+1}^y,$$

$$C_3 = \hat{u}_{i,j} F_{i,j+1}, \;\; q_3(\phi) = (1-\phi)^2 \hat{v}_{i,j} + 2\hat{u}_{i,j} \hat{v}_{i,j} (1-\phi)\phi + \phi^2 \hat{u}_{i,j}.$$

$$S(x_{i+1}, y) = \frac{\sum_{i=0}^{3} (1-\phi)^{3-i} \phi^i D_i}{q_4(\phi)}, \tag{10}$$

with

$$D_0 = \hat{v}_{i+1,j} F_{i+1,j}, \;\; D_1 = (2\hat{u}_{i+1,j} \hat{v}_{i+1,j} + \hat{v}_{i+1,j}) F_{i+1,j} + \hat{v}_{i+1,j} \hat{h}_j F_{i+1,j}^y,$$

$$D_2 = (2\hat{u}_{i+1,j} \hat{v}_{i+1,j} + \hat{u}_{i+1,j}) F_{i+1,j+1} - \hat{u}_{i+1,j} \hat{h}_j F_{i+1,j+1}^y, \;\; D_3 = \hat{u}_{i+1,j} F_{i+1,j+1},$$

$$q_4(\phi) = (1-\phi)^2 \hat{v}_{i+1,j} + 2\hat{u}_{i+1,j} \hat{v}_{i+1,j} (1-\phi)\phi + \phi^2 \hat{u}_{i+1,j}.$$

5. Choice of Derivatives

In most applications, the derivative parameters d_i, $F_{i,j}^x$, $F_{i,j}^y$ and $F_{i,j}^{xy}$ are not given and hence must be determined either from given data or by some other means. These methods are the approximation based on various mathematical theories. An obvious choice is mentioned here:

5.1 Arithmetic Mean Method

Arithmetic mean method is the three-point difference approximation based on arithmetic manipulation. This method is defined as:

5.1.2 Arithmetic Mean Method for 2D Data

$$d_0 = \Delta_0 + (\Delta_0 - \Delta_1) \frac{h_0}{(h_0 + h_1)},$$

$$d_n = \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2}) \frac{h_{n-1}}{(h_{n-1} + h_{n-2})},$$

$$d_i = \frac{\Delta_i + \Delta_{i-1}}{2},$$

$$i = 1, 2, 3, ..., n-1,$$

where

$$\Delta_i = \frac{f_{i+1} - f_i}{h_i}.$$

5.1.2. Arithmetic Mean Method for 3D Data

$$F_{0,j}^x = \Delta_{0,j} + (\Delta_{0,j} - \Delta_{1,j})\frac{h_0}{(h_0 + h_1)}, \quad F_{m,j}^x = \Delta_{m-1,j} + (\Delta_{m-1,j} - \Delta_{m-2,j})\frac{h_{m-1}}{(h_{m-1} + h_{m-2})}.$$

$$F_{i,j}^x = \frac{\Delta_{i,j} + \Delta_{i-1,j}}{2}. \quad i = 1, 2, \ldots, m-1; \quad j = 0, 1, 2, \ldots, n.$$

$$F_{i,0}^y = \hat{\Delta}_{i,0} + (\hat{\Delta}_{i,0} - \hat{\Delta}_{i,1})\frac{\hat{h}_0}{(\hat{h}_0 + \hat{h}_1)}, \quad F_{i,n}^y = \hat{\Delta}_{i,n-1} + (\hat{\Delta}_{i,n-1} - \hat{\Delta}_{i,n-2})\frac{\hat{h}_{n-1}}{(\hat{h}_{n-1} + \hat{h}_{n-2})}.$$

$$F_{i,j}^y = \frac{\hat{\Delta}_{i,j} + \hat{\Delta}_{i,j-1}}{2}. \quad i = 0, 1, 2 ..., m; \quad j = 1, 2, ..., n-1.$$

$$F_{i,j}^{xy} = \frac{1}{2}\left\{\frac{F_{i,j+1}^x - F_{i,j-1}^x}{\hat{h}_{j-1} + \hat{h}_j} + \frac{F_{i+1,j}^y - F_{i-1,j}^y}{h_{i-1} + h_i}\right\}. \quad i = 1, 2, ..., m-1; \quad j = 1, 2, ..., n-1.$$

where $\Delta_{i,j} = \frac{F_{i+1,j} - F_{i,j}}{h_i}$ and $\hat{\Delta}_{i,j} = \frac{F_{i,j+1} - F_{i,j}}{\hat{h}_j}$. These arithmetic mean methods are computationally economical and suitable for visualization of shaped data.

6. Visualization of 3D Positive Data

Let $(x_i, y_j, F_{i,j})$ be positive data defined over rectangular grid $I = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 0, 1, 2, \ldots, m-1$; $j = 0, 1, 2, \ldots, n-1$ s.t.

$$F_{i,j} > 0, \qquad \forall \quad i, j.$$

The bicubic partially blended surfaces patches (6) inherit all the properties of network of boundary curves [4]. The bicubic partially blended surface (6) is positive if boundary curves $S(x, y_j)$, $S(x, y_{j+1})$, $S(x_i, y)$ and $S(x_{i+1}, y)$ defined in (7), (8), (9) and (10) are positive.

$$S(x, y_j) > 0 \text{ if } \sum_{i=0}^{3} (1-\theta)^{3-i}\theta^i A_i > 0 \text{ and } q_1(\theta) > 0.$$

$$q_1(\theta) > 0 \text{ if } u_{i,j} > 0 \text{ and } v_{i,j} > 0.$$

$$\sum_{i=0}^{3} (1-\theta)^{3-i}\theta^i A_i > 0 \text{ if } A_i > 0, i = 0, 1, 2, 3.$$

$$A_i > 0, i = 0, 1, 2, 3 \text{ if } u_{i,j} > Max\left\{0, \frac{-h_i F_{i,j}^x}{2F_{i,j}}\right\}, \quad v_{i,j} > Max\left\{0, \frac{h_i F_{i+1,j}^x}{2F_{i+1,j}}\right\}.$$

$$S(x, y_{j+1}) > 0 \text{ if } \sum_{i=0}^{3} (1-\theta)^{3-i}\theta^i B_i > 0 \text{ and } q_2(\theta) > 0.$$

$$q_2(\theta) > 0 \text{ if } u_{i,j+1} > 0 \text{ and } v_{i,j+1} > 0.$$

$$\sum_{i=0}^{3} (1-\theta)^{3-i}\theta^i B_i > 0 \text{ if } B_i > 0, i = 0, 1, 2, 3.$$

$$B_i > 0, i = 0,1,2,3 \text{ if } u_{i,j+1} > Max\left\{0, \frac{-h_i F^x_{i,j+1}}{2F_{i,j+1}}\right\}, v_{i,j+1} > Max\left\{0, \frac{h_i F^x_{i+1,j+1}}{2F_{i+1,j+1}}\right\}.$$

$$S(x_i, y) > 0 \text{ if } \sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i C_i > 0 \text{ and } q_3(\phi) > 0.$$

$$q_3(\phi) > 0 \text{ if } \hat{u}_{i,j} > 0 \text{ and } \hat{v}_{i,j} > 0.$$

$$\sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i C_i > 0 \text{ if } C_i > 0, i = 0,1,2,3.$$

$$C_i > 0, i = 0,1,2,3 \text{ if } \hat{u}_{i,j} > Max\left\{0, \frac{-\hat{h}_j F^y_{i,j}}{2F_{i,j}}\right\}, \hat{v}_{i,j} > Max\left\{0, \frac{\hat{h}_j F^y_{i,j+1}}{2F_{i,j+1}}\right\}.$$

$$S(x_{i+1}, y) > 0 \text{ if } \sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i D_i > 0 \text{ and } q_4(\phi) > 0.$$

$$q_4(\phi) > 0 \text{ if } \hat{u}_{i+1,j} > 0 \text{ and } \hat{v}_{i+1,j} > 0.$$

$$\sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i D_i > 0 \text{ if } D_i > 0, i = 0,1,2,3.$$

$$D_i > 0, i = 0,1,2,3 \text{ if } \hat{u}_{i+1,j} > Max\left\{0, \frac{-\hat{h}_j F^y_{i+1,j}}{2F_{i+1,j}}\right\}, \quad \hat{v}_{i+1,j} > Max\left\{0, \frac{\hat{h}_j F^y_{i+1,j+1}}{2F_{i+1,j+1}}\right\}.$$

This leads to the following theorem:

Theorem 6.1. The bicubic partially blended rational function defined in (6) visualize positive data in the view of positive surface if in each rectangular patch $I_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, free parameters $u_{i,j}, v_{i,j}, u_{i,j+1}, v_{i,j+1}, \hat{u}_{i,j}, \hat{v}_{i,j}, \hat{u}_{i+1,j}$ and $\hat{v}_{i+1,j}$ satisfy the following conditions:

$$u_{i,j} = l_{i,j} + Max\left\{0, \frac{-h_i F^x_{i,j}}{2F_{i,j}}\right\}, l_{i,j} > 0, \quad v_{i,j} = m_{i,j} + Max\left\{0, \frac{h_i F^x_{i+1,j}}{2F_{i+1,j}}\right\}, m_{i,j} > 0.$$

$$\hat{u}_{i,j} = n_{i,j} + Max\left\{0, \frac{-\hat{h}_j F^y_{i,j}}{2F_{i,j}}\right\}, n_{i,j} > 0, \quad \hat{v}_{i,j} = o_{i,j} + Max\left\{0, \frac{\hat{h}_j F^y_{i,j+1}}{2F_{i,j+1}}\right\}, o_{i,j} > 0.$$

$$u_{i,j+1} = k_{i,j} + Max\left\{0, \frac{-h_i F^x_{i,j+1}}{2F_{i,j+1}}\right\}, k_{i,j} > 0, \quad v_{i,j+1} = t_{i,j} + Max\left\{0, \frac{h_i F^x_{i+1,j+1}}{2F_{i+1,j+1}}\right\}, t_{i,j} > 0.$$

$$\hat{u}_{i+1,j} = g_{i,j} + Max\left\{0, \frac{-\hat{h}_j F^y_{i+1,j}}{2F_{i+1,j}}\right\}, g_{i,j} > 0, \quad \hat{v}_{i+1,j} = w_{i,j} + Max\left\{0, \frac{\hat{h}_j F^y_{i+1,j+1}}{2F_{i+1,j+1}}\right\}, w_{i,j} > 0.$$

7. Visualization of 3D Constrained Data

Let $(x_i, y_j, F_{i,j})$ be data defined over the rectangular grid

$$I_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}], i = 0,1,2,...,m-1;$$
$$j = 0,1,2,...,n-1,$$

lying above the plane

$$Z = C\left[1 - \frac{x}{A} - \frac{y}{B}\right],$$

i.e.

$$F_{i,j} > Z_{i,j}, \forall i, j.$$

The corresponding surface generated by bicubic partially blended rational function (6) will also lie above the plane if each of the boundary curve $S(x, y_j)$, $S(x, y_{j+1})$, $S(x_i, y)$ and $S(x_{i+1}, y)$ lie above the plane.

The boundary curve $S(x, y_j)$ will lie above the plane if

$$S(x, y_j) > (1-\theta)Z_{i,j} + \theta Z_{i+1,j}. \tag{11}$$

Substituting value of $S(x, y_j)$ in above equation

$$\frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i A_i}{q_1(\theta)} > (1-\theta)Z_{i,j} + \theta Z_{i+1,j}. \tag{12}$$

$q_1(\theta) > 0$ if $u_{i,j} > 0$ and $v_{i,j} > 0$. Multiplying both sides of above equation by $q_1(\theta)$ and after some rearrangement (12) can be rewritten as $U_1(\theta) > 0$, where

$$U_1(\theta) = \sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i M_i, \tag{13}$$

with

$$M_0 = v_{i,j}(F_{i,j} - Z_{i,j}), \quad M_1 = (2u_{i,j}v_{i,j} + v_{i,j})F_{i,j} + v_{i,j}h_i F^x_{i,j} - 2u_{i,j}v_{i,j}Z_{i,j} - v_{i,j}Z_{i+1,j},$$
$$M_2 = (2u_{i,j}v_{i,j} + u_{i,j})F_{i+1,j} - u_{i,j}h_i F^x_{i+1,j} - 2u_{i,j}v_{i,j}Z_{i+1,j} - u_{i,j}Z_{i,j}, \quad M_3 = u_{i,j}(F_{i+1,j} - Z_{i+1,j}).$$

$$U_1(\theta) > 0 \text{ if } \sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i M_i > 0.$$

$$\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i M_i > 0 \text{ if } M_i > 0, i = 0,1,2,3.$$

$$M_i > 0, i = 0,1,2,3 \text{ if } u_{i,j} > Max\left\{0, \frac{-h_i F^x_{i,j} - F_{i,j} + Z_{i+1,j}}{2(F_{i,j} - Z_{i,j})}\right\}, \quad v_{i,j} > Max\left\{0, \frac{h_i F^x_{i+1,j} - F_{i+1,j} + Z_{i,j}}{2(F_{i+1,j} - Z_{i+1,j})}\right\}.$$

The boundary curve $S(x, y_{j+1})$ will lie above the plane if free parameters $u_{i,j+1}$ and $v_{i,j+1}$ satisfy the following conditions

$$u_{i,j+1} > Max\left\{0, \frac{-h_i F^x_{i,j+1} - F_{i,j+1} + Z_{i+1,j+1}}{2(F_{i,j+1} - Z_{i,j+1})}\right\}, \quad v_{i,j+1} > Max\left\{0, \frac{h_i F^x_{i+1,j+1} - F_{i+1,j+1} + Z_{i,j+1}}{2(F_{i+1,j+1} - Z_{i+1,j+1})}\right\}.$$

The boundary curve $S(x_i, y)$ will lie above the plane if

$$S(x_i, y) > (1-\phi)Z_{i,j} + \phi Z_{i,j+1}. \tag{14}$$

Substituting value of $S(x_i, y)$ in above equation

$$\frac{\sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i C_i}{q_3(\phi)} > (1-\phi)Z_{i,j} + \phi Z_{i,j+1}. \tag{15}$$

$q_3(\phi) > 0$ if $\hat{u}_{i,j} > 0$ and $\hat{v}_{i,j} > 0$. Multiplying both sides of above equation by $q_3(\phi)$ and after some rearrangement (15) can be rewritten as $U_2(\phi) > 0$, where

$$U_2(\phi) = \sum_{i=0}^{3} (1-\phi)^{3-i} \phi^i L_i, \tag{16}$$

with

$$L_0 = \hat{v}_{i,j}(F_{i,j} - Z_{i,j}), \quad L_1 = (2\hat{u}_{i,j}\hat{v}_{i,j} + \hat{v}_{i,j})F_{i,j} + \hat{v}_{i,j}\hat{h}_j F_{i,j}^y - 2\hat{u}_{i,j}\hat{v}_{i,j}Z_{i,j} - \hat{v}_{i,j}Z_{i,j+1},$$

$$L_2 = (2\hat{u}_{i,j}\hat{v}_{i,j} + \hat{u}_{i,j})F_{i,j+1} - \hat{u}_{i,j}\hat{h}_j F_{i,j+1}^y - 2\hat{u}_{i,j}\hat{v}_{i,j}Z_{i,j+1} - \hat{u}_{i,j}Z_{i,j}, \quad L_3 = \hat{u}_{i,j}(F_{i,j+1} - Z_{i,j+1}).$$

$$U_2(\phi) > 0 \text{ if } \sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i L_i > 0,$$

$$\sum_{i=0}^{3}(1-\phi)^{3-i}\phi^i L_i > 0 \text{ if } L_i > 0, i = 0,1,2,3.$$

$$L_i > 0, i = 0,1,2,3 \text{ if } \hat{u}_{i,j} > Max\left\{0, \frac{-\hat{h}_j F_{i,j}^y - F_{i,j} + Z_{i,j+1}}{2(F_{i,j} - Z_{i,j})}\right\}, \quad \hat{v}_{i,j} > Max\left\{0, \frac{\hat{h}_j F_{i,j+1}^y - F_{i,j+1} + Z_{i,j}}{2(F_{i,j+1} - Z_{i,j+1})}\right\}.$$

The boundary curve $S(x_{i+1}, y)$ will lie above the plane if free parameters $\hat{u}_{i+1,j}$ and $\hat{v}_{i+1,j}$ satisfy the following conditions:

$$\hat{u}_{i+1,j} > Max\left\{0, \frac{-\hat{h}_j F_{i+1,j}^y - F_{i+1,j} + Z_{i+1,j+1}}{2(F_{i+1,j} - Z_{i+1,j})}\right\}, \quad \hat{v}_{i+1,j} > Max\left\{0, \frac{\hat{h}_j F_{i+1,j+1}^y - F_{i+1,j+1} + Z_{i+1,j}}{2(F_{i+1,j+1} - Z_{i+1,j+1})}\right\}.$$

Therefore, we can conclude above discussion in the following theorem:

Theorem 7.1. The bicubic partially blended rational function (6) generates surface that lies on same side of plane as that of data if in each rectangular patch $I_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, free parameters $u_{i,j}, v_{i,j}, u_{i,j+1}, v_{i,j+1}, \hat{u}_{i,j}, \hat{v}_{i,j}, \hat{u}_{i+1,j}$ and $\hat{v}_{i+1,j}$ satisfy the following conditions:

$$u_{i,j} = \tilde{k}_{i,j} + Max\left\{0, \frac{-h_i F_{i,j}^x - F_{i,j} + Z_{i+1,j}}{2(F_{i,j} - Z_{i,j})}\right\}, \tilde{k}_{i,j} > 0, \quad v_{i,j} = \tilde{t}_{i,j} + Max\left\{0, \frac{h_i F_{i+1,j}^x - F_{i+1,j} + Z_{i,j}}{2(F_{i+1,j} - Z_{i+1,j})}\right\}, \tilde{t}_{i,j} > 0.$$

$$\hat{u}_{i,j} = \tilde{r}_{i,j} + Max\left\{0, \frac{-\hat{h}_j F_{i,j}^y - F_{i,j} + Z_{i,j+1}}{2(F_{i,j} - Z_{i,j})}\right\}, \tilde{r}_{i,j} > 0, \quad \hat{v}_{i,j} = \tilde{w}_{i,j} + Max\left\{0, \frac{\hat{h}_j F_{i,j+1}^y - F_{i,j+1} + Z_{i,j}}{2(F_{i,j+1} - Z_{i,j+1})}\right\}, \tilde{w}_{i,j} > 0.$$

$$u_{i,j+1} = \tilde{l}_{i,j} + Max\left\{0, \frac{-h_i F_{i,j+1}^x - F_{i,j+1} + Z_{i+1,j+1}}{2(F_{i,j+1} - Z_{i,j+1})}\right\}, \tilde{l}_{i,j} > 0,$$

$$v_{i,j+1} = \tilde{m}_{i,j} + Max\left\{0, \frac{h_i F_{i+1,j+1}^x - F_{i+1,j+1} + Z_{i,j+1}}{2(F_{i+1,j+1} - Z_{i+1,j+1})}\right\}, \quad \tilde{m}_{i,j} > 0.$$

$$\hat{u}_{i+1,j} = \tilde{n}_{i,j} + Max\left\{0, \frac{-\hat{h}_j F_{i+1,j}^y - F_{i+1,j} + Z_{i+1,j+1}}{2(F_{i+1,j} - Z_{i+1,j})}\right\}, \tilde{n}_{i,j} > 0,$$

$$\hat{v}_{i+1,j} = \tilde{o}_{i,j} + Max\left\{0, \frac{\hat{h}_j F_{i+1,j+1}^y - F_{i+1,j+1} + Z_{i+1,j}}{2(F_{i+1,j+1} - Z_{i+1,j+1})}\right\}, \tilde{o}_{i,j} > 0.$$

8. Demonstration

We shall illustrate the shape preserving schemes developed in Section 3, 6 and 7 with some examples. A positive data set is considered in Table 1.

Table 1.

x	1	2	4	10	28	30	32
y	20.8	8.8	3.2	0.1	3.9	6.2	9.6

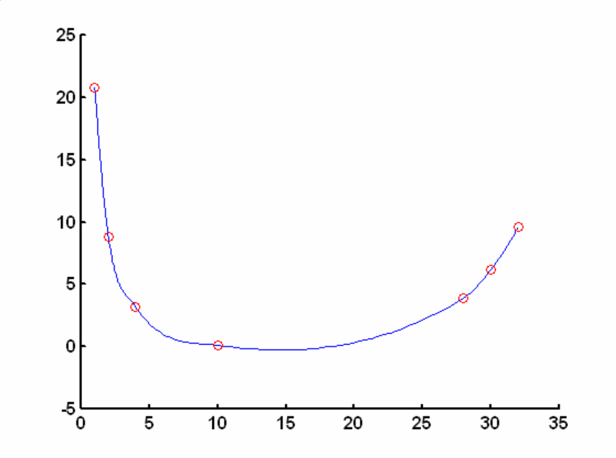

Fig. 1

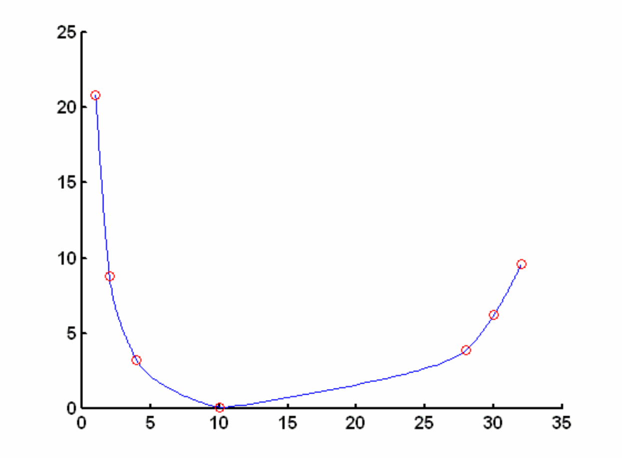

Fig. 2

Figure 1 is produced using Cubic Hermite Spline which failed to preserve positive shape of data. Figure 2 is produced by positivity preserving scheme developed by Zawwar and Jamaludin in [6].

The data set for second example shown in Table 2, is lying above the straight line: $y = x + 2$.

Table 2.

x	0	2	4	10	28	30	32
y	22.8	12.8	10.2	12.5	33.9	38.9	43.6

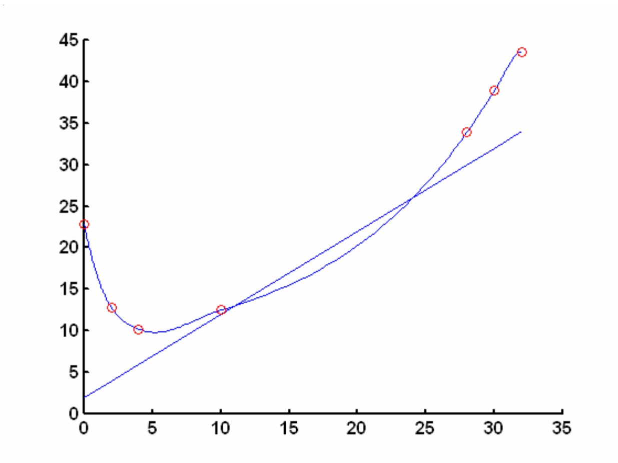

Fig. 3

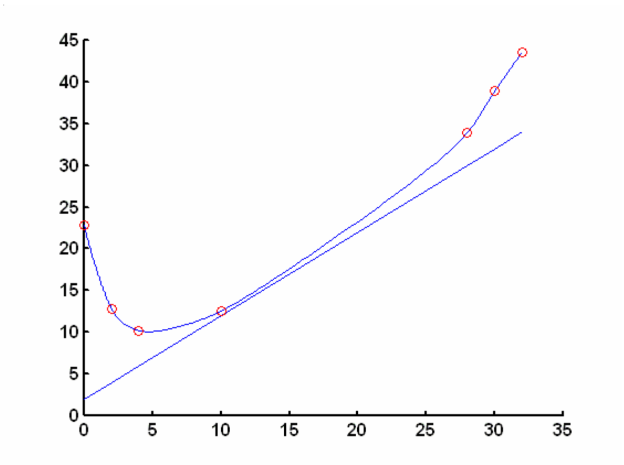

Fig. 4

Figure 3 is produced using Cubic Hermite Spline. This curve does not lie above the line $y = x + 2$. This flaw is recovered nicely in Figure 4 using scheme of Section 3 with $l_i = m_i = 0.1$.

The data set in Table 3 is lying above the straight line: $y = x/2 + 1$.

Table 3.

x	2	3	7	8	9	13	14
y	12	4.5	6.5	12	7.5	9.5	18

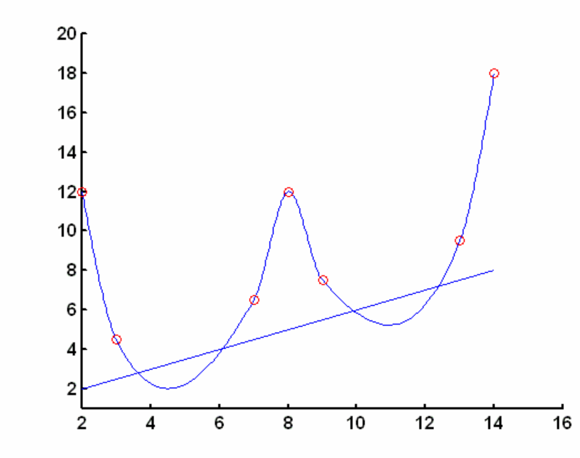

Fig. 5

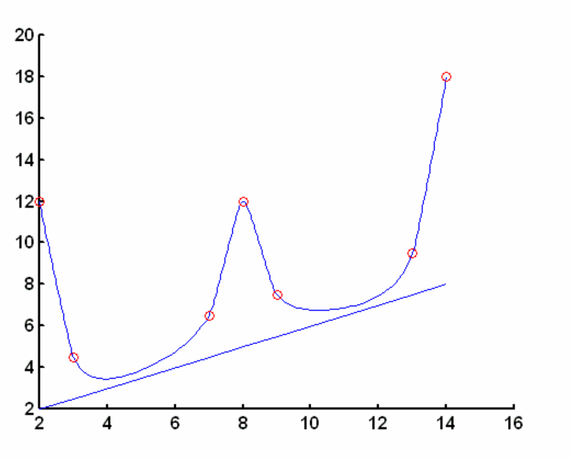

Fig. 6

Figure 5 is produced using Cubic Hermite Spline. This curve does not lie above the line $y = x/2+1$. This flaw is recovered nicely in Figure 6 using scheme of Section 3 with $l_i = m_i = 0.3$.

The positive data set in Table 4 is generated from the following function:

$$F(x, y) = 13.12 + 0.6215x - 0.37 y^{0.16} + 0.3965 y^{0.16}.$$

Table 4.

y/x	0.01	100	200	300
0.01	12.9510	94.0706	175.1983	256.3260
100	12.3615	157.3376	302.3281	447.3187
200	12.2718	166.9630	321.6697	476.3765
300	12.2145	173.1087	334.0191	494.9294

The positive surface generated by scheme in Section 6 is shown in Figure 7 with $l_{i,j} = m_{i,j} = n_{i,j} = o_{i,j} = k_{i,j} = t_{i,j} = g_{i,j} = w_{i,j} = 0.4$.

The positive data set in Table 5 is generated from the following function:

$$F(x, y) = 0.5\left(\left\| x\right| - \left| y\right\| - \left| x\right| - \left| y\right|\right) + 3.1, -3 \le x, y \le 3.$$

Table 5.

y/x	-3	-2	-1	1	2	3
-3	0.1	1.1	2.1	2.1	1.1	0.1
-2	1.1	1.1	2.1	2.1	1.1	1.1
-1	2.1	2.1	2.1	2.1	2.1	2.1
1	2.1	2.1	2.1	2.1	2.1	2.1
2	1.1	1.1	2.1	2.1	1.1	1.1
3	0.1	1.1	2.1	2.1	1.1	0.1

The positive surface generated by scheme in Section 6 with

$$l_{i,j} = m_{i,j} = n_{i,j} = o_{i,j} = k_{i,j} = t_{i,j} = g_{i,j} = w_{i,j} = 0.5$$

is shown in Figure 8.

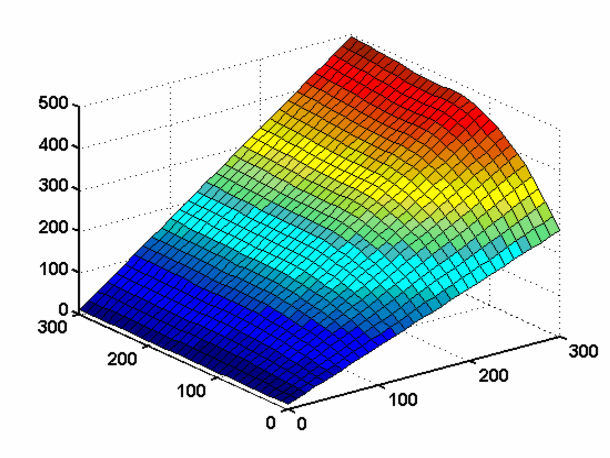

Fig. 7

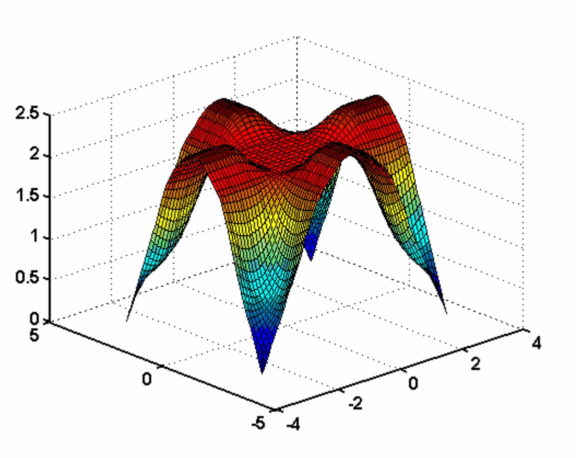

Fig. 8

The data set in Table 6 is of the following plane: $Z = \left(1 - \frac{x}{6} - \frac{y}{6}\right), 1 \le x, y \le 6.$

Table 6.

y/x	1	2	3	4	5	6
1	0.6667	0.5000	0.3333	0.1667	0.0000	-0.1667
2	0.5000	0.3333	0.1667	0.0000	-0.1667	-0.3333
3	0.3333	0.1667	0.0000	-0.1667	-0.3333	-0.5000
4	0.1667	0.0000	-0.1667	-0.3333	-0.5000	-0.6667
5	0.0000	-0.1667	-0.3333	-0.5000	-0.6667	-0.8333
6	-0.1667	-0.3333	-0.5000	-0.6667	-0.8333	-1.0000

The data in Table 7 is of the function: $F(x, y) = Sin^2 x + y^2 / 4 + 0.1, 1 \le x, y \le 6.$

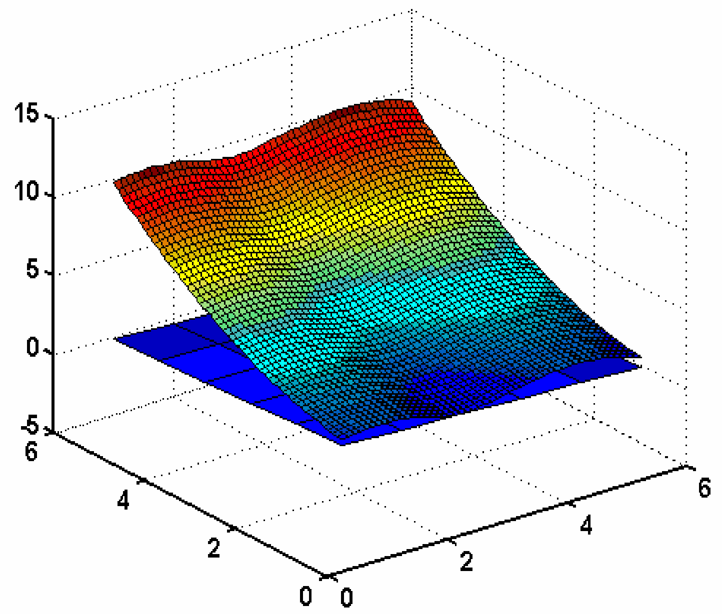

Fig. 9

Table 7.

y/x	1	2	3	4	5	6
1	1.0581	1.1768	0.3699	0.9228	1.2695	0.4281
2	1.8081	1.9268	1.1199	1.6728	2.0195	1.1781
3	3.0581	3.1768	2.3699	2.9228	3.2695	2.4281
4	4.8081	4.9268	4.1199	4.6728	5.0195	4.1781
5	7.0581	7.1768	6.3699	6.9228	7.2695	6.4281
6	9.8081	9.9268	9.1199	9.6728	10.0195	9.1781

This data is lying above the plane defined in Table 6. Figure 9 is produced by surface scheme developed in Section 7 with $\tilde{k}_{i,j} = \tilde{t}_{i,j} = \tilde{r}_{i,j} = \tilde{w}_{i,j} = \tilde{l}_{i,j} = \tilde{m}_{i,j} = \tilde{n}_{i,j} = \tilde{o}_{i,j} = 0.6.$ From figure it is clear that surface is lying above the plane as required.

9. Conclusion

In this paper the problem of positive interpolation of curves and surfaces is discussed. Simple constraints are developed on free parameters in the description of rational cubic and rational bicubic function to visualize positive data. Choice of the derivative parameters is left at the wish of the user as well. The method is very easy to implement as compared to methods already developed ([2],[5],[7]), local, computationally economical and visually pleasing.

10. References

[1] M. R. Asim and K. W. Brodlie. Curve drawing subject to positivity and more general constraints, *Computers and Graphics*, 2003, **27**: 469-485.

[2] K. W. Brodlie, P. Mashwama and S. Butt. Visualization of surface data to preserve positivity and other simple constraints. *Computers and Graphics*, 1995, **19**(4): 585-594.

[3] S. Butt and K. W. Brodlie. Preserving positivity using piecewise cubic interpolation. *Computers and Graphics*, 1993, **17**(1): 55-64.

[4] G. Casciola and L. Romani. Rational interpolants with tension parameters. *Curve and Surface Design*, Tom Lyche, Marie-Laurence Mazure and Larry L. Schumaker (eds), 2003, 41-50.

[5] M. Z. Hussain. Shape preserving curves and surfaces for computer graphics, Ph.D. Thesis, University of the Punjab, Lahore, Pakistan, 2002.

[6] M. Z. Hussain and J. M. Ali. Positivity preserving piecewise rational cubic interpolation. *Matematika*, 2006, **22**(1).

[7] A. Kouibia and M. Pasadas. Variational bivariate interpolating splines with positive constraints, *Appl. Numer. Math*, 2003, **44**: 507-526.

[8] C. Manni. On shape preserving C^2 Hermite interpolation. *BIT*, 2001, **14**: 127-148.

[9] E. Nadler. Non-negativity of bivariate quadratic functions on a triangle. *Computer Aided Geometric Design*, 1992, **9**: 195-205.

[10] A. R. M. Piah, T. N. T. Goodman and K. Unsworth. Positivity preserving scattered data interpolation. *Proc. 11th IMA Mathematics of Surfaces Conference,* Loughborough, U. K., September 5-7 2005, 336-349.

[11] M. Sarfraz, M. Z. Hussain and S. Butt. A rational Spline for visualizing positive data. *Proc. IEEE, International Conference on Information Visualization,* London, U. K., July 19-21 2000, 57-62.

[12] M. Sarfraz, S. Butt and M. Z. Hussain. Visualization of shaped data by a rational cubic Spline interpolation, *Computers and Graphics*, 2001, **25**: 833-845.

[13] J. W. Schmidt and W. Hess. Positivity interpolation with rational quadratic Splines. *Computing*, 1987, **38**: 261-267.

[14] J. W. Schmidt and W. Hess. Positivity of cubic polynomial on intervals and positive Spline interpolation, *BIT*, 1988, **28**: 340-352.