ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 1, No. 2, 2006, *pp*. 78-84

Slow Manifold Model and Simulation of the Lü system *

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(*Received Sep. 09, 2005, accepted Nov. 10, 2005*)

Abstract. Based on geometric singular perturbation theory, we discuss the existence of slow manifold model of some chaotic systems such as the Lü's system, the Lorenz system, the Chen system and the Chua's system. Equations of the first order approximate slow manifold are given by using standard geometric singular perturbation method. Some numerical simulation results are also presented.

Keywords: slow manifold model, chaotic systems, geometric singular perturbation

1. Introduction

In recent years, great progress has been made in research of nonlinear chaotic dynamics. In 1963, E. N. Lorenz discovered the Lorenz system. Its mathematical model is a system of nonlinear ordinary differential equations which has the following form

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = -xz + rx - y \\ \dot{z} = xy - bz \end{cases} \tag{1}$$

Taken σ=10,b=8/3, r=28, the Lorenz system has a chaotic attractor, see [1].

In 1999, Guanrong Chen, et al. discovered another chaotic attractor in studying anti-controlling chaos, and named it Chen's system. It also has the form of nonlinear ordinary differential equations

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = (r - \sigma)x - xz + ry \\ \dot{z} = xy - bz \end{cases} \tag{2}$$

Taken σ=35, b =3, r=28, the Chen system has a chaotic attractor, see [2].

In 2002, also in studying anti-controlling chaos, Jinhu Lü, et al. discovered a new chaotic system, Lü system; its mathematical model also is a system of nonlinear ordinary differential equations

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = -xz + ry \\ \dot{z} = xy - bz \end{cases} \tag{3}$$

Given σ=36, b=3, r=20, Lü system has a chaos attractor, see [3].

One can see that the three dynamical systems, all are nonlinear ordinary differential equation systems with degree three. And all x derivative in those systems are multiplied by a number σ. The only difference is the form of y derivative. According to a critical term $a_{12}a_{21}$ that put forward by Vanecek and Celkovsk'y in 1966(see [4]), the above systems belong to different kinds of type: the Lorenz system: $a_{12}a_{21}>0$; the Chen system: $a_{12}a_{21}<0$; the Lü system: $a_{12}a_{21}=0$. This reflected the well relation of these three systems. The Lü's

* This Research was supported by the National Natural Science Foundation of China (No. 90210004), the National Social Science Foundation of China (No. 02BJY051) and the Department of Education Foundation of Jiangsu Province (No. 03SJB790008).

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system connected the Lorenz system with the Chen system, and represented their transformation continuously. So they may have some common properties. It is known all that they have chaotic attractors. How about their invariant manifolds? If we find their invariant manifolds, we can get an approximate knowledge of their dynamical behavior.

Because all x derivatives in those systems are multiplied by a numberσ , while y and z derivatives are not, they look like slow-fast systems. Results in slow-fast systems are fruitful. Can some chaotic systems be seen as slow-fast systems so one can use results of slow-fast systems instead of numerical analysis?

In this paper, we take the three systems as slow-fast autonomous dynamic systems. In slow-fast auto dynamic systems, variables are separated into two groups. One is fast variables, another is slow variables. The behaviors of the whole system are determined by its slow variables. Slow system usually are simpler than their original ones. One can get a geometrical view of the whole dynamical behavior by geometric perturbation theory founded by Fenichel (see [5]).

Geometric singular perturbation theory is a good method for qualitative and quantitative analysis of slow-fast autonomous dynamical system (see [5,6]). Jones extended Fenichel's theory to solve the existence of solitary wave of some PDEs (see [7]). The existence results in this work are well known. Our motivation for this paper is to show that the Fenichel geometric methods not only give valuable insight into general ordinary differential equations, but also provide a treatment that generalizes to some chaotic systems. We do this attempt starting with the slow manifold of chaotic systems. There are many other techniques that can be used on the slow manifold problem. Ramdani, Rossetto et al gave the slow manifold of some singular system including the Lorenz and Chua's system in nonstandard method (see [8,9]). In this paper we follow the standard way.

Because all applications are based on the slow manifold, it is very important to investigate the expression of slow manifold.

2. Slow manifold model of Lü system and Simulation

To the Lü system.

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = -xz + ry \\ \dot{z} = xy - bz \end{cases}$$

It is easy to see that original O (0,0,0) is an equilibrium for any parameter in Lü system. When r>0, the original is an unstable saddles. When b>0, r>0, the system have two other equilibrium $A(\sqrt{br},\sqrt{br},r), B(-\sqrt{br},-\sqrt{br},r)$, and when σ =36,b=3, r=20, the Lü system has a chaotic attractor (see [3]). Here the system becomes

$$\begin{cases} \dot{x} = 36(y-x) \\ \dot{y} = -xz + 20y \\ \dot{z} = xy - 3z \end{cases} \tag{4}$$

Taking ε=1/36, then we can treat the system (4) as slow-fast autonomous system. Therefore we can precede the qualitative analysis by using geometric singular perturbation method.

The slow system is

$$\begin{cases} \varepsilon\dot{x} = y-x \\ \dot{y} = -xz + 20y \\ \dot{z} = xy - 3z \end{cases} \tag{5}$$

where *x* is the fast variable, while *y* and *z* are slow variables. The dualistic system of (5), namely fast system is

$$\begin{cases} x'_\tau = y-x \\ y'_\tau = \varepsilon(-xz + 20y) \\ z'_\tau = \varepsilon(xy - 3z) \end{cases} \tag{6}$$

where $\tau = t/\varepsilon$ is called the fast-time scale, and t is called the slow-time scale. As long as $\varepsilon \neq 0$, system (6) is equivalent to system (5).

In fast system, letting $\varepsilon \to 0$, we obtain a zero order approximate slow manifold M_0: $x = y$. Obviously, the dimension of M_0 is 2. On M_0, the Jacobin matrix of system (6) is

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which just contains Dim (M_0)=2 eigenvalues with zero real part (in fact, they are zero real roots), therefore M_0 is normal hyperbolic. Fenichel's persistence theorem states that, provided some hypotheses are satisfied, system (5) possesses a slow manifold that can be written as the graph of a function. The most critical hypotheses are satisfied in systems (5), so by Fenichel's theorem there exists a local invariable manifold M_ε for system (5).

M_ε presents many good properties, for example, M_ε is $O(\varepsilon)$ close to M_0, M_ε is local invariable, If M_0 is C^r $(0<r<\infty$) and can be described as the graph of a smooth function, so does M_ε.

The slow manifold M_ε given by Fenichel's theorem plays a central role in systems of the form (5). The dynamics on it is slow, while, due to its hyperbolic structure, nearby trajectories approach it exponentially. Also, the $O(\varepsilon)$ closeness of M_0 and M_ε is crucial for showing the existence of solutions to the three systems we present here .

In general, one cannot directly calculate the expression of M_ε. In the following we investigate the leading order expression of M_ε. From the invariable manifold theory by Fenichel, because of the $O(\varepsilon)$ closeness of M_0 and M_ε, the relation of variables x and y in M_ε is $x = y + O(\varepsilon)$. We expand it in ε series:

$$x = y + \varepsilon H(y,z) + O(\varepsilon^2). \tag{7}$$

Because M_ε is local invariable manifold of (5), so derivate (7) to get

$$\dot{x} = \dot{y} + \varepsilon \frac{\partial H}{\partial y}\dot{y} + \varepsilon \frac{\partial H}{\partial z}\dot{z} + O(\varepsilon^2) \tag{8}$$

and

$$\begin{aligned} \varepsilon\dot{x} &= \varepsilon\dot{y} + O(\varepsilon^2) \\ &= \varepsilon\{-[y + \varepsilon H(y,z) + O(\varepsilon^2)]z + 20y\} + O(\varepsilon^2) \\ &= \varepsilon(20y - yz) + O(\varepsilon^2) \end{aligned} \tag{9}$$

Plugging formula (7) into the first term of (5) one has

$$\begin{aligned} \varepsilon\dot{x} &= y - [y + \varepsilon H(y,z) + O(\varepsilon^2)] \\ &= -[\varepsilon H(y,z) + O(\varepsilon^2)] \end{aligned} \tag{10}$$

Comparing the coefficients of the same power of ε in (9) and (10), one gets

$$H(y,z)=yz-20y.$$

Thus the fist order expression of the slow manifold M_ε of the Lü's system (2), i.e. slow manifold model of the Lü's system (2) is

$$x=y+\varepsilon(yz-20y). \tag{11}$$

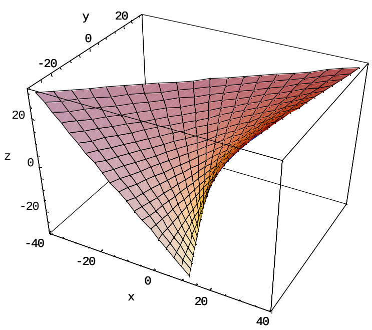

Fig. 1 Slow manifold of slow-fast Lü's system

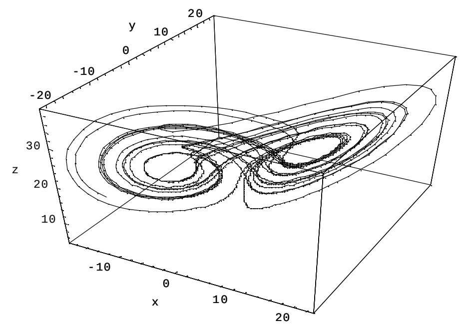

Fig. 2 Strange trajectories of slow-fast Lü's system

Through the numerical simulation by the software matlab, we get its graph is given in figure 1. Figure 2 shows the strange trajectories of system (3).

Using the same method, we can discuss the slow manifold model of the Lorenz system and the Chen system. The first order expression of the slow manifold M_ε of the Chen system is

$$x=y+\varepsilon(yz-21y) \quad (12)$$

Its graph is given in figure 3. Figure 4 shows the strange trajectories of system (2).

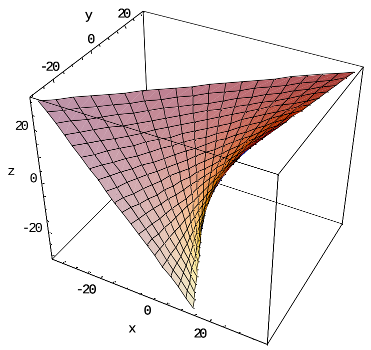

Fig. 3 Slow manifold of slow-fast Chen system

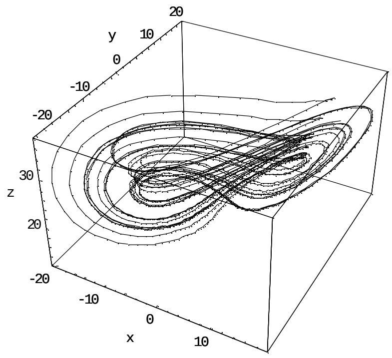

Fig. 4 Strange trajectories of slow-fast Chen system

The fist order expression of the slow manifold M_ε of the Lorenz system is

$$x=y+\varepsilon(yz-27y) \quad (13)$$

Its graph and trajectories can be seen in figure 5 and figure 6, respectively.

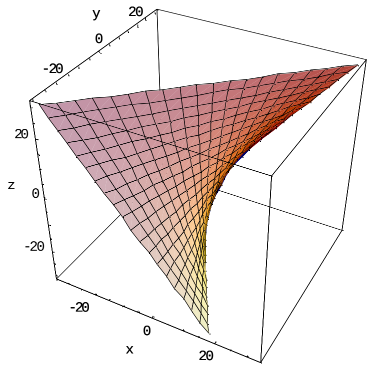

Fig. 5 Slow manifold of slow-fast Lorenz system

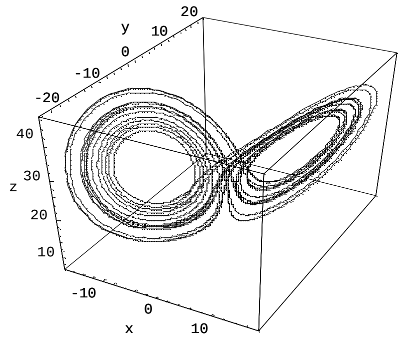

Fig. 6 Strange trajectories of slow-fast Lorenz system

Geometric singular perturbation theory only requires the limiting slow system and the limiting fast system, which are both dimension-two systems and much simpler than their original systems. No explicit information about their solution is required. We can make a brief image of system's dynamics. It is easy to proof that three equilibrium of the Lorenz system, $O(0,0,0)$, $A(\sqrt{br-1},\sqrt{br-1},r-1), B(-\sqrt{br-1},-\sqrt{br-1},r-1)$, all satisfy equation (13), which means that they are all in the slow manifold M_ε. Therefore we can use geometric singular perturbation theory to analysis the qualitative behavior and orbits of the system. Given initial data ($x(0)$, $y(0)$, $z(0)$), because the velocity of x, y, z is different, x is a fast variable, but y and z are slow variables, the fast movement takes places first, x changing very fast, but y and z remaining almost unchanged. Then x attains the half stability condition that is x reaches in the slow manifold. In the slow manifold, x, y and z all change slowly and the movement attains a certain equilibrium point (one of the *O, A, B*, for example *O*), but can't forever stay in the equilibrium point. Because x just attains half stability condition on slow manifold, it must lose stable and begin fast movement again. The fast movement ends very fast, the slow movement takes place again. Therefore the fast movement and the slow movement exercise alternately. The behavior of the system is most like this: it goes out from the point *O*, and then comes into the point *A*, then turns out from *A* into the point *O*, then leaves the point *O* and head for point *B*, turn out from point *B* again and come into the point *O*. A period begins after a period ending, going round and starting again, the back and forth is continuously. The circumvolution round the points A and B is on the plane approximately, however its type and numbers of turning are irregular, as a result it becomes the strange attractor like butterfly wings.

We analyze the relations and the difference of the slow manifold models among three kinds of systems as following.

Because the three systems are similar in many ways, the graphs of their slow manifolds also look like the same, which all are slightly folded planes. Because slow manifolds are exponentially attracting all trajectories starting in the attractor tend to the slightly folded plans, so one can see butterflies wings. Another interesting thing is that the coefficients in the expression of slow manifold is just as the difference of first power term coefficient of variables x and y in second equation.

3. Slow manifold model of the Chua's system and simulation

In order to comparing the method of this paper with the method of Ramdani and Rossetto, in the following we discuss the slow manifold of the Chua's system. The Chua's system is an electric circuit system. Because of its extensive applied foregroud, the Chua's system has become a new focus in research of controlling nonlinear chaotic electric circuit and investigation of the nerve network in recent years. The Chua's electric circuit system can emerge the state of the quantitative dynamics in any three order nonlinear system, among them including a subsection linear function of three parts with singular symmetry. The dynamics equation of the Chua's system can be expressed as the system of nonlinear differential equations as follows

$$\begin{cases} \varepsilon\dot{x} = y - x - f(x) \\ \dot{y} = x - y + z \\ \dot{z} = -\beta y \end{cases} \tag{14}$$

For $f(x)$, we still adopt the expression of Ramdani in [8], namely

$$f(x) = \begin{cases} bx + a - b & x \geq 1 \\ ax & -1 \leq x \leq 1 \\ bx - a + b & x \leq -1 \end{cases}$$

System (14) is a slow-fast autonomous system, where x is the fast variable, y and z are slow ones. Given ε=1/9, a= -8/7, b= -5/7, β =100/7, the fast system is

$$\begin{cases} x_\tau' = y - x - f(x) \\ y_\tau' = \varepsilon(x - y - z) \\ z_\tau' = -\varepsilon\beta y \end{cases} \tag{15}$$

Where $\tau = t/\varepsilon$ is a fast variable.

The equilibrium points of (14) are $O(0,0,0)$, $A(\frac{b-a}{1+b},0,\frac{a-b}{1+b}), B(\frac{a-b}{1+b},0,\frac{b-a}{1+b})$, when $|x| > 1$, the Jacobian matrix is

$$J_1 = \begin{pmatrix} -\frac{1+b}{\varepsilon} & \frac{1}{\varepsilon} & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}$$

At given parameter value, we can get its eigenvalues $\lambda_1 = -3.94, \lambda_{2,3} = 0.18 \pm 3.04i$, so A, B are unstable foci.

When $|x| < 1$, the Jacobian matrix is

$$J_1 = \begin{pmatrix} -\frac{1+b}{\varepsilon} & \frac{1}{\varepsilon} & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = 2.21, \lambda_{2,3} = -0.965 \pm 2.711i$, so A, B still are unstable foci.

In fast system (15), letting $\varepsilon \to 0$, we can get the zero order approximate slow manifold M_0:

$$x = \begin{cases} \frac{y-a+b}{1+b} & y \geq -(1+a) \\ \frac{y+a-b}{1+b} & y \leq 1+a \end{cases}$$

The dimension of the zero order approximate slow manifold M_0 is 2.

When y >=-(1+ a) or y<=1+ a, the Jacobian matrix of the system (15) on the zero order approximate slow manifold M_0 is

$$J = \begin{pmatrix} -1-b & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Its eigenvalues are two zero roots and -1- b, thus the zero order approximate slow manifold M_0 satisfies normal hyperbolic condition. We look for its fist order approximately slow manifold M_ε.

When y>=-(1+ a), let the equation of the slow manifold be

$$x = \frac{y-a+b}{1+b} + \varepsilon h(y,z) + O(\varepsilon^2)$$

Derive the above formula

$$\dot{x} = \frac{1}{1+b}\dot{y} + \varepsilon\frac{\partial H}{\partial y}\dot{y} + \varepsilon\frac{\partial H}{\partial z}\dot{z} + O(\varepsilon^2)$$

$$= \frac{1}{1+b}((x-y+z) + \varepsilon\frac{\partial H}{\partial y}\dot{y} + \varepsilon\frac{\partial H}{\partial z}\dot{z} + O(\varepsilon^2)$$

$$\varepsilon\dot{x} = \frac{1}{1+b}(\frac{y-a+b}{1+b} - y + z)\varepsilon + O(\varepsilon^2)$$

On the other hand,

$$\varepsilon\dot{x} = y - x - f(x)$$

$$= y - x - bx - a + b$$

$$= -(1+b)H(y,z)\varepsilon + O(\varepsilon^2)$$

$$\therefore H(y,z) = -\frac{1}{(1+b)^2}(\frac{y-a+b}{1+b} - y + z)$$

$$= -\frac{1}{(1+b)^3}(b - a - by + (1+b)z)$$

$$\therefore M_\varepsilon : x = \frac{y-a+b}{1+b} - \varepsilon\frac{1}{(1+b)^3}(b - a - by + (1+b)z)$$

When y<=1+ a, by similar calculating we can get

$$M_\varepsilon : x = \frac{y+a-b}{1+b} - \varepsilon\frac{1}{(1+b)^3}(a - b - by + (1+b)z)$$

So the slow manifold model is divided into two parts and its equation is

$$M_\varepsilon : x = \begin{cases} \frac{y-a+b}{1+b} - \varepsilon \frac{1}{(1+b)^3}(b-a-by+(1+b)z) & y \geq -(1+a) \\ \frac{y+a-b}{1+b} - \varepsilon \frac{1}{(1+b)^3}(a-b-by+(1+b)z) & y \leq 1+a \end{cases}$$

Through the numerical simulation by the software matlab, we get its graph is two disjunctive half planes which can be seen in figure 7.

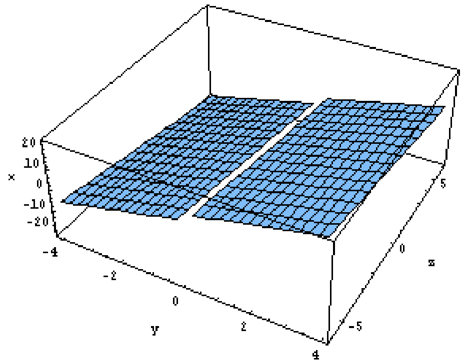

Figure 7 Slow manifold model of the Chua's system

4. Conclusion

We give the slow manifold model that is concrete and terse of the Lü's system by geometric singular perturbation theory. The amount of calculation of our method is quit little in this paper, because we only need to calculate the eigenvalues of degenerate fast sub-system and then make power expanding. Because the degenerate system is a simple system, the analytic expression of the slow manifold M_ε is very clear. Thus it is easy to distinguish the relation between equilibrium and slow manifold, and it is also easy to make qualitative analysis of the orbits theoretically and numerical simulation.

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