Spectral Theory of Differential Operators and Energy Levels of Subatomic Particles

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Abstract. Motivated by the Bohr atomic model, in this article we establish a mathematical theory to study energy levels, corresponding to bounds states, for subatomic particles. We show that the energy levels of each subatomic particle are finite and discrete, and corresponds to negative eigenvalues of the related eigenvalue problem. Consequently there are both upper and lower bounds of the energy levels for all subatomic particles. In particular, the energy level theory implies that the frequencies of mediators such as photons and gluons are also discrete and finite. Both the total number N of energy levels and the average energy level gradient (for two adjacent energy levels) are rigorously estimated in terms of certain physical parameters. These estimates show that the energy level gradient is extremely small, consistent with the fact that it is hard to notice the discrete behavior of the frequency of subatomic particles.

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1 Introduction

This article is part of a research program initiated in the last few years by the authors to derived experimentally verifiable laws of Nature based only on a few fundamental first principles, guided by experimental and observation evidences. The new theory we have established gives rise to solutions and explanations to a number of longstanding mysteries in modern theoretical physics. This work is synthesized in a recent by by the authors [8].

Basically, we have discovered three fundamental principles: the principle of interaction dynamics (PID) [5], the principle of representation invariance (PRI) [6], and the principle of symmetry-breaking (PSB) for unification [8].

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PID takes the variation of the action under the energy-momentum conservation constraints, and is required by the dark matter and dark energy phenomena for gravity), by the quark confinements (for strong interaction), and by the Higgs field (for the weak interaction).

PRI requires that the gauge theory be independent of the choices of the representation generators. These representation generators play the same role as coordinates, and in this sense, PRI is a coordinate-free invariance/covariance, reminiscent of the Einstein principle of general relativity. In other words, PRI is purely a logic requirement for the gauge theory.

PSB offers an entirely different route of unification from the Einstein unification route which uses large symmetry group. The three sets of symmetries — the general relativistic invariance, the Lorentz and gauge invariances, as well as the Galileo invariance — are mutually independent and dictate in part the physical laws in different levels of Nature. For a system coupling different levels of physical laws, part of these symmetries must be broken.

This article is motivated 1) by the classical atomic energy level theory, 2) the weakton model of elementary particles [7], and 3) the new field theory for the four fundamental interactions [5,6].

The classical atomic energy level theory demonstrates that there are finite number of energy levels for an atom given by $E_n = E_0 + \lambda_n$, $n = 1, \cdots, N$, where λ_n are the negative eigenvalues of the Schrödinger operator, representing the bound energies of the atom, holding the orbital electrons, due to the electromagnetism.

The weakton model of elementary particles and the unified field theory are developed recently by the authors [5–7]. The field theory is based on two recently postulated principles by the authors: the principle of interaction dynamics (PID) and the principle of representation invariance (PRI). Intuitively, PID takes the variation of the action under energy-momentum conservation constraint. PID offers a completely different and natural way of introducing Higgs fields, and is also required by the quark confinement. For gravity, we show that PID is the direct consequence of Einsteins principle of general relativity and the presence of dark matter and dark energy. PRI requires that the $SU(N)$ gauge theory be independent of representations of $SU(N)$. PRI has remarkably rich physical consequences.

The main objectives of this article are 1) to introduce the energy levels for all subatomic particles, 2) to develop a mathematical theory to study energy levels, and 3) to derive physical implications and predictions of the theory.

Hereafter we explore the key ingredients and the main results in this article.

FIRST, we develop a mathematical theory for estimating the number of negative eigenvalues for a class of differential operators, and consequently the theory is applied to study the energy levels for subatomic particles.

SECOND, the constituents of subatomic particles are spin-$\frac{1}{2}$ fermions, which are bound

together by either weak or strong interactions. Hence the starting point of the study is the layered weak and strong potentials derived recently by the authors [6], which play the similar role as the Coulomb potential for the electromagnetic force which bounds the orbital electrons moving around the nucleons.

THIRD, dynamic equations of massless particles are the Weyl equations, and the dynamic equations for massive particles are the Dirac equations. The bound energies of all subatomic particles are the negative eigenvalues of the corresponding Dirac and Weyl operators, and the bound states are the corresponding eigenfunctions.

The Weyl equations were introduced by H. Weyl in 1929 to describe massless spin-$\frac{1}{2}$ free particles [11], which is now considered as the basic dynamic equations of neutrinos [3, 4, 9]; see also [2]. For a bound state of a massless particle under either weak or strong interaction potentials, we derive then a new set of dynamic equations (3.11) based on quantum mechanics principles, which are still called the Weyl equations; see Section 3 for the detailed derivation.

FOURTH, with bound state equations for both massless and massive particles, we derive the corresponding spectral equations for the bound states. With the mathematical theory developed in this article on the related eigenvalue problems, we show that the energy levels of each subatomic particle are finite and discrete:

$$0 < E_1 < \cdots < E_N < \infty,$$

and each energy level E_n corresponds to a negative eigenvalue λ_n of the related eigenvalue problem. Physically, λ_n represents the bound energy of the particle, and are related to the energy level E_n with the following relation:

$$E_n = E_0 + \lambda_n, \quad \lambda_n < 0 \qquad \text{for } 1 \le n \le N. \tag{1.1}$$

Here E_0 is the intrinsic potential energy of the constituents of a subatomic particle such as the weaktons.

FIFTH, one important consequence of the above derived energy level theory is that there are both upper and lower bounds of the energy levels for all sub-atomic particles, and the largest and smallest energy levels are given by

$$0 < E_{min} = E_0 + \lambda_1 < E_{max} = E_0 + \lambda_N < \infty. \tag{1.2}$$

In particular, it follows from the energy level theory that the frequencies of mediators such as photons and gluons are also discrete and finite, and are given by $\omega_n = E_n / \hbar$ ($n = 1, \cdots, N$). In the Planck classical quantum assumption that the energy is discrete for a fixed frequency, and the frequency is continuous. Our results are different in two aspects. One is that the energy levels have an upper bound. Two is that the frequencies are also discrete and finite. Hence the derived results in this article will have significant implications for field quantizations.

SIXTH, based on the mathematical theory developed in this article, we have rigorously derived estimates on the total number N of energy levels for subatomic particles, which can be summarized as follows:

$$
N = \begin{cases}
\left(\dfrac{4}{\lambda_1} \dfrac{\rho_0^2 A_w}{\rho_w} \dfrac{m_w c}{\hbar} \dfrac{g_w^2}{\hbar c} \right)^{3/2} & \text{for charged leptons and quarks,} \\[4mm]
\left(\dfrac{4}{\lambda_1} \dfrac{\rho_1^2 A_s}{\rho_q} \dfrac{m_q c}{\hbar} \dfrac{g_s^2}{\hbar c} \right)^{3/2} & \text{for baryons,} \\[4mm]
\left(\dfrac{A_w}{\beta_1} \dfrac{\rho_\gamma}{\rho_w} \dfrac{g_w^2}{\hbar c} \right)^{3} & \text{for photons.}
\end{cases}
$$

Here g_w and g_s are the weak and strong charges, m_w is the mass of the constituent weakton, m_q is the mass of the constituent quark, ρ_w and ρ_q are the radii of the constituent weakton and quark respectively, $\rho_0, \rho_1, \rho_\gamma$ are the radii of the weak attraction, the strong attraction and the photon respectively, and A_w, A_s are nondimensional constants.

SEVENTH, by estimating the average energy level gradient (for two adjacent energy levels), we are able to show that

$$\triangle E \sim 10^{-58} \text{eV},$$

which is too small to be practically measurable. This is consistent with the fact that it is hard to notice the discrete behavior of the frequency of subatomic particles.

EIGHTH, it is now classical that the electromagnetic, weak and strong interactions are described by gauge fields. Consequently, they all enjoy a common property that moving charges generate magnetism. Namely, in the same spirit as the electric charge e producing magnetism, the weak and strong charges g_w, g_s can also produce similar effects, which we also call magnetism.

For a massless particle, the term

$$g \vec{\sigma} \cdot \text{curl} \vec{A}$$

in the Weyl equations reflects the magnetic force generated by the weak or strong interaction, where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli matrix operator, and $\vec{A}$ is the special component of the combined weak or strong potential $A_\mu = (A_0, \vec{A})$ using PRI first introduced in [6].

For a massive particle, magnetic effect is reflected by the term

$$\vec{\mu}^k \cdot \text{curl} \vec{A}$$

in the Dirac equations, where $\vec{\mu}^k = \hbar g \vec{\sigma} / 2 m_k$ represents magnetic moment, generated by either weak or strong interaction.

NINTH, we are able to establish a relationship between the spectrum and the bound state of subatomic particles in terms of related physical parameters. For example, for an

electron, the eigenvalue λ_e and and the certain quantities of the corresponding bound state ψ_e enjoy the following relation:

$$\lambda_e = \frac{1}{2}m_w v^2 + \frac{2g_w^2}{r_e} - \frac{2A_w g_w^2}{\rho_w}\kappa_e,$$

where $\frac{1}{2}m_w v^2$ is the kinetic energy of each constituent weakton in the electron, r_e is the radius of the electron, and κ_e is a new parameter for the electron; see (5.10).

This article is divided into two parts. The first part consists of Sections 2 and 3, and deals with the mathematical theory for eigenvalue problems originated from the Dirac and Weyl equations for bound states of subatomic particles. Section 2 studies spectrum of general elliptic differential operators, and Section 3 addresses the spectrum of the Weyl operators.

Part 2 consists of Sections 4 and 5, and studies the energy levels and spectrum of the Dirac and Weyl equations applied to subatomic particles. Section 4 focuses on the spectral problems, and Section 5 studies energy levels of subatomic particles.

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Part I

Mathematics

2 Spectrum of Elliptic operators

2.1 Physical background

Based on the Bohr atomic model, an atom consists of a proton and its orbital electron, bounded by electromagnetic energy. Due to the quantum effect, the orbital electron is in proper discrete energy levels:

$$0 < E_1 < \cdots < E_N, \tag{2.1}$$

which can be expressed as

$$E_n = E_0 + \lambda_n \qquad (\lambda_n < 0), \tag{2.2}$$

where λ_n $(1 \le n \le N)$ are the negative eigenvalues of a symmetric elliptic operator. Here E_0 stands for the intrinsic energy, and λ_n stands for the bound energy of the atom, holding the orbital electrons, due to the electromagnetism. Hence there are only N energy levels E_n for the atom, which are certainly discrete.

To see this, let Z be the atomic number of an atom. Then the potential energy for electrons is given by

$$V(r) = -\frac{Ze^2}{r}.$$

With this potential, the wave function ψ of an orbital electron satisfies the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m_0}\nabla^2\psi+\frac{Ze^2}{r}\psi=0. \tag{2.3}$$

Let ψ take the form

$$\psi=e^{-i\lambda t/\hbar}\varphi(x),$$

where λ is the bound energy. Putting ψ into (2.3) leads to

$$-\frac{\hbar^2}{2m_0}\nabla^2\varphi-\frac{Ze^2}{r}\varphi=\lambda\varphi.$$

Since the orbital electrons are bound in the interior of the atom, the following condition holds true:

$$\varphi=0 \qquad \text{for } |x|>r_0,$$

where r_0 is the radius of an atom. Thus, if ignoring the electromagnetic interactions between orbital electrons, then the bound energy of an electron is a negative eigenvalue of the following elliptic boundary problem

$$\begin{aligned} -\frac{\hbar^2}{2m_0}\nabla^2\varphi-\frac{Ze^2}{r}\varphi&=\lambda\varphi &&\text{for } x\in B_{r_0},\\ \varphi&=0 &&\text{for } x\in\partial B_{r_0}, \end{aligned} \tag{2.4}$$

where B_{r_0} is a ball with the atom radius r_0.

According to the spectral theory for elliptic operators, the number of negative eigenvalues of (2.4) is finite. Hence, it is natural that the energy levels in (2.2) are finite and discrete.

In fact, we shall demonstrate in this article that all subatomic particles possess discrete and finite energy levels as electrons. The study is based on a unified field model for interactions of Nature [5,6] and an elementary particle theory, called the weakton model in [6,7], both developed recently by the authors. Based on these two models, two or three of weaktons and quarks are bound together to form a subatomic particle under the weak and strong interactions. Thus, naturally these particles have a sequence of energy levels, which are described by the negative eigenvalues of some linear differential operators.

2.2 Mathematical preliminaries

Consider the eigenvalue problem of linear elliptic operators given by

$$\begin{aligned} -D^2\psi+A\psi&=\lambda\psi &&\text{for } x\in\Omega,\\ \psi&=0 &&\text{for } x\in\partial\Omega, \end{aligned} \tag{2.5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\psi = (\psi_1, \cdots, \psi_m)^T : \Omega \to \mathbb{C}^m$ is the unkwon wave function,

$$D = \nabla + i\vec{B}, \quad \vec{B} = (B_1, \cdots, B_n), \tag{2.6}$$

and A, B_k $(1 \le k \le n)$ are $m \times m$ Hermitian matrix-valued functions.

Let λ_0 be an eigenvalue of (2.5). The corresponding eigenspace:

$$E_{\lambda_0} = \left\{ \psi \in L^2(\Omega, \mathbb{C}^m) \mid \psi \text{ satisfy (2.5) with } \lambda = \lambda_0 \right\}$$

is finite dimensional, and its dimension

$$\sigma = \dim E_{\lambda_0}$$

is called the multiplicity of λ_0. Physically, σ is also called degeneracy. Usually, we count the multiplicity σ of λ_0 as σ eigenvalues:

$$\lambda_1 = \cdots = \lambda_\sigma = \lambda.$$

The following spectral theorem is classical.

Theorem 2.1. *Assume that the functions in the matrices A and B_j $(1 \le j \le n)$ are square integrable. Then the following assertions hold true:*

1. *All eigenvalues of (2.5) are real with finite multiplicities, and form an infinite consequence as follows:*
   $$-\infty < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots, \quad \lambda_n \to \infty \text{ as } n \to \infty,$$
   where λ_n is counting the multiplicity.

2. *The eigenfunctions ψ_n corresponding to λ_n are orthogonal to each other:*
   $$\int_\Omega \psi_n^\dagger \psi_l dx = 0 \qquad \forall n \ne l.$$
   In particular, $\{\psi_n\}$ form an orthogonal basis of $L^2(\Omega, \mathbb{C}^m)$.

3. *There are only finite number of negative eigenvalues in $\{\lambda_n\}$,*
   $$-\infty < \lambda_1 \le \cdots \le \lambda_N < 0, \tag{2.7}$$
   and the number N of negative eigenvalues depends on A, B_j $(1 \le j \le n)$ and the domain Ω.

Remark 2.1. For the energy levels of particles, we are mainly interested in the negative eigenvalues of (2.5) and the estimates for the number N in (2.7).

2.3 Conditions for existence of negative eigenvalues

The following theorem provides a necessary and sufficient condition for the existence of negative eigenvalues of (2.5), and a criterion to estimate the number of negative eigenvalues.

Theorem 2.2. *For the eigenvalue problem* (2.5)*, the following assertions hold true:*

1. *Equations* (2.5) *have negative eigenvalues if and only if there is a function* $\psi \in H_0^1(\Omega, \mathbb{C}^m)$, *such that*

$$\int_\Omega [(D\psi)^\dagger(D\psi) + \psi^\dagger A\psi]dx < 0, \tag{2.8}$$

where D is as in (2.6)*.*

2. *If there are K linear independent functions* $\psi_1, \cdots, \psi_K \in H_0^1(\Omega, \mathbb{C}^m)$, *such that*

$$\psi \text{ satisfies } (2.8) \quad \text{for any } \psi \in E^K = \text{span } \{\psi_1, \cdots, \psi_K\}, \tag{2.9}$$

then the number N of negative eigenvalues is larger than K, i.e., $N \geq K$.

Proof. Assertion (1) follows directly from the following classical formula for the first eigenvalue λ_1 of (2.5):

$$\lambda_1 = \min_{\psi \in H_0^1(\Omega, \mathbb{C}^m)} \frac{1}{||\psi||_{L^2}} \int_\Omega \left[(D\psi)^\dagger(D\psi) + \psi^\dagger A\psi\right] dx. \tag{2.10}$$

We now prove Assertion (2) by contradiction. Assume that it is not true, then $K > N$. By Theorem 2.1, the K functions ψ_j in (2.9) can be expended as

$$\psi_j = \sum_{i=1}^{N} \alpha_{ji} e_i + \sum_{l=1}^{\infty} \beta_{jl} \varphi_l \qquad \text{for } 1 \leq j \leq K, \tag{2.11}$$

where e_i $(1 \leq i \leq N)$ and φ_l are eigenfunctions corresponding to negative and nonnegative eigenvalues. Since $K > N$, there exists a K-th order matrix P such that

$$P\alpha = \begin{pmatrix} 0 & \cdots & 0 \\ & * & \end{pmatrix}, \tag{2.12}$$

where

$$\alpha = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1N} \\ \vdots & & \vdots \\ \alpha_{K1} & \cdots & \alpha_{KN} \end{pmatrix} \qquad \text{with } \alpha_{ij} \text{ as in } (2.11).$$

Thus, under the transformation P,

$$\widetilde{\psi} = P \begin{pmatrix} \psi \\ 0 \end{pmatrix} \in E^K, \qquad \psi = (\psi_1, \cdots, \psi_N)^T, \tag{2.13}$$

where E^K is as in (2.9).

However, by (2.11) and (2.12), the first term $\widetilde{\psi}_1$ in (2.13) can be expressed in the form

$$\widetilde{\psi}_1 = \sum_{l=1}^{\infty} \theta_l \varphi_l \in E^K. \tag{2.14}$$

Note that φ_l are the eigenfunctions corresponding to the nonnegative eigenvalues of (2.5). Hence we have

$$\int_{\infty} [(D\widetilde{\psi}_1)^{\dagger}(D\widetilde{\psi}_1) + \widetilde{\psi}_1^{\dagger} A \widetilde{\psi}_1] dx = \int_{\Omega} \widetilde{\psi}_1^{\dagger}(-D^2\widetilde{\psi}_1 + A\widetilde{\psi}_1) dx \tag{2.15}$$
$$= \sum_{l=1}^{\infty} |\theta_l|^2 \lambda_l > 0.$$

Here $\lambda_l \geq 0$ are the nonnegative eigenvalues of (2.5). Hence we derive, from (2.14) and (2.15), a contradiction with the assumption in Assertion (2). Thus, the proof of the theorem is complete. $\square$

2.4 Estimates for the number of negative eigenvalues

For simplicity, it is physically sufficient for us to consider the eigenvalue problem of the Laplace operators, given by

$$\begin{cases} -\nabla^2 \psi + V(x)\psi = \lambda \psi & \text{for } x \in B_r, \\ \psi = 0 & \text{for } x \in \partial B_r, \end{cases} \tag{2.16}$$

where $B_r \subset \mathbb{R}^n$ is a ball with radius r.

In physics, V represents a potential function and takes negative value in a bound state, ensuring by Theorem 2.2 that (2.16) possesses negative eigenvalues.

Here, for the potential function $V(x)$, we assume that

$$V(\rho x) \simeq \rho^{\alpha} V_0(x) \qquad (\alpha > -2), \tag{2.17}$$

where $V_0(x)$ is defined in the unit ball B_1, and

$$\Omega = \{x \in B_1 \mid V_0(x) < 0\} \neq \varnothing. \tag{2.18}$$

Let $\theta > 0$ be defined by

$$\theta = \inf_{\psi \in L^2(\Omega, \mathbb{C}^m)} \frac{1}{||\psi||_{L^2}} \int_{\Omega} |V(x)| \, |\psi|^2 dx. \tag{2.19}$$

The main result in this section is the following theorem, which provides a relation between N, θ and r, where N is the number of negative eigenvalues of (2.16). Let λ_1 be the first eigenvalue of the equation

$$\begin{cases} -\Delta e = \lambda e & \text{for } x \in \Omega, \\ e = 0 & \text{for } x \in \partial \Omega, \end{cases} \tag{2.20}$$

where $\Omega \subset B_1$ is as defined by (2.18).

Theorem 2.3. *Under the assumptions of (2.17) and (2.18), the number N of the negative eigenvalues of (2.16) satisfies the following approximative relation*

$$N \simeq \left(\frac{\theta r^{2+\alpha}}{\lambda_1} \right)^{n/2}, \tag{2.21}$$

provided that $\theta r^{2+\alpha} / \lambda_1 \gg 1$ is sufficiently large, where r and θ are as in (2.16) and (2.19), and λ_1 is the first eigenvalue of (2.20).

Remark 2.2. The estimates (2.21) is also valid for Problem (2.16) in a shell region

$$B = \left\{ x \in \mathbb{R}^n \,|\, r_0 < |x| < r_1 \right\}, \tag{2.22}$$

if $r_0 \ll r_1$. The spectral equations for subatomic particles are defined in such shell-regions as (2.22).

To prove Theorem 2.3, we need to introduce a lemma, which is due to H. Weyl [10]; see also [1].

Lemma 2.1 (H. Weyl). *Let λ_N be the N-th eigenvalue of the m-th order elliptic operator*

$$\begin{aligned} (-1)^m \Delta^m e &= \lambda e \quad \text{for } x \in \Omega \subset \mathbb{R}^n, \\ D^k e|_{\partial\Omega} &= 0 \qquad \text{for } 0 \le k \le m-1, \end{aligned} \tag{2.23}$$

then λ_N has the asymtptotical relation

$$\lambda_N \sim \lambda_1 N^{2m/n}, \tag{2.24}$$

where λ_1 is the first eigenvalue of (2.23).

Proof of Theorem 2.3. The ball B_r can be written as

$$B_r = \left\{ y = rx \mid x \in B_1 \right\}.$$

Note that $\partial/\partial y = r^{-1} \partial/\partial x$, (2.16) can be equivalently expressed as

$$\begin{aligned} -\Delta\varphi + r^2 V(rx)\varphi &= \beta\varphi \quad \text{for } x \in B_1, \\ \varphi &= 0 \qquad \text{for } x \in \partial B_1, \end{aligned} \tag{2.25}$$

and the eigenvalue λ of (2.16) is

$$\lambda = \frac{1}{r^2}\beta, \quad \text{where } \beta \text{ is the eigenvalue of (2.25).}$$

Hence the number of negative eigenvalues of (2.16) is the same as that of (2.25), and we only need to prove (2.21) for (2.25).

By (2.17), the equation (2.25) is approximatively in the form

$$
\begin{cases}
-\Delta\varphi + r^{2+\alpha}V_0(x)\varphi = \beta\varphi & \text{for } x \in B_1, \\
\varphi = 0 & \text{for } x \in \partial B_1.
\end{cases}
\tag{2.26}
$$

Based on Assertion (2) in Theorem 2.2, we need to find N linear independent functions $\varphi_n \in H_0^1(B_1)$ $(1 \le n \le N)$ satisfying

$$
\int_{B_1} \left[|\nabla\varphi|^2 + r^{2+\alpha}V_0(x)\varphi^2 \right] dx < 0,
\tag{2.27}
$$

for any $\varphi \in \text{span}\{\varphi_1, \cdots, \varphi_N\}$ with $||\varphi||_{L^2} = 1$.

To this end, we take the eigenvalues $\{\lambda_n\}$ and eigenfunctions $\{e_n\}$ of (2.20) such that

$$
0 < \lambda_1 \le \cdots \le \lambda_N < \lambda_{N+1},
$$

and

$$
\lambda_N < \theta r^{2+\alpha} \le \lambda_{N+1}.
\tag{2.28}
$$

For the eigenfunctions e_n, we make the extension

$$
\varphi_n = \begin{cases}
e_n & \text{for } x \in \Omega, \\
0 & \text{for } x \in B_1/\Omega.
\end{cases}
$$

It is known that φ_n is weakly differentiable, and $\varphi_n \in H_0^1(\Omega)$. These functions φ_n $(1 \le n \le N)$ are what we need. Let

$$
\varphi = \sum_{n=1}^N \alpha_n \varphi_n, \quad ||\varphi||_{L^2} = 1.
$$

By Assertion (2) in Theorem 2.1, φ_n $(1 \le n \le N)$ are orthonormal:

$$
\int_{B_1} \varphi_i\varphi_j dx = \int_\Omega e_ie_j dx = \delta_{ij}.
$$

Therefore we have

$$
||\varphi||_{L^2} = \sum_{n=1}^N \alpha_n^2 = 1.
\tag{2.29}
$$

Thus the integral in (2.27) is

$$\int_{B_1}[|\nabla\varphi|^2+r^{2+\alpha}V_0(x)\varphi^2]dx$$

$$=\int_\Omega-\left(\sum_{n=1}^N\alpha_ne_n\right)\left(\sum_{n=1}^N\alpha_n\Delta e_n\right)dx+r^{2+\alpha}\int_\Omega V_0(x)\varphi^2dx$$

$$=\sum_{n=1}^N\alpha_n^2\lambda_n+r^{2+\alpha}\int_\Omega V_0(x)\varphi^2dx$$

$$\leq\sum_{n=1}^N\alpha_n^2\lambda_n-\theta r^{2+\alpha}\qquad\text{by (2.19)}$$

$$<0\qquad\qquad\text{by (2.28) and (2.29).}$$

It follows from Theorem 2.2 that there are at least N negative eigenvalues for (2.26). When $\theta r^{2+\alpha}\gg1$ is sufficiently large, the relation (2.28) implies that

$$\lambda_N\simeq\theta r^{2+\alpha}. \tag{2.30}$$

The result (2.21) follows from (2.24) and (2.30). The proof is complete. $\qquad\square$

Remark 2.3. In Part 2, we shall see that for particles with mass m, the parameters in (2.21) are

$$\alpha=0,\quad r=1,\quad \theta=4m\rho_1^2g^2/\hbar^2\rho,$$

where $g=g_w$ or g_s is the weak or strong interaction charge, ρ is the particle radius, and ρ_1 is the weak or strong attracting radius. Hence the number of energy levels of massive particles is given by

$$N=\left[\frac{4}{\lambda_1 A}\frac{\rho_1^2}{\rho}\frac{mc}{\hbar}\frac{g^2}{\hbar c}\right]^{3/2},$$

where λ_1 is the first eigenvalue of $-\Delta$ in the unit ball B_1, and A is a constant.

3 Spectrum of Weyl operators

3.1 Wave equations for massless particles

First, we recall a basic postulate of quantum mechanics.

Basic Postulate of Quantum Mechanics. *For a quantum system with observable Hermit operators $L_1,\cdots,L_m$, if the physical quantities l_k corresponding to L_k $(1\leq k\leq m)$ satisfy a relation:*

$$H(l_1,\cdots,l_m)=0, \tag{3.1}$$

then the following equation induced from (3.1)

$$H(L_1,\cdots,L_m)\psi=0 \tag{3.2}$$

describes the system provided the operator $H(L_1, \cdots, L_m)$ in (3.2) is Hermitian.

The classical quantum mechanical equations, such as the Schrödinger equation, the Klein-Gordon equation and the Dirac equations, are based on this Basic postulate.

It is known that the basic equations describing a free and massless particles are the Weyl equations:

$$\frac{\partial \psi}{\partial t} = c(\vec{\sigma} \cdot \vec{\partial})\psi, \tag{3.3}$$

where $\psi = (\psi^1, \psi^2)^T$ is a two-component Weyl spinor, $\vec{\partial} = (\partial_1, \partial_2, \partial_3)$ is the gradient operator, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli matrix operator, and the Pauli matrices σ_k $(1 \leq k \leq 3)$ are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Weyl equations (3.3) also enjoy the Basic Postulate. In fact, the energy and momentum operators E and P for the Weyl spinor are given by

$$E = i\hbar \frac{\partial}{\partial t}, \quad P = i\hbar(\vec{\sigma} \cdot \vec{\partial}). \tag{3.4}$$

By the de Broglie relation

$$E = \hbar\omega, \quad P = \hbar/\lambda, \quad c = \omega\lambda,$$

we get that

$$E = cP, \tag{3.5}$$

and (3.3) follows from (3.4) and (3.5).

For a massless particle system in a bound state by either weak or strong interaction, (3.4) is replaced by

$$E = i\hbar \frac{\partial}{\partial t} - V, \quad P = i\hbar(\vec{\sigma} \cdot \vec{D}), \tag{3.6}$$

where $V = gA_0$ is the potential, g is the weak or strong charge,

$$A_\mu = (A_0, A_1, A_2, A_3)$$

represents the combined gauge potential for either weak or strong interaction derived using PRI [6], and the differential operator $\vec{D} = (D_1, D_2, D_3)$ is defined by

$$D_j = \partial_j + i\frac{g}{\hbar c}A_j.$$

By the weakton model developed in [6,7], the mediators such as photons and gluons consist of two massless weaktons, which are bound in a small ball B_r by the weak and strong interactions. Hence the Weyl spinor ψ of each weakton is restricted in a small ball, i.e.

$$\psi = 0 \qquad \forall x \notin B_r.$$

which implies the boundary condition

$$\psi|_{\partial B_r} = 0. \tag{3.7}$$

However, in mathematics the boundary problem for the Weyl equations generated by (3.5) and (3.6) given by

$$i\hbar \frac{\partial \psi}{\partial t} = i\hbar c (\vec{\sigma} \cdot \vec{D}) \psi + V(x) \psi$$

$$\psi|_{\partial B_r} = 0 \tag{3.8}$$

is in general not well-posed. The main reason is that equations (3.8) are first-order differential equations and the Dirichlet boundary condition is over-determined.

Hence, for the massless particle system with the boundary condition (3.7), we have to consider the relation

$$PE = cP^2, \tag{3.9}$$

which is of first order in t. It is known that the operator PE is Hermitian if

$$PE = EP.$$

It implies PE should be in the form

$$PE = PE_0 - \frac{1}{2}(PV + VP), \quad E_0 = i\hbar \frac{\partial}{\partial t}. \tag{3.10}$$

Thus the boundary problem for a massless system is generated by (3.9) with (3.10), and is given in the following general form

$$(\vec{\sigma} \cdot \vec{D}) \frac{\partial \psi}{\partial t} = c(\vec{\sigma} \cdot \vec{D})^2 \psi - \frac{ig}{2\hbar} \left[(\vec{\sigma} \cdot \vec{D}) A_0 + A_0 (\vec{\sigma} \cdot \vec{D}) \right] \psi,$$

$$\psi|_{\partial \Omega} = 0, \tag{3.11}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $\vec{D}$ is as in (3.6).

3.2 Spectral theory of Weyl operators

The equations (3.11) describe bound states of a massless particle system. The solutions ψ of the bound state equations (3.11) take in the following form

$$\psi = e^{-i\lambda t/\hbar} \varphi, \quad \varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}.$$

Then equations (3.11) are reduced to the following eigenvalue problem

$$-\hbar c \left(\vec{\sigma} \cdot \vec{D} \right)^2 \varphi + \frac{ig}{2} \{ (\vec{\sigma} \cdot \vec{D}), A_0 \} \varphi = i\lambda (\vec{\sigma} \cdot \vec{D}) \varphi,$$

$$\varphi|_{\partial \Omega} = 0, \tag{3.12}$$

where $\{(\vec{\sigma}\cdot\vec{D}),A_0\}$ is the anti-commutator defined by

$$\{(\vec{\sigma}\cdot\vec{D}),A_0\}=(\vec{\sigma}\cdot\vec{D})A_0+A_0(\vec{\sigma}\cdot\vec{D}).$$

Definition 3.1. *A real number λ and a two-component wave function $\varphi\in H_0^1(\Omega,\mathbb{C}^2)$ are called the eigenvalue and eigenfunction of (3.12), if (λ,φ) satisfies (3.12) and*

$$\int_\Omega \varphi^\dagger[i(\vec{\sigma}\cdot\vec{D})\varphi]dx>0. \tag{3.13}$$

Remark 3.1. The physical significance of (3.13) is that the kinetic energy $E=cP$ is positive, i.e. $E>0$.

Remark 3.2. The equation (3.12) is essentially an eigenvalue problem of the first order differential operator:

$$i\hbar c(\vec{\sigma}\cdot\vec{D})+gA_0,$$

which is called the Weyl operator. In addition, the operator $(\vec{\sigma}\cdot\vec{D})^2$ is elliptic and can be rewritten as

$$(\vec{\sigma}\cdot\vec{D})^2=D^2-\frac{g}{\hbar c}\vec{\sigma}\cdot\text{curl }\vec{A}. \tag{3.14}$$

The ellipticity of (3.14) ensures the existence of eigenvalues of (3.12).

Note also that $\vec{A}=(A_1,A_2,A_3)$ represents the magnetic-component of the weak or strong interaction. Hence, in (3.13), the term

$$g\vec{\sigma}\cdot\text{curl}\vec{A}$$

stands for magnetic energy of the weak or the strong interaction.

Theorem 3.1. *For (3.12), the following assertions hold true:*

1. *The eigenvalues of (3.12) are real, with finite multiplicities, and satisfy*

$$-\infty<\lambda_1\leq\cdots\leq\lambda_n\leq\cdots,\quad \lambda_n\to\infty \qquad for\ n\to\infty.$$

2. *The eigenfunctions are orthogonal in the sense that*

$$\int_\Omega \psi_n^\dagger\left[i(\vec{\sigma}\cdot\vec{D})\psi_l\right]dx=0 \qquad \forall n\neq l. \tag{3.15}$$

3. *The number of negative eigenvalues is finite*

$$-\infty<\lambda_1\leq\cdots\leq\lambda_N<0.$$

4. *Equations (3.12) have negative eigenvalues if and only if there exists a function $\varphi\in H_0^1(\Omega,\mathbb{C}^2)$ satisfying (3.13) such that*

$$\int_\Omega\left[\hbar c|(\vec{\sigma}\cdot\vec{D})\varphi|^2+\frac{ig}{2}\varphi^\dagger\left\{(\vec{\sigma}\cdot\vec{D}),A_0\right\}\varphi\right]dx<0. \tag{3.16}$$

Proof. It is clear that the operator

$$L = i(\vec{\sigma} \cdot \vec{D}) \colon H_0^1(\Omega, \mathbb{C}^2) \to L^2(\Omega, \mathbb{C}^2),$$

is a Hermitian operator. Consider a functional $F \colon H_0^1(\Omega, \mathbb{C}^2) \to \mathbb{R}$ defined by

$$F(\psi) = \int_\Omega \left[\hbar c |L\psi|^2 + \frac{ig}{2} \psi^\dagger \{ (\vec{\sigma} \cdot \vec{D}), A_0 \} \psi \right] dx.$$

By (3.14), the operator $L^2 = -(\vec{\sigma} \cdot \vec{D})^2$ is elliptic. Hence F has the following lower bound on S:

$$S = \left\{ \psi \in H_0^1(\Omega, \mathbb{C}^2) \,\Big|\, \int_\Omega \psi^\dagger L\psi \, dx = 1 \right\}.$$

Namely,

$$\min_{\psi \in S} F(\psi) > -\infty.$$

Based on the Lagrange multiplier theorem of constraint minimization, the first eigenvalue λ_1 and the first eigenfunction $\psi_1 \in S$ satisfy

$$\lambda_1 = F(\psi_1) = \min_{\psi \in S} F(\psi). \tag{3.17}$$

In addition, if

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$$

are the first m eigenvalues with eigenfunctions φ_k, $1 \le k \le m$, then we have

$$\lambda_{m+1} = F(\psi_{m+1}) = \min_{\psi \in S, \psi \in H_m^\perp} F(\psi), \tag{3.18}$$

where $H_m = \text{span}\{\psi_1, \cdots, \psi_m\}$, and $H_m^\perp$ is the orthogonal complement of H_m in the sense of (3.15).

Assertions (1)-(4) follow from (3.17) and (3.18), and the proof is complete. $\qquad\square$

3.3 Estimates on number of negative eigenvalues

If the interaction potential $A_\mu = \{-K, 0, 0, 0\}$ and $K > 0$ is a constant, then the equation (3.12) becomes

$$\begin{aligned} -\Delta\varphi &= i(\lambda + K)(\vec{\sigma} \cdot \vec{\partial})\varphi & &\text{in } \Omega \subset \mathbb{R}^n, \\ \varphi &= 0 & &\text{on } \partial\Omega. \end{aligned} \tag{3.19}$$

It is clear that the number N of negative eigenvalues of (3.19) satisfies that

$$\beta_N < K \le \beta_{N+1}, \tag{3.20}$$

and β_k is the k-th eigenvalue of the equation

$$-\Delta\varphi_k = i\beta_k(\vec{\sigma}\cdot\vec{\partial})\varphi_k,$$
$$\varphi_k|_{\partial\Omega} = 0. \tag{3.21}$$

For (3.21) we have the Weyl asymptotic relation as

$$\beta_N \sim \beta_1 N^{1/n}. \tag{3.22}$$

Hence from (3.20) and (3.21) we obtain an estimate on the number N of of negative eigenvalues:

$$N \simeq \left(\frac{K}{\beta_1}\right)^n, \tag{3.23}$$

where β_1 is the first eigenvalue of (3.21).

Remark 3.3. For the mediators such as the photons and gluons, the number N in (3.23) is

$$N = \left(\frac{A}{\beta_1}\frac{\rho_1}{\rho}\frac{g_w^2}{\hbar c}\right)^3,$$

where ρ_1, ρ are as in Remark 2.3, β_1 is the first eigenvalue of (3.21) on the unit ball $\Omega = B_1$, and A is a constant.

Part II
Physics

4   Spectral equations of subatomic particles

4.1   Constituents of subatomic particles

We first recall the weakton model introduced in [6]. The starting point of the model is the puzzling decay and reaction behavior of subatomic particles. For example, the electron radiations and the electron-positron annihilation into photons or quark-antiquark pair clearly shows that there must be interior structure of electrons, and the constituents of an electron contribute to the making of photon or the quark in the hadrons formed in the process. In fact, all sub-atomic decays and reactions show clearly that there must be interior structure of charged leptons, quarks and mediators. A careful examination of these subatomic decays and reactions leads us to propose a weakton model in [6], which is briefly recapitulated as follows. The weakton model is basic for the energy level theory.

Weaktons. There are six elementary particles, which we call weaktons, and their anti-particles:

$$w^*, \quad w_1, \quad w_2, \quad \nu_e, \quad \nu_\mu, \quad \nu_\tau,$$
$$\bar{w}^*, \quad \bar{w}_1, \quad \bar{w}_2, \quad \bar{\nu}_e, \quad \bar{\nu}_\mu, \quad \bar{\nu}_\tau, \tag{4.1}$$

where ν_e, ν_μ, ν_τ are the three generation neutrinos, and w^*, w_1, w_2 are three new particles, called w-weaktons. These are massless, spin-$\frac{1}{2}$ particles with one unit of weak charge g_w. Both w^* and $\bar{w}^*$ are the only weaktons carrying strong charge g_s.

Six classes of subatomic particles.

1. Charged Leptons:
$$e^\pm, \mu^\pm, \tau^\pm.$$

2. Quarks:
$$u, d, s, c, b, t,$$
$$\bar{u}, \bar{d}, \bar{s}, \bar{c}, \bar{b}, \bar{t}.$$

3. Baryons:
$$p, n, \Lambda, \sigma^\pm, \Sigma^0, \Delta^{++}, \Delta^\pm, \Delta^0, \Xi^\pm, \Xi^0, \text{etc.}$$

4. Mesons:
$$\pi^\pm, \pi^0, K^\pm, K^0, \eta, \rho^\pm, \rho^0, K^{*,\pm}, K^{*0}, \omega, \text{etc.}$$

5. Internal Bosons:
$$W^\pm, Z^0, H^\pm, H^0 \ (H^{\pm,0} \text{ the Higgs particles}).$$

6. Mediators:
$$\gamma, g, \nu, \phi_\gamma, \phi_g \ (\text{the dual bosons of } \gamma \text{ and } g).$$

Constituents of subatomic particles.

1. The weakton constituents of charged leptons and quarks are given by

$$
\begin{aligned}
e &= \nu_e w_1 w_2, & \mu &= \nu_\mu w_1 w_2, & \tau &= \nu_\tau w_1 w_2, \\
u &= w^* w_1 \bar{w}_1, & c &= w^* w_2 \bar{w}_2, & t &= w^* w_2 \bar{w}_2, \\
d &= w^* w_1 w_2, & s &= w^* w_1 w_2, & b &= w^* w_1 w_2,
\end{aligned}
\tag{4.2}
$$

where c, t and d, s, b are distinguished by their spin arrangements as follows:

$$
\begin{aligned}
u &= w^* w_1 \bar{w}_1 (\Uparrow\downarrow, \Downarrow\uparrow, \uparrow\downarrow\uparrow, \downarrow\uparrow\downarrow), & c &= w^* w_2 \bar{w}_2 (\Uparrow\downarrow, \Downarrow\uparrow), \\
t &= w^* w_2 \bar{w}_2 (\uparrow\downarrow\uparrow, \downarrow\uparrow\downarrow), & d &= w^* w_1 w_2 (\Uparrow\downarrow, \Downarrow\uparrow), \\
s &= w^* w_1 w_2 (\uparrow\Downarrow, \downarrow\Uparrow), & b &= w^* w_1 w_2 (\uparrow\downarrow\uparrow, \downarrow\uparrow\downarrow).
\end{aligned}
$$

2. Hadrons include baryons and mesons. Baryons consist of three quarks, and mesons consist of a quark and an antiquark:

$$\text{Baryons} = qqq, \qquad \text{Mesons} = q\bar{q}. \tag{4.3}$$

3. Internal bosons consist of

$$W^+ = \bar{w}_1 \bar{w}_2 (\uparrow\uparrow, \downarrow\downarrow),$$
$$W^- = w_1 w_2 (\uparrow\uparrow, \downarrow\downarrow), \tag{4.4}$$
$$Z^0 = \cos\theta_w w_2 \bar{w}_2 + \sin\theta_w w_1 \bar{w}_1 (\uparrow\uparrow, \downarrow\downarrow),$$

and the dual particles

$$H^+ = \bar{w}_1 \bar{w}_2 (\uparrow\downarrow, \downarrow\uparrow),$$
$$H^- = w_1 w_2 (\uparrow\downarrow, \downarrow\uparrow), \tag{4.5}$$
$$H^0 = \cos\theta_w w_2 \bar{w}_2 + \sin\theta_w w_1 \bar{w}_1 (\uparrow\downarrow, \downarrow\uparrow).$$

where $\theta_w = 28.76°$ is the Weinberg angle.

4. Mediators:

$$\gamma = \cos\theta_w w_1 \bar{w}_1 - \sin\theta_w w_2 \bar{w}_2 (\uparrow\uparrow, \downarrow\downarrow), \qquad g = w^* \bar{w}^* (\uparrow\uparrow, \downarrow\downarrow),$$
$$\phi_\gamma = \cos\theta_w w_1 \bar{w}_1 - \sin\theta_w w_2 \bar{w}_2 (\uparrow\downarrow, \downarrow\uparrow), \quad \phi_g = w^* \bar{w}^* (\uparrow\downarrow, \downarrow\uparrow),$$
$$\nu = \sum_l \alpha_l \nu_l \bar{\nu}_l (\downarrow\uparrow), \qquad\qquad\qquad\qquad \sum_l \alpha_l^2 = 1. \tag{4.6}$$

4.2 Bound potentials of subatomic particles

The main bound energy holding the weaktons and quarks to form subatomic particles is due to the weak and strong interactions. According to the unified field theory developed in [6], the sources of the weak and strong forces are the strong and weak charges:

$$g_w \quad \text{the weak charge,}$$
$$g_s \quad \text{the strong charge.}$$

The gauge theories for strong and weak interaction involve 3 weak interaction gauge potentials W_μ^a ($a = 1, 2, 3$), and 8 strong interaction gauge potentials S_μ^k ($k = 1, \cdots, 8$). The PRI applied to both gauge theories leads to the following 4-dimensional potentials:

$$W_\mu = \alpha_a W_\mu^a = (W_0, W_1, W_2, W_3),$$
$$S_\mu = \beta_k S_\mu^k = (S_0, S_1, S_2, S_3), \tag{4.7}$$

where W_μ^a ($1 \le a \le 3$) and S_μ^k ($1 \le k \le 8$) are the $SU(2)$ and $SU(3)$ gauge potentials, α_a ($1 \le a \le 3$) and β_k ($1 \le k \le 8$) are the $SU(2)$ and $SU(3)$ tensors. The acting forces are given by

$$\text{Weak force} = -g_w \nabla W_0,$$
$$\text{Strong force} = -g_s \nabla S_0, \tag{4.8}$$

and the magnetic forces are

$$\text{Weak magnetism} = -g_w \text{curl } \vec{W},$$
$$\text{Strong magnetism} = -g_s \text{curl } \vec{S},$$
(4.9)

where $(W_0, \vec{W})$ and $(S_0, \vec{S})$ are as in (4.7).

By (4.8) and (4.9), the weak and strong potentials (4.7) generate the bound states (4.2)-(4.6) of subatomic particles, with corresponding interactions given as follows:

Charged Leptons and Quarks:	weak interaction,
Hadrons:	strong interaction,
Internal Bosons:	weak interaction,
Gluons g and dual gluons Φ_g:	weak and strong interactions,
Other Mediators:	weak interaction.

(4.10)

Weak interaction potentials. The layered weak interaction potentials are

$$W_0 = g_w(\rho)e^{-r/r_0}\left[\frac{1}{r} - \frac{A_w}{\rho}\left(1 + \frac{2r}{r_0}\right)e^{-r/r_0}\right],$$
$$g_w(\rho) = g_w\left(\frac{\rho_w}{\rho}\right)^3,$$
(4.11)

where ρ_w and ρ are the radii of the constituent weakton and the particle, A_w is a constant depending on the types of particles, and r_0 is the radius of weak interaction:

$$r_0 \simeq 10^{-16}\text{cm}.$$

Based on (4.11), the weak potential generated by two particles with N_1, N_2 weak charges and radii ρ_1, ρ_2 is given by

$$\Phi_w = N_1 N_2 g_w(\rho_1)g_w(\rho_2)e^{-r/r_0}\left[\frac{1}{r} - \frac{\bar{A}_w}{\sqrt{\rho_1\rho_2}}\left(1 + \frac{2r}{r_0}\right)e^{-r/r_0}\right],$$
(4.12)

where $\bar{A}_w$ is a constant depending on the two particles, and $g_w(\rho_k)$ $(k = 1, 2)$ are as in (4.11).

According to the standard model, the coupling constant G_w of the weak interaction and the Fermi constant G_f have the following relation

$$G_w^2 = \frac{8}{\sqrt{2}}\left(\frac{m_w c}{\hbar}\right)^2 G_f,$$
$$G_f = 10^{-5}\hbar c / \left(\frac{m_p c}{\hbar}\right)^2,$$
(4.13)

where m_w and m_p are the masses of the $W^\pm$ particles and the proton. By the gauge theory, we have

$$g_w(\rho_n) = G_w. \tag{4.14}$$

It follows from (4.13) and (4.14) that [†]

$$g_w^2(\rho_n) = G_w^2 = \frac{8}{\sqrt{2}} \left(\frac{m_w}{m_p} \right)^2 \times 10^{-5} \hbar c.$$

Then, by (4.11), we deduce the weak charge g_w as

$$g_w^2 = 0.63 \times \left(\frac{\rho_n}{\rho_w} \right)^6 \hbar c, \tag{4.15}$$

where ρ_n is the nucleon radius.

Strong interaction potentials. The layered strong interaction potentials are

$$S_0 = g_s(\rho) \left[\frac{1}{r} - \frac{A_s}{\rho} \left(1 + \frac{r}{R} \right) e^{-r/R} \right],$$
$$g_s(\rho) = g_s \left(\frac{\rho_w}{\rho} \right)^3, \tag{4.16}$$

where A_s is a constant depending on the particle type, and R is the attracting radius of strong interactions given by

$$R = \begin{cases} 10^{-16} \text{cm} & \text{for } w^* \text{ and quarks,} \\ 10^{-13} \text{cm} & \text{for hadrons.} \end{cases}$$

By the Yukawa theory, we deduce that

$$g_s^2 = \frac{2}{3(8\sqrt{e} - e)} \left(\frac{\rho_n}{\rho_w} \right)^6 g^2, \tag{4.17}$$

where e is the base of the natural logarithm, g is the classical coupled constant of strong interactions between nucleons, and experimentally $g^2 \sim 1 - 10 \hbar c$.

The strong interaction potential between two particles with N_1, N_2 charges g_s and radii ρ_1, ρ_2 is given by

$$\Phi_s = N_1 N_2 g_s(\rho_1) g_s(\rho_2) \left[\frac{1}{r} - \frac{\bar{A}_s}{\sqrt{\rho_1 \rho_2}} \left(1 + \frac{r}{R} \right) e^{-r/R} \right], \tag{4.18}$$

[†]In our previous work [6], we called $g_w(\rho_n)$ and $g_s(\rho_n)$ as the basic units for weak and strong charges. However, with the weakton theory of elementary particles [6,7], it is more natural to $g_w(\rho_w)$ and $g_s(\rho_w)$ as the basic units of weak and strong charges.

where $g_s(\rho_k)$ $(k=1,2)$ are as in (4.16), and $\bar{A}_s$ is a constant depending on the types of two particle involved.

The weak and strong forces F_w and F_s between two particles are given by

$$F_w = -\nabla\Phi_w, \qquad F_s = -\nabla\Phi_s.$$

4.3 Wave equations of bound states

We infer from (4.2)-(4.6) that a subatomic particle consists of two or three fermions with wave functions:

$$\Psi^1, \cdots, \Psi^N \qquad \text{for } N=2,3. \tag{4.19}$$

Because the bound energy of each fermion is approximately the superposition of the remaining $N-1$ particles. Thus the bound potential for each fermion takes the form

$$A_\mu = \begin{cases} (N-1)W_\mu & \text{for weak interaction,} \\ (N-1)S_\mu & \text{for strong interaction,} \end{cases} \tag{4.20}$$

where W_μ, S_μ are as in (4.7).

Dynamic equations of bound states for massless particles. The wave equations for the massless case are the modified Weyl equations given by (3.11).

Dynamic equations of bound states for massive particles. Let the masses of the constituent fermions be given as follows

$$m = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_N \end{pmatrix}.$$

Then the N wave functions (4.19) satisfy the following Dirac equation

$$\left(i\gamma^\mu D_\mu - \frac{mc}{\hbar} \right)\Psi = 0, \tag{4.21}$$

where $\Psi = (\Psi^1, \cdots, \Psi^N)^T$, and

$$D_\mu = \partial_\mu + i\frac{g}{\hbar c}A_\mu \qquad \text{with } A_\mu \text{ be given in (4.20).}$$

We know that each wave function Ψ^k in (4.19) is a four-component Dirac spinor:

$$\Psi^k = (\Psi_1^k, \Psi_2^k, \Psi_3^k, \Psi_4^k)^T \qquad \text{for } 1\leq k\leq N.$$

Hence equations (4.21) take the following equivalent form

$$\begin{aligned} \left(i\hbar\frac{\partial}{\partial t} - gA_0 - c^2 m_k \right)\begin{pmatrix} \Psi_1^k \\ \Psi_2^k \end{pmatrix} &= ic\hbar(\vec{\sigma}\cdot\vec{D})\begin{pmatrix} \Psi_3^k \\ \Psi_4^k \end{pmatrix}, \\ \left(i\hbar\frac{\partial}{\partial t} - gA_0 + c^2 m_k \right)\begin{pmatrix} \Psi_3^k \\ \Psi_4^k \end{pmatrix} &= ic\hbar(\vec{\sigma}\cdot\vec{D})\begin{pmatrix} \Psi_1^k \\ \Psi_2^k \end{pmatrix}, \end{aligned} \tag{4.22}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli matrix operator.

4.4 Spectral equations of bound states

We now derive spectral equations for both massive and massless bound states from (4.22) and (3.11).

A. Massive bound states. Consider the case where $m \neq 0$. Let the solutions of (4.22) be written in the form as

$$\Psi^k = e^{-i(\lambda + m_k c^2)t/\hbar} \psi^k.$$

Then equations (4.22) become

$$(\lambda - g A_0) \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix} = ic\hbar(\vec{\sigma} \cdot \vec{D}) \begin{pmatrix} \psi_3^k \\ \psi_4^k \end{pmatrix}, \tag{4.23}$$

$$(\lambda - g A_0 + 2m_k c^2) \begin{pmatrix} \psi_3^k \\ \psi_4^k \end{pmatrix} = ic\hbar(\vec{\sigma} \cdot \vec{D}) \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix}, \tag{4.24}$$

for $1 \leq k \leq N$. The equations (4.24) can be rewritten as

$$\begin{pmatrix} \psi_3^k \\ \psi_4^k \end{pmatrix} = \frac{i\hbar}{2m_k c} \left(1 + \frac{\lambda - g A_0}{2m_k c^2} \right)^{-1} (\vec{\sigma} \cdot \vec{D}) \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix}. \tag{4.25}$$

In physics, λ is the energy, and $\lambda - g A_0$ is the kinetic energy

$$\lambda - g A_0 = \frac{1}{2} m_k v^2.$$

For the case[‡] where $v^2/c^2 \simeq 0$, (4.25) can be approximately expressed as

$$\begin{pmatrix} \psi_3^k \\ \psi_4^k \end{pmatrix} = \frac{i\hbar}{2m_k c} (\vec{\sigma} \cdot \vec{D}) \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix}.$$

Inserting this equation into (4.23), we deduce that

$$(\lambda - g A_0) \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix} = -\frac{\hbar^2}{2m_k} (\vec{\sigma} \cdot \vec{D})^2 \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix}. \tag{4.26}$$

Now, we need to give the expression of $(\vec{\sigma} \cdot \vec{D})^2$. To this end, note that the Pauli matrices satisfy

$$\sigma_k^2 = 1, \; \sigma_k \sigma_j = -\sigma_j \sigma_k = i\varepsilon_{jkl}\sigma_l.$$

[‡]Note that the general case can be studied as well involving variable coefficient elliptic operators, and the same results hold true.

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Hence we obtain

$$(\vec{\sigma} \cdot \vec{D})^2 = \left(\sum_{k=1}^{3} \sigma_k D_k \right)^2 = D^2 + i\vec{\sigma} \cdot (\vec{D} \times \vec{D}). \tag{4.27}$$

With $\vec{D} = \nabla + i\frac{g}{\hbar c}\vec{A}$, we derive that

$$\vec{D} \times \vec{D} = i\frac{g}{\hbar c}\left[\nabla \times \vec{A} + \vec{A} \times \nabla \right].$$

Note that as an operator we have

$$\nabla \times \vec{A} = \mathrm{curl}\vec{A} - \vec{A} \times \nabla.$$

Hence we get

$$\vec{D} \times \vec{D} = i\frac{g}{\hbar c}\mathrm{curl}\vec{A}.$$

Thus, (4.27) can be written as

$$(\vec{\sigma} \cdot \vec{D})^2 = D^2 - \frac{g}{\hbar c}\vec{\sigma} \cdot \mathrm{curl}\vec{A}. \tag{4.28}$$

This is the expression given in (3.14).

By (4.28), the spectral equations (4.26) are in the form

$$\left[-\frac{\hbar^2}{2m_k}D^2 + gA_0 \right] \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix} + \vec{\mu}^k \cdot \mathrm{curl}\vec{A} \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix} = \lambda \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix}, \tag{4.29}$$

where $D = (D_1, D_2, D_3)$, (A_0, A_1, A_2, A_3) is as in (4.20), and

$$\vec{\mu}^k = \frac{\hbar g}{2m_k}\vec{\sigma}, \quad D_j = \partial_j + i\frac{g}{\hbar c}A_j \qquad \text{for } 1 \le j \le 3. \tag{4.30}$$

The equations (4.29) are supplemented with the Dirichlet boundary conditions:

$$(\psi_1^k, \psi_2^k)|_{\partial\Omega} = 0 \qquad \text{where } \Omega \subset \mathbb{R}^3. \tag{4.31}$$

B. Massless bound states. For massless systems, the spectral equations are given by (3.12), which, by (4.28), can be rewritten as

$$\left[-\hbar c D^2 + g\vec{\sigma} \cdot \mathrm{curl}\vec{A} \right] \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix} + \frac{ig}{2}\left\{ (\vec{\sigma} \cdot \vec{D}), A_0 \right\} \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix}$$

$$= i\lambda \left(\vec{\sigma} \cdot \vec{D} \right) \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix} \qquad \text{for } 1 \le k \le N,$$

$$(\psi_1^k, \psi_2^k)|_{\partial\Omega} = 0, \tag{4.32}$$

where $\{A,B\} = AB + BA$ is the anti-commutator.

C. Magnetism of weak and strong interactions. In the equations (4.29)-(4.30) and (4.32) for bound states, we see that there are terms

$$\vec{\mu}^k \cdot \text{curl} \vec{A} \qquad \text{for massive particle systems,} \qquad (4.33)$$

$$g\vec{\sigma} \cdot \text{curl} \vec{A} \qquad \text{for massless particle systems,} \qquad (4.34)$$

where $\vec{\mu}^k = \frac{\hbar g}{2m}\vec{\sigma}$ represents magnetic moment, and the term in (4.34) is magnetic force generated by either weak or strong interaction. In other words, in the same spirit as the electric charge e producing magnetism, the weak and strong charges g_w, g_s can also produce similar effects, which we also call magnetism.

Indeed, all three interactions –electromagnetic, weak and strong interactions– enjoy a common property that moving charges generate magnetism, due mainly to the fact that they are all gauge fields with gauge groups $U(1)$, $SU(2)$ and $SU(3)$, respectively.

5 Energy levels of subatomic particles

5.1 Charged leptons and quarks

In (4.2), charged leptons and quarks are made up of three weaktons, with masses caused by the deceleration of the constituent weaktons. Let the masses of the constituent weaktons be m_1, m_2, m_3, and the wave functions of these weaktons be given by

$$\psi^k = \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix} \qquad \text{for } k = 1,2,3.$$

Here ψ_1^k and ψ_2^k represent the left-hand and right-hand states. The bound states are due to the weak interaction, and the potential in (4.20) takes the form

$$A_\mu = 2W_\mu = 2(W_0, \vec{W}).$$

By (4.29)-(4.31), the spectral equations for charged leptons and quarks are

$$
\begin{aligned}
-\frac{\hbar^2}{2m_k}\left(\partial_j + i\frac{2g_w}{\hbar c}W_j\right)^2 \psi^k + 2\left(g_w W_0 + \vec{\mu}_w^k \cdot \text{curl } \vec{W}\right)\psi^k \\
= \lambda \psi^k \qquad \text{in } \rho_w < |x| < \rho, \\
\psi^k|_{\partial B_{\rho_w}} = 0, \quad \psi^k|_{\partial B_\rho} = 0,
\end{aligned}
\tag{5.1}
$$

for $1 \le k \le 3$, where ρ_w is the weakton radius, ρ is the attracting radius of weak interaction, W_0 is given by (4.11), and

$$\vec{\mu}_w^k = \frac{\hbar g_w}{2m_k}\vec{\sigma} \quad \text{is the weak magnetic moment.}$$

   With (5.1), we are in position now to derive a few results on energy levels for charged leptons and quarks.

A. Bound states and energy levels. We know that the negative eigenvalues and eigenfunctions of (5.1) correspond to the bound energies and bound states. Let

$$-\infty < \lambda_1 \le \cdots \le \lambda_N < 0 \tag{5.2}$$

be all negative eigenvalues of (5.1) with eigenfunctions

$$\psi_1, \cdots, \psi_N \qquad \text{for } \psi_n = (\psi_n^1, \psi_n^2, \psi_n^3)^T.$$

Each bound state ψ_n satisfies

$$\int_{B_\rho} |\psi_n^k|^2 dx = 1 \qquad \text{for } 1 \le k \le 3, \quad 1 \le n \le N. \tag{5.3}$$

Then we deduce from (5.1) and (5.3) that

$$\lambda_n = \frac{\hbar}{2m_k} \int_{B_\rho} \left| \left(\nabla + i \frac{g_w}{\hbar c} \vec{W} \right) \psi_n^k \right|^2 dx \tag{5.4}$$
$$+ 2 \int_{B_\rho} \psi_n^{k\dagger} (\vec{\mu}_w^k \cdot \text{curl} \vec{A}) \psi_n^k dx + 2 \int_{B_\rho} g_w W_0 |\psi_n|^2 dx.$$

In the right-hand side of (5.4), the first term stands for the kinetic energy, the second term stands for the weak magnetic energy, and the third term stands for the weak potential energy. By (4.11), the potential energy in (5.4) is negative. Hence the bound energy can be written as

$$\lambda_n = \text{kinetics energy} + \text{magnetic energy} + \text{potential energy}. \tag{5.5}$$

B. Masses. Each particle corresponds to an energy level E_n, and its mass M_n and λ_n satisfy the following relation

$$M_n = \sum_{i=1}^3 m_i + \frac{3g_w^2}{\rho_w c^2} + \frac{\lambda_n}{c^2}, \tag{5.6}$$

where g_w^2/ρ_w represents the intrinsic energy of the constituent weakton.

C. Parameters of electrons. We recall the weak interaction potential (4.11) for the weaktons, which is written as

$$W_0 = g_w \left[\frac{1}{r} - \frac{A_w}{\rho_w} \left(1 + \frac{2r}{r_0} \right) e^{-r/r_0} \right] e^{-r/r_0}. \tag{5.7}$$

where A_w is the constant for weaktons.

Assume that the masses of three constituent weaktons are the same, and we ignore the magnetism, i.e. let $\vec{W} = 0$. Then (5.1) is reduced to

$$
\begin{aligned}
-\frac{\hbar^2}{2m_w}\Delta\psi + 2g_w W_0\psi = \lambda\psi \quad &\text{for } \rho_w < |x| < \rho, \\
\psi = 0 \quad\quad\quad\quad\quad\quad\quad\quad &\text{for } |x| = \rho_w, \rho.
\end{aligned}
\tag{5.8}
$$

We shall apply (5.7) and (5.8) to derive some basic parameters and their relations for bound states of electrons.

Let λ_e and ψ_e be a spectrum and the corresponding bound state of an electron, which satisfy (5.8). It follows from (5.7) and (5.8) that

$$
\lambda_e = \frac{1}{2}m_w v^2 + \frac{2g_w^2}{r_e} - \frac{2A_w g_w^2}{\rho_w}\kappa_e,
\tag{5.9}
$$

where $\frac{1}{2}m_w v^2$ is the kinetic energy of each constituent weakton in the electron, r_e is the radius of the naked electron, and κ_e is a new parameter for the electron:

$$
\begin{aligned}
r_e &= \int_{B_\rho} \frac{1}{r} e^{-r/r_0}|\psi_e|^2 dr, \\
\kappa_e &= \int_{B_\rho}\left(1 + \frac{2r}{r_0}\right)e^{-2r/r_0}|\psi_e|^2 dx, \\
\left(\frac{m_w v}{\hbar}\right)^2 &= \int_{B_\rho} |\nabla\psi_e|^2 dx.
\end{aligned}
\tag{5.10}
$$

These three parameters are related with the energy levels of an electron, i.e. with the λ_n and ψ_n.

The most important case is the lowest energy level state. We shall use the spherical coordinate to discuss the first eigenvalue λ_1 of (5.8). Let the first eigenfunction ψ_e be in the form

$$
\psi_e = \varphi_0(r)Y(\theta,\varphi).
$$

Then φ_0 and Y satisfy

$$
\begin{aligned}
-\frac{\hbar^2}{2m_w}\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr})\varphi_0 + 2g_s W_0\varphi_0 + \frac{\beta_k}{r^2}\varphi_0 &= \lambda_e\varphi_0, \\
\varphi_0(\rho_w) = \varphi_0(\rho) &= 0,
\end{aligned}
\tag{5.11}
$$

and

$$
\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_k = \beta_k Y_k,
\tag{5.12}
$$

where $\beta_k = k(k+1)$, $k = 0,1,\cdots$.

Because λ_e is the minimal eigenvalue, it implies that $\beta_k = \beta_0 = 0$ in (5.11). The eigenfunction Y_0 of (5.12) is given by

$$Y_0 = \frac{1}{\sqrt{4\pi}}.$$

Thus ψ_e is as follows

$$\psi_e = \frac{1}{\sqrt{4\pi}} \varphi_0(r), \tag{5.13}$$

and λ_e and φ_0 are the first eigenvalue and eigenfunction of the following equation

$$-\frac{\hbar^2}{2m_w} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) \varphi_0 + 2g_w W_0 \varphi_0 = \lambda_1 \varphi_0,$$
$$\varphi_0(\rho_w) = \varphi_0(\rho) = 0. \tag{5.14}$$

In this case, the parameters in (5.10) are simplified as

$$r_e = \int_0^\rho r \varphi_0^2(r) dr,$$
$$K_e = \int_0^\rho r^2 \left(1 + \frac{2r}{r_0} \right) e^{-2r/r_0} \varphi_0^2(r) dr, \tag{5.15}$$
$$\left(\frac{m_w \nu}{\hbar} \right)^2 = \int_0^\rho r^2 \left(\frac{d\varphi_0}{dr} \right)^2 dr.$$

5.2 Baryons

By (4.3), baryons consist of three quarks

$$\text{Baryon} = qqq, \tag{5.16}$$

and by (4.2), quarks consist of three w-weaktons

$$q = w^* ww \quad \text{and} \quad q = w^* w \bar{w}.$$

Each quark possesses one strong charge g_s and three weak charges $3g_w$. It looks as if the bound energy of baryons is provided by both weak and strong interactions. However, since the weak interactions is short-ranged, i.e.

$$\text{range of weak force} \leq 10^{-16} \text{cm},$$

and the radii of baryons are

$$r > 10^{-16} \text{cm}.$$

Hence the main force to hold three quarks together is the strong interaction. Let m_1, m_2, m_3 be the masses of the three constituent quarks in (5.16), and $\psi^k = (\psi_1^k, \psi_2^k)^T$ $(1 \leq k \leq 3)$ be the wave functions. Then the spectral equations (4.29)-(4.31) for baryons are in the form

$$
\begin{aligned}
&-\frac{\hbar^2}{2m_k} \left(\nabla + i\frac{2g_s}{\hbar c}\vec{S} \right)^2 \psi^k + 2g_s S_0 \psi^k + 2\vec{\mu}_s^k \cdot \mathrm{curl}\vec{S}\psi^k \\
&= \lambda \psi^k \qquad \text{for } 1 \leq k \leq 3, \ \rho_0 < |x| < \rho_1, \\
&\quad \psi^k = 0 \qquad \text{for } |x| = \rho_0, \rho_1,
\end{aligned}
\tag{5.17}
$$

where ρ_0 is the quark radius, ρ_1 is the strong attracting radius, $S_\mu = (S_0, \vec{S})$ as in (4.7) is a 4-dimensional strong potential, and $\vec{\mu}_s^k = \hbar g_w \vec{\sigma}/2m_k$ is the strong magnetic moment. The wave functions ψ^k are normalized:

$$
\int_\Omega |\psi^k| dx = 1 \qquad \text{with } \Omega = \{x \in \mathbb{R}^3 \,|\, \rho_0 < |x| < \rho_1\}.
$$

By (4.16), the strong interaction potential for quarks is taken in the form

$$
S_0 = g_s \left(\frac{\rho_w}{\rho_0} \right)^3 \left[\frac{1}{r} - \frac{A_s}{\rho_0} \left(1 + \frac{r}{r_0} \right) e^{-r/r_0} \right],
\tag{5.18}
$$

where $r_0 = 10^{-16}$cm.

Since the proton has a long lifetime, we are interested in its physical parameters. As in the case of electrons, we also have the following formula for proton:

$$
\begin{aligned}
\lambda_p &= \frac{1}{2} m_q v^2 + \left[\frac{2g_s^2}{r_p} - \frac{2A_s g_s^2}{\rho_0} \kappa_p \right] \left(\frac{\rho_w}{\rho_0} \right)^6, \\
r_p &= \int_0^{\rho_1} r \varphi_p^2(r) dr, \\
\kappa_p &= \int_0^{\rho_1} r^2 \left(1 + \frac{r}{r_0} \right) e^{-r/r_0} \varphi_p^2(r) dr, \\
\left(\frac{m_q v}{\hbar} \right)^2 &= \int_0^{\rho_1} r^2 \left(\frac{d\varphi_p}{dr} \right)^2 dr,
\end{aligned}
\tag{5.19}
$$

where m_q is the quark mass, r_p is the proton radius, A_s as in (5.18), and λ_p, φ_p are the first eigenvalue and eigenfunctions of the following equation

$$
\begin{aligned}
&-\frac{\hbar^2}{2m_q} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) \varphi_p + 2g_s S_0 \varphi_p = \lambda_p \varphi_p, \\
&\quad \varphi_p = 0 \qquad \text{for } r = \rho_0, \rho_1.
\end{aligned}
\tag{5.20}
$$

5.3 Mediators

By (4.6), the constituents of mediators consist of two weaktons. In this section, we only consider gluons and photon.

Gluons. The weakton constituents of gluons are given by

$$g = w^* \bar{w}^*. \tag{5.21}$$

Based on the weakton model, w^* contains a weak charge g_w and a strong charge g_s. By (4.15) and (4.17), g_w and g_s have the same order. Therefore, the interactions for gluons (5.21) are both weak and strong forces, i.e. the 4-dimensional potential $A_\mu = (A_0, \vec{A})$ is given by

$$g A_0 = g_w W_0 + g_s S_0, \quad g \vec{A} = g_w \vec{W} + g_s \vec{S}. \tag{5.22}$$

In (5.21), we only need to consider the bound states for a single weakton, i.e. $N = 1$ in (4.32). Then the spectral equations are written as

$$
\begin{aligned}
-\hbar c D^2 \psi &+ \left[g_w \vec{\sigma} \cdot \text{curl} \vec{W} + g_s \vec{\sigma} \cdot \text{curl} \vec{S} \right] \psi + \frac{i}{2} g_w \left\{ (\vec{\sigma} \cdot \vec{D}), W_0 \right\} \psi \\
&+ \frac{i}{2} g_s \left\{ (\vec{\sigma} \cdot \vec{D}), S_0 \right\} \psi = i\lambda (\vec{\sigma} \cdot \vec{D}) \psi \qquad \text{for } \rho_w < |x| < \rho_g, \\
\psi &= 0 \qquad \text{for } |x| = \rho_w, \rho_g,
\end{aligned}
\tag{5.23}
$$

where ρ_w, ρ_g are the radii of weaktons and gluons, and

$$\vec{D} = \nabla + \frac{ig}{\hbar c} \vec{A}, \qquad \text{where } g, \vec{A} \text{ are given in (5.22).}$$

Photons. The photon consists of a pair of weakton and anti-weakton:

$$\gamma = w\bar{w}.$$

Because the weaktons w and $\bar{w}$ only contain a weak charge, the interaction in γ is the weak force, i.e.

$$A_\mu = W_\mu = (W_0, \vec{W}).$$

In this case, the spectral equations for a photon are in the form

$$
\begin{aligned}
-\hbar c (\nabla + \frac{ig_w}{\hbar c} \vec{W})^2 \psi &+ g_w \vec{\sigma} \cdot \text{curl} \vec{W} \psi + \frac{i}{2} g_w \left\{ [\vec{\sigma} \cdot (\nabla + \frac{ig_w}{\hbar c} \vec{W})], W_0 \right\} \psi \\
&= i\lambda [\vec{\sigma} \cdot (\nabla + \frac{ig_w}{\hbar c} \vec{W})] \psi \qquad \text{for } \psi_w < |x| < \rho_\gamma, \\
\psi &= 0 \qquad \text{on } |x| = \rho_w, \rho_\gamma,
\end{aligned}
\tag{5.24}
$$

where ρ_γ is the photon radius.

5.4   Number of energy levels

Based on the spectral theorems, Theorems 2.1 and 3.1, the number of energy levels of a subatomic particle is finite. Hence all subatomic particles are in finite and discrete energy states. By (2.21) and (3.23), we can derive estimates for the numbers of energy levels for various subatomic particles.

Leptons and quarks. To estimate the numbers of energy levels, we always ignore the effect of magnetism, i.e., we set $\vec{W} = 0$ in (5.1). In this case the spectral equations (5.1) are reduced to

$$-\frac{\hbar^2}{2m_w}\Delta\psi + 2g_w W_0\psi = \lambda\psi \quad \text{for } \rho_w < |x| < \rho_0,$$
$$\psi = 0 \qquad\qquad\qquad \text{on } |x| = \rho_w, \rho_0, \tag{5.25}$$

where m_w is the mass of the constituent weaktons.

Formulas in (4.11) for weaktons can be approximatively written as

$$W_0 = -\frac{g_w A_w}{\rho_w}. \tag{5.26}$$

Take a scaling transformation

$$x = \rho_0 x^1, \tag{5.27}$$

where ρ_0, as in (5.25), is the attracting radius of the weak interaction. Note that the weakton radius ρ_w is much smaller than ρ_0,

$$\rho_w \ll \rho_0, \tag{5.28}$$

Hence, by (5.26)-(5.27), the problem (5.25) can be approximately expressed in the form

$$-\Delta\psi + \frac{4m_w A_w \rho_0^2}{\hbar^2 \rho_w}g_w^2\psi = \lambda\psi \quad \text{for } |x| < 1,$$
$$\psi = 0 \qquad\qquad\qquad\qquad \text{on } |x| = 1. \tag{5.29}$$

It is clear that the parameters in (2.21) are given by

$$\theta = \frac{4m_w A_w \rho_0^2}{\hbar^2 \rho_w}g_w^2, \quad r = 1, \quad \alpha = 0.$$

Thus the number N of the energy levels for charged leptons and quarks is approximately given by

$$N = \left[\frac{4}{\lambda_1}\frac{\rho_0^2 A_w}{\rho_w}\frac{m_w c}{\hbar}\frac{g_w^2}{\hbar c}\right]^{3/2}, \tag{5.30}$$

where λ_1 is the first eigenvalue of $-\Delta$ in the unit ball $B_1 \subset \mathbb{R}^3$.

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Baryons. For baryons, similar to (5.29) we can transform the spectral equations (5.17) into the following form

$$-\Delta\psi + \frac{4m_q A_s \rho_1^2}{\hbar^2 \rho_q} g_s^2 \psi = \lambda \psi \quad \text{for } |x| < 1,$$
$$\psi = 0 \qquad \qquad \text{on } |x| = 1, \tag{5.31}$$

where ρ_q is the quark radius, ρ_1 is the attracting radius of strong interaction, m_q is the quark mass, and A_s is the constant as in (5.18).

Then, by (5.31), we derive an estimate of N:

$$N = \left[\frac{4}{\lambda_1} \frac{\rho_1^2 A_s}{\rho_q} \frac{m_q c}{\hbar} \frac{g_s^2}{\hbar c} \right]^{3/2}, \tag{5.32}$$

where λ_1 is the same as in (5.30).

Photons. For simplicity, here we only consider the photon. Then the spectral equations (5.24) are reduced to

$$-\Delta\varphi = i \left(\lambda + \frac{A_w \rho_\gamma}{\hbar c \rho_w} g_w^2 \right) (\vec{\sigma} \cdot \vec{\partial}) \varphi \quad \text{for } |x| < 1,$$
$$\varphi = 0 \qquad \qquad \text{on } |x| = 1, \tag{5.33}$$

where ρ_γ is the photon radius.

Then, by (5.33), we see that the parameter K in (3.23) is

$$K = \frac{A_w \rho_\gamma}{\rho_w} \frac{g_w^2}{\hbar c}.$$

Thus we derive from (3.23) an estimate for the number of photon energy levels:

$$N = \left[\frac{A_w}{\beta_1} \frac{\rho_\gamma}{\rho_w} \frac{g_w^2}{\hbar c} \right]^3, \tag{5.34}$$

where β_1 is the first eigenvalue of (3.21) in $B_1 \subset \mathbb{R}^3$.

5.5 Physical conclusions

To compute the numbers N in (5.30), (5.32) and (5.34), we need to recall the values of g_w^2 and g_s^2. By (4.15) we have

$$\frac{g_w^2}{\hbar c} = 0.63 \times \left(\frac{\rho_n}{\rho_w} \right)^6, \tag{5.35}$$

and in (4.17) taking $g^2 = 5\hbar c$, then we have

$$\frac{g_s^2}{\hbar c} = 0.32 \times \left(\frac{\rho_n}{\rho_w} \right)^6, \tag{5.36}$$

where ρ_n is the radius of nucleons.

Based on the energy level theory for subatomic particles established above, we deduce the following physical conclusions.

First, the energy levels for each subatomic particle are discrete, and the number of energy levels is finite:

$$0 < E_1 < \cdots < E_N < \infty,$$

and each energy level E_n corresponds to an eigenvalue λ_n of the related eigenvalue problem as (5.23) or (5.24). Physically, λ_n represents the bound energy of the particle, and is related to the energy level E_n with the following relation:

$$E_n = E_0 + \lambda_n \qquad \text{for } 1 \leq n \leq N. \tag{5.37}$$

Here E_0 is the intrinsic potential energy of the constituent weaktons, which is given by g_w^2 / ρ_w.

Second, by (5.37), we deduce immediately the following upper and lower bounds of the energy levels of sub-atomic particles:

$$E_0 + \lambda_1 \leq E_n < E_0 \qquad \text{for } 1 \leq n \leq N. \tag{5.38}$$

It is clear that the largest and smallest energy levels are given by

$$E_{max} = E_0 + \lambda_N, \qquad E_{min} = E_0 + \lambda_1. \tag{5.39}$$

The total energy level difference is

$$E_{max} - E_{min} = \lambda_N - \lambda_1,$$

and the average energy level gradient (for two adjacent energy levels) is approximately given by

$$\triangle E = \frac{E_{max} - E_{min}}{N} \sim \frac{|\lambda_1|}{N}.$$

In particular, for photons, by $|\lambda_1| \sim K\hbar c / \rho_\gamma$, the energy gradient can be estimated by

$$\triangle E \sim \frac{\hbar c}{\rho_\gamma K^2} \sim \frac{1}{\rho_\gamma} \frac{\hbar c}{\left(\frac{g_w^2}{\hbar c}\right)^2} \sim 10^{-58} \text{eV},$$

if we take the following approximate values of the related parameters:

$$\rho_\gamma = 10^{-19} \text{cm}, \quad \frac{\rho_n}{\rho_w} \sim 10^6.$$

One important consequence of finite and discrete energy levels for sub-atomic particles is that the frequency is also discrete and finite, and are given by $\omega_n = E_n / \hbar$ ($n = 1, \cdots, N$). This result shall play an important role in field quantizations.

Acknowledgments

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