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Astrophysical Dynamics and Cosmology

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Abstract. *First*, the essence of a physical theory for a multilevel system is through coupling different physical laws in different levels by a symmetry-breaking principle, rather than through a unification using larger symmetry. In astrophysical dynamics, the symmetry-breaking mechanism and the coupling are achieved by prescribing the coordinate system so that the laws of fluid dynamics and heat conductivity are coupled with gravitational field equations. Another important ingredient in modeling fluid motion in astrophysics is to use the momentum density field to replace the velocity field as the state function of cosmic objects. *Second*, by applying the new symmetry-breaking mechanism and the new coupled astrophysical dynamics model, we rigorously prove a basic theorem on blackholes: Assume the validity of the Einstein theory of general relativity, then black holes are closed, innate and incompressible. *Third*, we prove a theorem on structure of universes. Assume the Einstein theory of general relativity, and the principle of cosmological principle that the universe is homogeneous and isotropic. Then we show that 1) all universes are bounded, are not originated from a Big-Bang, and are static; and 2) The topological structure of our Universe can only be the 3D sphere. Also, thanks to the basic properties of blackholes, we show that our results on our Universe resolve such fundamental problems as dark matter and dark energy, redshifts and CMB. *Fourth*, we discovered that both supernovae explosion and AGN jets, as well as many astronomical phenomena, are due to combined relativistic, magnetic and thermal effects. The radial temperature gradient causes vertical Bénard convection cells, and the relativistic viscous force (via electromagnetic, the weak and the strong interactions) gives rise to an huge explosive radial force near the Schwarzschild radius, leading e.g. to supernovae explosion and AGN jets.

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1 Introduction

The goal of this article is to examine fundamental issues in astrophysics and cosmology, based on Einstein theory of general relativity and the cosmological principle, leading to new theories on astrophysical dynamics and cosmology as outlined below.

Black holes

One main objective of this paper is to study the nature of black holes and the structure and formation of our Universe. The concept of black holes was originated by the Schwarzschild solution of the Einstein gravitational field equations, in an exterior of a central symmetric matter field:

$$ds^2 = -\left(1-\frac{R_s}{r}\right)c^2dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2, \tag{1.1}$$

where

$$R_s = \frac{2MG}{c^2} \tag{1.2}$$

is the Schwarzschild radius. Here M is the mass of the matter in the centrally symmetric ball of radius R. It is well-known that when $R \le R_s$, the spherical 3D ball B_{R_s} is a black hole.

One main result we establish rigorously in this article is to show that

$$\boxed{\text{black holes are closed.}}$$

Namely, no energy can cross either side of the black hole surface $S_{R_s} = \{x \in \mathbb{R}^3 \mid |x| = R_s\}$. It is classical to know that no particles can escape from a black hole when they are in the Schwarzschild radius. Then in the exterior of a black hole, we have the energy-momentum conservation:

$$\frac{\partial E}{\partial \tau} + \text{div } P = 0, \tag{1.3}$$

where τ is the proper time, E and P are the energy and momentum densities. Then by (1.3), together with the fact that no matter can escape from inside of the black hole, we can easily show that

$$\lim_{r\to R_s^+} P_r = 0,$$

which implies that black holes are closed: no energy can penetrate the Schwarzschild surface. Here P_r is the radial component of the momentum density P.

The second main result on black holes we prove in this article is that

$$\boxed{\text{black holes are innate.}}$$

Namely, no new black holes can be generated from a massive object through a cosmic process. In particular, we show in this article that black holes can not be created by supernovae explosions. In other words, black holes can neither be created nor be annihilated, and the total number of black holes in the Universe is conserved. This results leads to a new theory on the origin of stars and galaxies.

Matter in a black hole will not be attracted to its center, forming a singularity. In fact, a black hole is filled with the matter in the interior.

It is worth to remark that the classical view on black holes is that a black hole is a huge attracting "sink" that matters can not come out from inside the black hole, but can be attracted into it, causing the growth of the black radius and cosmic instability. The results that we proved on black holes show that a black hole is incompressible and non-expanding: matters can neither come in or out of it. Black hole is closed body, and is either a 3D open ball or a 3D sphere with Schwarzschild radius determined by the mass.

The Universe

Modern cosmology adopts the view that our Universe is formed through the Big-Bang or Big-Bounce; see among others [1, 2, 7, 8]. One main objective of this article is to rigorously derive a new theory on the geometrical and topological structure and the nature of our Universe, based mainly on fundamental principles of general relativity.

Our general observation is that based on the blackhole theorem, Theorem 4.1, our Universe can only be a closed spherical Universe without boundary S^3. We proceed as follows.

Let E and M be the total energy and mass of the Universe:

$$E=\text{kinetic}+\text{electromagnetic}+\text{thermal}+\Psi, \qquad M=E/c^2, \tag{1.4}$$

where Ψ is the energy of all interaction fields. The total mass M dictates the Schwarzschild radius R_s.

If our Universe were born to the Big-Bang, assuming at the initial stage, all energy is concentrated in a ball with radius $R_0<R_s$, by the theory of black holes, then the energy contained B_{R_0} must generate a black hole in $\mathbb{R}^3$ with fixed radius R_s as defined by (1.2).

Assume at certain stage, the Universe were contained in ball of a radius R with $R_0< R<R_s$, then we can prove that the Universe must contain a sub-black hole with radius r given by

$$r=\sqrt{\frac{R}{R_s}}R.$$

Based on this property, the expansion of the Universe, with increasing R to R_s, will give rise to an infinite sequence of black holes with one embedded to another. Apparently, this scenario is clearly against the observations of our Universe, and demonstrates the following:

$$\boxed{\text{Our Universe cannot be originated from a Big-Bang.}}$$

Also, according to the basic cosmological principle that the universe is homogeneous and isotropic in three-dimensional space [8], given the energy density $\rho_0>0$ of the universe, based on the Schwarzschild radius, the universe will always be bounded in black hole, which is an open ball of radius:

$$R_s = \sqrt{\frac{3c^2}{8\pi G\rho_0}}.$$

This shows immediately the following conclusion:

$$\boxed{\text{There is no unbounded universe.}}$$

We have shown that a black hole is unable to expand and shrink. We arrive immediately from the above analysis the following conclusion:

$$\boxed{\text{Our Universe must be static, and not expanding.}}$$

Notice that the isotropy requirement in the cosmological principle excludes the globular open universe scenario. Consequently, we have shown that

$$\boxed{\text{our Universe must be a closed 3D sphere } S^3.}$$

Redshift problem. Then the natural and important question one has to answer is the consistency with astronomical observations, including the cosmic edge, the flatness, the horizon, the redshift, and the cosmic microwave background (CMB) problems. These problems can now be easily understood based on the structure of the Universe and the blackhole theorem we derived. Hereafter we focus only on the redshift and the CMB problems.

The most fundamental problem is the redshift problem. Observations clearly show that light coming from a remote galaxy is redshifted, and the farther away the galaxy is, the larger the redshift. In modern astronomy and cosmology, it is customary to characterize the redshift by a dimensionless quantity z in the formula

$$1+z = \frac{\lambda_{\text{observ}}}{\lambda_{\text{emit}}}, \tag{1.5}$$

where λ_{observ} and λ_{emit} represent the observed and emitting wavelenths.

There are three sources of redshifts: the Doppler effect, the cosmological redshift, and the gravitational redshift. If the Universe is not considered as a black hole, then the gravitational redshift and the cosmological redshift are both too small to be significant. Hence, modern astronomers have to think that the large port of the redshift is due to the Doppler effect.

However, due to black hole properties of our Universe, the black hole and cosmological redshifts cannot be ignored. Due to the horizon of the sphere, for an arbitrary point in the spherical Universe, its opposite hemisphere relative to the point is regarded as a black hole. Hence, g_{00} can be approximatively taken as the Schwarzschild solution for distant objects as follows

$$-g_{00}(r)=\alpha(r)\left(1-\frac{R_s}{\widetilde{r}}\right), \qquad \alpha(0)=2, \qquad \alpha(R_s)=1, \qquad \alpha'(r)<0,$$

where $\widetilde{r}=2R_s-r$ for $0\leq r<R_s$ is the distance from the light source to the opposite radial point, and r is the distance from the light source to the point. We derive then the following redshift formula, which is consistent with the observed redshifts:

$$1+z=\frac{1}{\sqrt{\alpha(r)(1-\frac{R_s}{\widetilde{r}})}}=\frac{\sqrt{2R_s-r}}{\sqrt{\alpha(r)(R_s-r)}} \qquad \text{for } 0<r<R_s. \tag{1.6}$$

CMB problem. In 1965, two physicists A. Penzias and R. Wilson discovered the low-temperature cosmic microwave background (CMB) radiation, which fills the Universe, and it has been regarded as the smoking gun for the Big-Bang theory. However, for the unique scenario of our Universe we derived, it is the most natural thing that there exists a CMB, because the Universe has always been there as a black-body and CMB is a result of blackbody equilibrium radiation.

Dark matter and dark energy. In view of the geometric structure of our Universe we derived, the observable cosmic mass M and the total mass M_{total}, including both M and the non-observable mass caused by space curvature energy, enjoy the following relation:

$$M_{\text{total}}=\begin{cases} 3\pi M/2 & \text{for the spherical universe,} \\ 3\pi M/4 & \text{for the globular universe.} \end{cases} \tag{1.7}$$

The difference $M_{\text{total}}-M$ can be regarded as the dark matter. Astronomical observations have shown that the measurable mass M is about one fifth of total mass M_{total}, which appears to be in better agreement with the spherical universe case, and supports that our Universe is a closed 3D sphere.

Also, the static Universe has to possess a negative pressure to balance the gravitational attracting force. The negative pressure is actually the effect of the gravitational repelling force, attributed to dark energy.

Equivalently, the above interpretation of dark matter and dark energy is consistent with the theory based on the new gravitational field equations developed by the authors [5]:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-\frac{8\pi G}{c^4}T_{\mu\nu}-\frac{1}{2}\left(\nabla_\mu\Psi_\nu+\nabla_\nu\Psi_\mu\right), \tag{1.8}$$

based on 1) the Einstein principle of general relativity, 2) the principle of interaction dynamics (PID), which is the direct consequence of the presence of the dark energy and dark matter. It is clear now that gravity can display both attractive and repulsive effect, caused by the duality between the *attracting* gravitational field $\{g_{\mu\nu}\}$ and the *repulsive* dual vector field $\{\Phi_\mu\}$, together with their nonlinear interactions governed by the field equations. Consequently, dark energy and dark matter phenomena are simply a property of gravity.

Symmetry-breaking principle

We know that different physical systems obey different physical principles. One common view of modeling a multilevel large system is through unification based on larger symmetry. We believe, however, that the essence of a physical theory for such a multilevel system is through coupling different physical laws in different levels by a symmetry-breaking principle. In other words, symmetry-breaking is a general principle when we deal with a physical system coupling different subsystems in different levels. This principle consists of two aspects as follows:

- The three sets of symmetries — the general relativistic invariance, the Lorentz and the gauge invariances, and the Galileo invariance — are mutually independent and dictate in part the physical laws in different levels of the physical world.

- For a system coupling different levels of physical laws, part of these symmetries must be broken.

In astrophysics, we encounter a difficulty that the Newtonian Second Law for fluid motion and the diffusion law for heat conduction are Galilean invariant, and are not compatible with the principle of general relativity. There are no basic principles and rules for combining relativistic systems and the Galilean systems together to form a consistent system. The reason is that in a Galilean system, time and space are independent, and physical fields are 3-dimensional; while in a relativistic system, time and space are related, and physical fields are 4-dimensional. The symmetry-breaking mechanism and the coupling in this case are achieved by prescribing the coordinate system

$$x^\mu=(x^0,x), \quad x^0=ct \text{ and } x=(x^1,x^2,x^3),$$

such that the metric is in the form:

$$ds^2=-g_{00}c^2dt^2+g_{ij}(x,t)dx^idx^j. \tag{1.9}$$

Here g_{ij} $(1\leq i,j\leq 3)$ are the spatial metric. Then the laws of fluid dynamics and heat conductivity will be established on the 3D space manifold with metric g_{ij} $(1\leq i,j\leq 3)$. It is then clear that by fixing the coordinate system to ensure that the metric is in the form (1.9), the system breaks the symmetry of general coordinate transformations, and we call such symmetry-breaking as relativistic-symmetry breaking.

It is important to remark that gravitational field equations give rise only to the law of gravity, and they do not contain fluid dynamics and thermodynamics laws.

Another important ingredient in modeling fluid motion in astrophysics is to use the momentum density field $P(x,t)$ to replace the velocity field $v(x,t)$ as the state function of cosmic objects. The main reason is that the momentum density field P is the energy flux containing the mass, the heat, and all interaction energy flux, and can be regarded as a continuous field. The advantage for momentum form of fluid motion is obvious. For example, it is clear that equation (1.3) plays an important role in deriving some key properties of black holes.

Supernovae explosion and AGN jets

Relativistic, magnetic and thermal effects are main ingredients in astrophysical fluid dynamics, and are responsible for many astronomical phenomena. The thermal effect is described by the Rayleigh number Re:

$$\mathrm{Re} = \frac{mGr_0r_1\beta}{\kappa\nu}\frac{T_0-T_1}{r_1-r_0}, \tag{1.10}$$

where T_0 and T_1 are the temperatures at the bottom and top of an annular shell region $r_0<r<r_1$; see e.g. (2.68) for other notations used here.

Based on our theory of black holes, including in particular the incompressibility and closedness of black holes, the relativistic effect is described by the δ-factor:

$$\delta = \frac{2mG}{c^2r_0}. \tag{1.11}$$

where m is the mass of the core $0<r<r_0$.

Consider e.g. an active galactic nucleus (AGN), which occupies a spherical annular shell region $R_s<r<R_1$, where R_s is the Schwarzschild radius of the black hole core of the galaxy. Then $r_0=R_s$ and the δ-factor is $\delta=1$. The relativistic effect is then reflected in the radial force

$$F_r = \frac{\nu}{2\alpha}\frac{\partial}{\partial r}\left(\frac{1}{\alpha}\frac{d\alpha}{dr}P_r\right), \qquad \alpha = \left(1-\frac{R_s}{r}\right)^{-1},$$

which gives rise to a huge explosive force:

$$\frac{\nu}{1-R_s/r}\frac{R_s^2}{r^4}P_r. \tag{1.12}$$

The relativistic effect is also reflected in the electromagnetic energy:

$$\frac{\nu_0}{2\alpha}\frac{\partial}{\partial r}\left(\frac{1}{\alpha}\frac{d\alpha}{dr}H_r\right),$$

which consists of a huge explosive electromagnetic energy:

$$\frac{\nu_0}{1-R_s/r}\frac{R_s^2}{r^4}H_r. \tag{1.13}$$

The basic mechanism for the formation of AGN jets is that the radial temperature gradient causes vertical Bénard convection cells. Each Bénard convection cell has a vertical exit, where the circulating gas is then pushed by the radial force and erupts leading to a jet. Each Bénard convection cell also an entrance, where the external gas is attracted into the nucleus, is cycloaccelerated by the radial force as well, goes down to the interior boundary $r=R_s$, and then is pushed toward to the exit. Thus the circulation cells form jets in their exits and accretions in their entrances.

This mechanism can also be applied to supernovae explosion. When a very massive red giant completely consumes its central supply of nuclear fuels, its core collapses. Its radius r_0 begins to decrease, and consequently the δ-factor increases. The huge mass m and the rapidly reduced radius r_0 make the δ-factor approaching one. The thermal convection gives rise to an outward radial circulation momentum flux P_r. Then the radial force as in (1.12) will lead to the supernova explosion. Also, $P_r=0$ at $r=r_0$, where r_0 is the radius of blackhole core of supernovae. Consequently, the supernova's huge explosion preserves a smaller ball, yielding a neutron star.

2 Astrophysical Fluid Dynamics

The main objectives of this section are 1) to introduce a symmetry-breaking mechanism to couple fluid dynamical equations with the gravitational field equation, and 2) fluid dynamical models for astrophysical and cosmological objects such as stars, galaxies, and clusters of galaxies.

2.1 Symmetry-breaking principle

Different physical systems obey different physical principles. The four fundamental interactions of Nature, the quantum systems, the fluid dynamics and heat conduction obey the following symmetry principles:

$$
\begin{aligned}
&\text{the general relativity for gravity,}\\
&\text{the Lorentz and gauge invariances for the other three interactions,}\\
&\text{the Lorentz invariance for quantum systems,}\\
&\text{the Galilean invariance for fluid and heat conductions.}
\end{aligned}
\tag{2.1}
$$

The corresponding fields and systems in (2.1) are governed by the following physical laws and first principles:

$$
\begin{aligned}
&\text{PID and PRI for four fundamental interactions,}\\
&\text{PLD for quantum systems,}\\
&\text{the Newton Second Law for fluids,}\\
&\text{diffusion laws for heat conductivity,}
\end{aligned}
\tag{2.2}
$$

Here PID stands for the principle of interaction dynamics, and PRI stands for the principle of representation invariance, both discovered recently by the authors. PLD stands for the principle of Lagrangian dynamics.

Astrophysics is the only field that involving all the fields in (2.1) and (2.2). Consequently, one needs to couple the basic laws in (2.2) to model astrophysical dynamics.

One radical difficulty we encounter now is that the Newtonian Second Law for fluid motion and the diffusion law for heat conduction are not compatible with the principle of general relativity. Also, there are no basic principles and rules for combining relativistic systems and the Galilean systems together to form a consistent system. The reason is that in a Galilean system, time and space are independent, and physical fields are 3-dimensional; while in a relativistic system, time and space are related, and physical fields are 4-dimensional.

The distinction between relativistic and Galilean systems gives rise to an obstacle for establishing a consistent model of astrophysical dynamics, coupling all the physical systems in (2.1) and (2.2).

In the unified field theory based on PID, the key ingredient for coupling gravity with the other three fundamental interactions is achieved through spontaneous gauge symmetry breaking. Here we propose that the coupling between the relativistic and the Galilean systems through relativistic-symmetry breaking.

In fact, the model given by (2.69)-(2.70) follows from this symmetry-breaking principle, where we have to chose the coordinate system

$$x^\mu=(x^0,x), \quad x^0=ct \text{ and } x=(x^1,x^2,x^3),$$

such that the metric is in the form:

$$ds^2=-\left(1+\frac{2}{c^2}\psi(x,t)\right)c^2dt^2+g_{ij}(x,t)dx^idx^j. \tag{2.3}$$

Here g_{ij} $(1\le i,j\le 3)$ are the spatial metric, and

$$\psi= \text{ the gravitational potential.} \tag{2.4}$$

With this metric (2.3)-(2.4), we can establish the fluid and heat equations as in (2.72). It is then clear that by fixing the coordinate system to ensure that the metric is in the form (2.3)-(2.4), the system breaks the symmetry of general coordinate transformations, and we call such symmetry-breaking as relativistic-symmetry breaking.

We believe the symmetry-breaking is a general phenomena when we deal with a physical system coupling different subsystems in different levels. The unified field theory for the four fundamental interactions is a special case, which couples the general relativity, the Lorentz and the gauge symmetries. Namely, the symmetry of general relativity needs to be linked to both the Lorentz invariance and the gauge invariance in two aspects as follows:

- In the Dirac equations for the fermions:

$$i\gamma^\mu D_\mu \psi - \frac{c}{\hbar} m\psi = 0,$$

  γ^μ have to obey two different transformations.

- The gauge-symmetry breaks in the gravitational field equations coupling the other interactions:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} + \frac{1}{2}(\widetilde{D}_\mu \Phi_\nu + \widetilde{D}_\nu \Phi_\mu), \tag{2.5}$$

  where

$$\widetilde{D}_\mu = D_\mu + \frac{k_1}{\hbar c} e A_\mu + \frac{k_2}{\hbar c} g_w W_\mu + \frac{k_3}{\hbar c} g_s S_\mu, \tag{2.6}$$

  D_μ is the covariant derivative, k_i $(1 \le i \le 3)$ are parameters, A_μ, W_μ, S_μ are the total electromagnetic, weak and strong interaction potentials. It is the terms A_μ, W_μ, S_μ in (2.6) that break the gauge symmetry of (2.5).

In summary, we are ready to postulate a general symmetry-breaking principle.

Principle of Symmetry-Breaking 2.1. *1) The three sets of symmetries,*

$$\begin{aligned} &\textit{the general relativistic invariance},\\ &\textit{the Lorentz and gauge invariances, and}\\ &\textit{the Galileo invariance,} \end{aligned} \tag{2.7}$$

are mutually independent and dictate in part the physical laws in different levels of the physical world.

2) for a system coupling different levels of physical laws, part of these symmetries must be broken.

2.2 Fluid dynamic equations on Riemannian manifolds

To consider astrophysical fluid dynamics, we first need to discuss the Navier-Stokes equations on Riemannian manifolds.

Let $(\mathcal{M}, g_{ij})$ be an n-dimensional Riemannian manifold. The fluid motion on $\mathcal{M}$ are governed by the Navier-Stokes equations given by

$$\begin{aligned} &\frac{\partial u}{\partial t} + (u\cdot\nabla)u = \nu\Delta u - \frac{1}{\rho}\nabla p + f \qquad \text{for } x \in \mathcal{M},\\ &\text{div} u = 0, \end{aligned} \tag{2.8}$$

where $u=(u^1,\cdots,u^n)$ is the velocity field, p is the pressure, f is the external force, ρ is the mass density, ν is the dynamic viscosity, and the differential operator Δ is the Laplace-Beltrami operator defined as $\Delta u=(\Delta u^1,\cdots,\Delta u^n)$ with

$$\Delta u^i = \operatorname{div}(\nabla u^i)+g^{ij}R_{jk}u^k, \tag{2.9}$$

$$\operatorname{div}(\nabla u^i)=g^{kl}\left[\frac{\partial}{\partial x^l}\left(\frac{\partial u^i}{\partial x^k}+\Gamma^i_{kj}u^j\right)+\Gamma^i_{lj}\left(\frac{\partial u^j}{\partial x^k}+\Gamma^j_{ks}u^s\right)-\Gamma^j_{kl}\left(\frac{\partial u^i}{\partial x^j}+\Gamma^i_{js}u^s\right)\right]. \tag{2.10}$$

Here R_{ij} is the Ricci curvature tensor and Γ^i_{kj} the Levi-Civita connection:

$$R_{ij}=\frac{1}{2}g^{kl}\left(\frac{\partial^2 g_{kl}}{\partial x^i\partial x^j}+\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}-\frac{\partial^2 g_{kj}}{\partial x^i\partial x^l}-\frac{\partial^2 g_{li}}{\partial x^j\partial x^k}\right)+g^{kl}g_{rs}\left(\Gamma^r_{kl}\Gamma^s_{ij}-\Gamma^r_{il}\Gamma^s_{kj}\right), \tag{2.11}$$

$$\Gamma^i_{kj}=\frac{1}{2}g^{il}\left(\frac{\partial g_{kl}}{\partial x^j}+\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{kj}}{\partial x^l}\right). \tag{2.12}$$

The nonlinear convection term $(u\cdot\nabla)u$ in (2.8) is defined by

$$\begin{aligned}&(u\cdot\nabla)u=(u^iD_iu^1,\cdots,u^iD_iu^n),\\&u^iD_iu^k=u^i\frac{\partial u^k}{\partial x^i}+\Gamma^k_{ij}u^iu^j\end{aligned} \tag{2.13}$$

the pressure term is

$$\nabla p=\left(g^{1k}\frac{\partial p}{\partial x^k},\cdots,g^{nk}\frac{\partial p}{\partial x^k}\right), \tag{2.14}$$

the divergence of u is

$$\operatorname{div}u=\frac{\partial u^k}{\partial x^k}+\Gamma^k_{jk}u^j=\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}u^k)}{\partial x^k}, \tag{2.15}$$

and $g= \det(g_{ij})$.

By (2.9) and (2.13)-(2.15), the Navier-Stokes equations (2.8) can be equivalently written as

$$\begin{aligned}&\frac{\partial u^i}{\partial t}+u^k\frac{\partial u^i}{\partial x^k}+\Gamma^i_{kj}u^ku^j=\nu\left[\operatorname{div}(\nabla u^i)+g^{ij}R_{jk}u^k\right]-\frac{1}{\rho}g^{ij}\frac{\partial p}{\partial x^j}+f^i,\\&\frac{\partial u^k}{\partial x^k}+\Gamma^k_{jk}u^j=0.\end{aligned} \tag{2.16}$$

Remark 2.2. In the Navier-Stokes equations (2.8), the Laplace operator Δ can be taken in two forms:

$$\Delta=d\delta+\delta d \qquad \text{the Laplace-Beltrami operator,} \tag{2.17}$$

$$\Delta= \operatorname{div}\cdot\nabla \qquad \text{the Laplace operator.} \tag{2.18}$$

Here we choose (2.17) instead of (2.18) to represent the viscous term in (2.8). The reason is that the Laplace-Beltrami operator

$$(d\delta+\delta d)u^i = \operatorname{div}\cdot\nabla u^i + g^{ij}R_{jk}u^k$$

gives rise to an additional term $g^{ij}R_{jk}u^k$. In fluid dynamics, the term $\mu\operatorname{div}\cdot\nabla u$ represents the viscous (frictional) force, and the term $g^{ij}R_{jk}u^k$ is the force generated by space curvature and gravitational interaction. Hence physically, it is more natural to take (2.17) instead of (2.18). $\square$

Remark 2.3. In the fluid dynamic equations (2.16), the symmetry of general relativity breaks, and the space and time are treated independently. $\square$

2.3 Schwarzschild and Tolman-Oppenheimer-Volkoff (TOV) metrics

We recall in this section the classical Schwarzschild and TOV metrics for centrally symmetric gravitational fields.

Schwarzschild metric

Many stars in the Universe are spherically-shaped, generating centrally symmetric gravitational fields. It is known that the Riemannian metric of a spherically symmetric gravitation field takes the following form:

$$ds^2 = -e^u c^2 dt^2 + e^v dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2.19}$$

where (r,θ,φ) is the spherical coordinate system, and $u=u(r,t)$ and $v=v(r,t)$ are functions of r and t, which are determined by the gravitational field equations.

In the exterior of a ball, the Einstein field equations become

$$R_{\mu\nu}=0 \tag{2.20}$$

Since the gravitational resource is static, u and v depend only on r. It is then easy to derive the Schwarzschild metric in the vacuum exterior of a centrally symmetric matter field with mass m:

$$ds^2 = -\left(1-\frac{2mG}{c^2 r}\right)c^2dt^2 + \left(1-\frac{2mG}{c^2 r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2. \tag{2.21}$$

We have in particular

$$e^{-v} = e^u = 1-\frac{2mG}{c^2 r}.$$

TOV metric

The Schwarzschild metric (2.21) describes the exterior gravitational fields of a centrally symmetric ball. For the interior gravitational fields, the metric is given by the TOV solution.

Let m be the mass of a centrally symmetric ball, and R be the radius of this ball. In the interior of the ball, the variable r satisfies $0 \leq r < R$. Let the ball be a static liquid sphere consisting of idealized fluid, an approximation of stars. The energy-momentum tensor of an idealized fluid is in the form

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu},$$

where p is the pressure, ρ is the density, and u^μ is the 4-velocity. For a static fluid, u^μ is given by

$$u^\mu = \frac{1}{\sqrt{-g_{00}}}(1,0,0,0).$$

Hence, the (1,1)-type of the energy-momentum tensor is in the form

$$T_\mu^\nu = \begin{pmatrix} -c^2\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

The Einstein field equations can be written as

$$e^{-v}\left(\frac{1}{r^2} - \frac{v'}{r}\right) - \frac{1}{r^2} = -\frac{8\pi G}{c^2}\rho, \tag{2.22}$$

$$e^{-v}\left(\frac{1}{r^2} + \frac{u'}{r}\right) - \frac{1}{r^2} = \frac{8\pi G}{c^4} p, \tag{2.23}$$

$$e^{-v}\left(u'' - \frac{1}{2}u'v' + \frac{1}{2}u'^2 + \frac{1}{r}u' - \frac{1}{r}v'\right) = \frac{16\pi G}{c^2} p, \tag{2.24}$$

$$p' + \frac{1}{2}(p + c^2\rho)u' = 0. \tag{2.25}$$

By the Bianchi identity, only three equations of (2.22)-(2.25) are independent. Here we also regard p and ρ as unknown functions. Therefore, for the four unknown functions u, v, p, ρ, we have to add an equation of state to the system of (2.22)-(2.25):

$$\rho = f(p), \tag{2.26}$$

and the function f will be given according to physical conditions.

On the surface $r=R$ of the ball, $p=0$ and u and v are given in terms of the Schwarzschild solution:

$$p(R) = 0, \quad u(R) = -v(R) = \ln\left(1 - \frac{2Gm}{Rc^2}\right). \tag{2.27}$$

We are now in position to discuss the solutions of problem (2.22)-(2.27). Let

$$M(r)=\frac{c^2 r}{2G}(1-e^{-v}). \tag{2.28}$$

Then the equation (2.22) can be rewritten as

$$\frac{1}{r^2}\frac{dM}{dr}=4\pi\rho,$$

whose solution is given by

$$M(r)=\int_0^r 4\pi r^2 \rho dr \qquad \text{for } 0<r<R. \tag{2.29}$$

By (2.29), we see that $M(r)$ is the mass, contained in the ball B_r. It follows from (2.28) that

$$e^{-v}=1-\frac{2GM(r)}{c^2 r}. \tag{2.30}$$

Inserting (2.30) in (2.23) we obtain

$$u'=\frac{1}{r(c^2r-2MG)}\left[\frac{8\pi G}{c^2}pr^3+2GM(r)\right]. \tag{2.31}$$

Putting (2.31) into (2.25) we get

$$p'=-\frac{(p+c^2\rho)}{2r(c^2r-2MG)}\left[\frac{8\pi G}{c^2}pr^3+2GM(r)\right]. \tag{2.32}$$

Thus, it suffices for us to derive the solution p,M and ρ from (2.26)-(2.28) and (2.32), and then v and u will follow from (2.30)-(2.31) and (2.27).

The equation (2.32) is called the TOV equation, which was derived to describe the structure of neutron stars.

We note that (2.30) is the interior metric of a blackhole provided that $2GM(r)/c^2r=1$. Thus the TOV solution (2.30) gives a rigorous proof of the following theorem for the existence of black holes.

Theorem 2.4. *If the matter field in a ball B_R of radius R is spherically symmetric, and the mass M_R and the radius R satisfy*

$$\frac{2GM_R}{c^2 R}=1,$$

then the ball must be a blackhole.

An idealized model is that the density is homogeneous, i.e. (2.26) is given by

$$\rho=\rho_0 \qquad \text{a constant.}$$

In this case, we have

$$M(r)=\frac{4\pi}{3}\rho_0 r^3 \qquad \text{for } 0\le r\le R,$$
$$\rho_0=\frac{3}{4\pi}\frac{m}{R^3}.$$

Thus we obtain the following solution of (2.30)-(2.32) with (2.27):

$$p(r)=\rho_0\left[\frac{\left(1-\frac{2Gmr^2}{c^2R^3}\right)^{1/2}-\left(1-\frac{2Gm}{c^2R}\right)^{1/2}}{3\left(1-\frac{2Gm}{c^2R}\right)^{1/2}-\left(1-\frac{2Gmr^2}{c^2R^3}\right)^{1/2}}\right], \tag{2.33}$$

$$e^u=\left[\frac{3}{2}\left(1-\frac{2Gm}{c^2R}\right)^{1/2}-\frac{1}{2}\left(1-\frac{2Gmr^2}{c^2R^3}\right)^{1/2}\right]^2, \tag{2.34}$$

$$e^v=\left[1-\frac{2Gmr^2}{c^2R^3}\right]^{-1}. \tag{2.35}$$

The functions (2.33)-(2.35) are the TOV solution. By (2.19), the solution (2.34) and (2.35) yields the metric

$$\begin{aligned} ds^2=&-\left[\frac{3}{2}\left(1-\frac{2Gm}{c^2R}\right)^{1/2}-\frac{1}{2}\left(1-\frac{2Gmr^2}{c^2R^3}\right)^{1/2}\right]^2 c^2dt^2\\ &+\left[1-\frac{2Gmr^2}{c^2R^3}\right]^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2), \end{aligned} \tag{2.36}$$

which is called the TOV metric.

2.4 Differential operators in spherical coordinates

In Subsection 2.2, we gave the Navier-Stokes equations on general Riemannian manifolds. For astrophysical fluid dynamics, we mainly concern the equations on $3D$ spheres. Hence in this subsection we discuss the basic differential operators (2.9)-(2.15) under the spherical coordinate systems (θ,φ,r).

For a $3D$ sphere M, the Riemannian metric is given by

$$ds^2=\alpha(r)dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2 \tag{2.37}$$

where $\alpha(r)>0$ represents the relativistic effects:

$$\alpha=\begin{cases} 1 & \text{no relativistic effect,}\\ \left(1-\dfrac{2Gm}{c^2r}\right)^{-1} & \text{for the Schwarzschild metric (2.21),}\\ \left(1-\dfrac{2Gmr^2}{c^2R^3}\right)^{-1} & \text{for the TOV metric (2.36).}\end{cases} \tag{2.38}$$

In (2.37) we have

$$g_{11}=\alpha(r),\qquad g_{22}=r^2,\qquad g_{33}=r^2\sin^2\theta,\qquad g_{ij}=0 \text{ for } i\neq j.$$

By (2.12) we can get the Levi-Civita connection as

$$\begin{aligned}
&\Gamma^2_{21}=\Gamma^2_{12}=\frac{1}{r}, && \Gamma^2_{33}=-\sin\theta\cos\theta, && \Gamma^3_{31}=\Gamma^3_{13}=\frac{1}{r},\\
&\Gamma^3_{32}=\Gamma^3_{32}=\frac{\cos\theta}{\sin\theta}, && \Gamma^1_{22}=-\frac{r}{\alpha}, && \Gamma^1_{33}=-\frac{r}{\alpha}\sin^2\theta,\\
&\Gamma^1_{11}=\frac{1}{2\alpha}\frac{d\alpha}{dr}, && \Gamma^k_{ij}=0 \ \text{ for others.}
\end{aligned}\tag{2.39}$$

We deduce from (2.39) the explicit form of the Ricci curvature tensor (2.11):

$$R_{11}=-\frac{1}{\alpha r}\frac{d\alpha}{dr},\quad R_{22}=\frac{1}{\alpha}-\frac{r}{2\alpha^2}\frac{d\alpha}{dr}-1,\quad R_{33}=R_{22}\sin^2\theta,\quad R_{ij}=0\quad \forall i\neq j.\tag{2.40}$$

Based on (2.39) and (2.40) we can obtain the expressions of the differential operators (2.9)-(2.15) as follows:

1) The Laplace-Beltrami operator $\Delta u^k=(\Delta u_r,\Delta u_\theta,\Delta u_\varphi)$:

$$\begin{aligned}
\Delta u_\theta=&\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\theta}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\varphi^2}\right.\\
&\left.+\frac{1}{\alpha}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)+\frac{2}{r}\frac{\partial u_r}{\partial\theta}-\frac{2\cos\theta}{\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}-\frac{1}{\sin^2\theta}u_\theta\right]\\
&+\frac{1}{\alpha r^2}\left[2\frac{\partial}{\partial r}(ru_\theta)-\frac{\alpha'}{2\alpha}\frac{\partial}{\partial r}(r^2u_\theta)\right],
\end{aligned}\tag{2.41}$$

$$\begin{aligned}
\Delta u_\varphi=&\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\varphi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u_\varphi}{\partial\varphi^2}+\frac{1}{\alpha}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right)\right.\\
&\left.+\frac{2\cos\theta}{\sin\theta}\frac{\partial u_\varphi}{\partial\theta}-2u_\varphi+\frac{2\cos\theta}{\sin^3\theta}\frac{\partial u_\theta}{\partial\varphi}+\frac{2}{r\sin^2\theta}\frac{\partial u_r}{\partial\varphi}\right]\\
&+\frac{1}{\alpha r^2}\left[2\frac{\partial}{\partial r}(ru_\varphi)-\frac{\alpha'}{2\alpha}\frac{\partial}{\partial r}(r^2u_\varphi)\right],
\end{aligned}\tag{2.42}$$

$$\begin{aligned}
\Delta u_r=&\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u_r}{\partial\varphi^2}+\frac{1}{\alpha}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right)\right]\\
&-\frac{2}{\alpha r^2}\left[u_r+r\frac{\cos\theta}{\sin\theta}u_\theta+r\frac{\partial u_\theta}{\partial\theta}+r\frac{\partial u_\varphi}{\partial\varphi}-\frac{r^2}{2}\frac{\partial}{\partial r}\left(\frac{\alpha'}{2\alpha}u_r\right)\right].
\end{aligned}\tag{2.43}$$

2) By (2.13) and (2.39), $(u\cdot\nabla)u^k$ can be written as

$$u^kD_ku_\theta = u_r\frac{\partial u_\theta}{\partial r}+u_\theta\frac{\partial u_\theta}{\partial\theta}+u_\varphi\frac{\partial u_\theta}{\partial\varphi}+\frac{2}{r}u_\theta u_r-\sin\theta\cos\theta u_\varphi^2 \tag{2.44}$$

$$u^kD_ku_\varphi = u_r\frac{\partial u_\varphi}{\partial r}+u_\theta\frac{\partial u_\varphi}{\partial\theta}+u_\varphi\frac{\partial u_\varphi}{\partial\varphi}+\frac{2\cos\theta}{\sin\theta}u_\theta u_\varphi+\frac{2}{r}u_\varphi u_r, \tag{2.45}$$

$$u^kD_ku_r = u_r\frac{\partial u_r}{\partial r}+u_\theta\frac{\partial u_r}{\partial\theta}+u_\varphi\frac{\partial u_r}{\partial\varphi}-\frac{r}{\alpha}(u_\theta^2+\sin^2\theta u_\varphi^2-\frac{\alpha'}{2r}u_r^2). \tag{2.46}$$

3) The gradient operator:

$$\nabla p=\left(\frac{1}{\alpha}\frac{\partial p}{\partial r},\quad \frac{1}{r^2}\frac{\partial p}{\partial\theta},\quad \frac{1}{r^2\sin^2\theta}\frac{\partial p}{\partial\varphi}\right). \tag{2.47}$$

4) By (2.15) and $\sqrt{g}=r^2\sin\theta\sqrt{\alpha}$, the divergent operator $\mathrm{div}u$ is

$$\mathrm{div}u=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta u_\theta)+\frac{\partial u_\varphi}{\partial\varphi}+\frac{1}{r^2\sqrt{\alpha}}\frac{\partial}{\partial r}(r^2\sqrt{\alpha}u_r). \tag{2.48}$$

Remark 2.5. The expressions (2.41)-(2.48) are the differential operators appearing in the fluid dynamic equations describing the stellar fluids. However, we need to note that the two components u_θ and u_φ are the angular velocities of θ and φ, i.e.

$$u_\theta=\frac{d\theta}{dt},\quad u_\varphi=\frac{d\varphi}{dt}.$$

In classical fluid dynamics, the velocity field $v=(v_\theta,v_\varphi,v_r)$ is the line velocity. The relation of u and v is given by

$$u_\theta=\frac{1}{r}v_\theta,\quad u_\varphi=\frac{1}{r\sin\theta}v_\varphi,\quad u_r=v_r. \tag{2.49}$$

Hence, inserting (2.49) into (2.8) with the expressions (2.41)-(2.48), we derive the Navier-Stokes equations in the usual spherical coordinate form as follows

$$\begin{aligned}
&\frac{\partial v_\theta}{\partial t}+(u\cdot\nabla)v_\theta=\nu\Delta v_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+f_\theta,\\
&\frac{\partial v_\varphi}{\partial t}+(u\cdot\nabla)v_\varphi=\nu\Delta v_\varphi-\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi}+f_\varphi,\\
&\frac{\partial v_r}{\partial t}+(u\cdot\nabla)v_r=\nu\Delta v_r-\frac{1}{\rho\alpha}\frac{\partial p}{\partial r}+f_r,\\
&\mathrm{div}\, v=0,
\end{aligned} \tag{2.50}$$

where

$$\begin{aligned}
\Delta v_\theta =&\widetilde{\Delta} v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{1}{2\alpha^2 r}\frac{d\alpha}{dr}\frac{\partial}{\partial r}(rv_\theta),\\
\Delta v_\varphi =&\widetilde{\Delta} v_\varphi+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\varphi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi}-\frac{v_\varphi}{r^2\sin^2\theta}-\frac{1}{2\alpha^2 r}\frac{d\alpha}{dr}\frac{\partial}{\partial r}(rv_\varphi),\\
\Delta v_r =&\widetilde{\Delta} v_r-\frac{2}{\alpha r^2}\left(v_r+\frac{\partial v_\theta}{\partial\theta}+\frac{\cos\theta}{\sin\theta}v_\theta+\frac{1}{\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}\right)+\frac{1}{2\alpha}\frac{\partial}{\partial r}\left(\frac{1}{\alpha}\frac{d\alpha}{dr}v_r\right),
\end{aligned}\tag{2.51}$$

$\widetilde{\Delta}$ is the Laplace operator for scalar fields given by

$$\widetilde{\Delta} T=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial T}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\varphi^2}+\frac{1}{\alpha r^2}\frac{\partial}{\partial r}(r^2\frac{\partial T}{\partial r}),\tag{2.52}$$

the nonlinear term $(u\cdot\nabla)v$ is

$$\begin{aligned}
(v\cdot\nabla)v_\theta=&\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta v_r}{r}-\frac{\cos\theta v_\varphi^2}{r\sin\theta},\\
(v\cdot\nabla)v_\varphi=&\frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial\theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}+v_r\frac{\partial v_\varphi}{\partial r}+\frac{v_\varphi v_r}{r}+\frac{\cos\theta v_\varphi v_\theta}{r\sin\theta},\\
(v\cdot\nabla)v_r=&\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial v_r}{\partial\varphi}+v_r\frac{\partial v_r}{\partial r}-\frac{1}{\alpha r}(v_\theta^2+v_\varphi^2)+\frac{1}{2\alpha}\frac{d\alpha}{dr}v_r^2,
\end{aligned}\tag{2.53}$$

and the divergent term div v reads

$$\text{div }\ v=\frac{1}{r\sin\theta}\frac{\partial(\sin\theta v_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}+\frac{1}{r^2\sqrt{\alpha}}\frac{\partial(r^2\sqrt{\alpha}v_r)}{\partial r}.\tag{2.54}$$

2.5 Momentum representation

The Universe, galaxies and galactic clusters are composed of stars and interstellar nebulae. Their velocity fields are not continuous. Hence it is not appropriate that we model cosmic objects using continuous velocity field $v(x,t)$ as in the Navier-Stoks equations or by discrete position variables $x_k(t)$ as in the N-body problem.

The idea is that we use the momentum density field $P(x,t)$ to replace the velocity field $v(x,t)$ as the state function of cosmic objects. The main reason is that the momentum density field P is the energy flux containing the mass, the heat, and all interaction energy flux, and can be regarded as a continuous field. The aim of this section is to establish the momentum form of astrophysical fluid dynamics model.

The physical laws governing the dynamics of cosmic objects are as follows

$$\begin{aligned}
&\text{Theory of General Relativity,}\\
&\text{Newtonian Second Law,}\\
&\text{Heat Conduction Law,}\\
&\text{Energy-Momentum Conservation,}\\
&\text{Equation of State.}
\end{aligned}\tag{2.55}$$

The mathematical expressions of these laws are given respectively in the following.

1) Gravitational field equations.

$$\begin{aligned}&R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-\frac{8\pi G}{c^4}T_{\mu\nu}+\frac{1}{2}(\widetilde{D}_\mu\Phi_\nu+\widetilde{D}_\nu\Phi_\mu),\\&\widetilde{D}_\mu=D_\mu+\frac{e}{\hbar c}A_\mu,\end{aligned}\tag{2.56}$$

where A_μ is the electromagnetic potential, the time components of $g_{\mu\nu}$ are as

$$g_{00}=-\left(1+\frac{2}{c^2}\psi\right),\qquad\qquad g_{0k}=g_{k0}=0\qquad\text{for } 1\leq k\leq 3,$$

and ψ is the gravitational potential.

2) Fluid dynamic equations. The Newton Second Law can be expressed as

$$\frac{dP}{d\tau}=\text{ Force},\tag{2.57}$$

where τ is the proper time given by

$$d\tau=\sqrt{-g_{00}}dt,\tag{2.58}$$

P is the momentum density field, formally defined by

$$\frac{dx}{d\tau}=\frac{1}{\rho}P,$$

with ρ being the energy density,

$$\frac{dP}{d\tau}=\frac{\partial P}{\partial\tau}+\frac{\partial P}{\partial x^k}\frac{dx^k}{d\tau}=\frac{\partial P}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)P,$$

and

$$\begin{aligned}&\nu\Delta P+\mu\nabla(\mathrm{div}P) &&\text{the frictional force,}\\&-\nabla p &&\text{the pressure gradient,}\\&\frac{c^2}{2}\rho(1-\beta T)\nabla g_{00}=-\rho(1-\beta T)\nabla\psi &&\text{the gravitational force.}\end{aligned}$$

Hence, the momentum form of the fluid dynamic equations (2.57) is written as

$$\frac{\partial P}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)P=\nu\Delta P+\mu\nabla(\mathrm{div}P)-\nabla p-\rho(1-\beta T)\nabla\psi,\tag{2.59}$$

where the differential operators Δ,∇ and $(P\cdot\nabla)$ are with respect to the space metric g_{ij} $(1\leq i,j\leq 3)$ determined by (2.56), as defined in (2.9)-(2.15).

3) Heat conduction equation:

$$\frac{\partial T}{\partial \tau}+\frac{1}{\rho}(P\cdot\nabla)T=\kappa\widetilde{\Delta}T+Q, \tag{2.60}$$

where $\widetilde{\Delta}$ is defined as

$$\widetilde{\Delta}T=-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}g^{ij}\frac{\partial T}{\partial x^j}),$$

and $g=\det(g_{ij}),1\leq i,j\leq 3$.

4) Energy-momentum conservation:

$$\frac{\partial \rho}{\partial \tau}+\mathrm{div}P=0, \tag{2.61}$$

where ρ is the energy density:

$$\rho = \text{mass + electromagnetism + potential + heat.}$$

5) Equation of state:

$$p=f(\rho,T). \tag{2.62}$$

Remark 2.6. Both physical laws (2.57) and (2.61) are the more general form than the classical ones:

$$\begin{aligned} &m\frac{dv}{d\tau}=\text{Force} && \text{the Newton Second Law,}\\ &\frac{\partial m}{\partial \tau}+\mathrm{div}(mv)=0 && \text{the continuity equation,} \end{aligned} \tag{2.63}$$

where m is the mass density. Hence the momentum representation equations (2.59)-(2.61) can be applicable in general. The momentum P represents the energy density flux, consisting essentially of

$$P=mv+\text{ radiation flux+heat flux.}$$

Hence in astrophysics, the momentum density P is a better candidate than the velocity field v, to serve as the continuous-media type of state function. □

2.6 Astrophysical Fluid Dynamics Equations

Dynamic equations of stellar atmosphere

Different from planets, stars are fluid spheres. Like the Sun, most of stars possess atmospheric layers. The atmospheric dynamics of stars is an important topic, and we are now ready to present the stellar atmospheric model.

The spatial domain is a spherical shell:

$$\mathcal{M}=\{x\in\mathbb{R}^3 \mid r_0<r<r_1\}.$$

The stellar atmosphere consists of rarefied gas. For example, the solar corona has mass density about $\rho_m=10^{-9}\rho_0$ where ρ_0 is the density of the earth atmosphere. Hence we use the Schwarzschild solution in (2.38) as the metric:

$$\alpha(r)=\left(1-\frac{2mG}{c^2r}\right)^{-1}, \quad r_0>\frac{2mG}{c^2}. \tag{2.64}$$

where m is the total mass of the star, and the condition $r_0>2mG/c^2$ ensures that the star is not a black hole.

The stellar atmospheric model is the momentum form of the astrophysical fluid dynamical equations defined on the spherical shell $\mathcal{M}$:

$$\begin{aligned}
&\frac{\partial P}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)P=\nu\Delta P+\mu\nabla(\mathrm{div}P)-\nabla p-\frac{mG\rho}{r^2}(1-\beta T)\vec{k},\\
&\frac{\partial T}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)T=\kappa\widetilde{\Delta}T,\\
&\frac{\partial\rho}{\partial\tau}+\mathrm{div}P=0,
\end{aligned} \tag{2.65}$$

where $P=(P_r,P_\theta,P_\varphi)$ is the momentum density field, T is the temperature, p is the pressure, ρ is the energy density, ν and μ is the viscosity coefficient, β is the coefficient of thermal expansion, κ is the thermal diffusivity, α is as in (2.64), $\Delta P,(P\cdot\nabla)P,\widetilde{\Delta}T,$div P are as in (2.51)-(2.54), and

$$(P\cdot\nabla)T=\frac{P_\theta}{r}\frac{\partial T}{\partial\theta}+\frac{P_\varphi}{r\sin\theta}\frac{\partial T}{\partial\varphi}+P_r\frac{\partial T}{\partial r}. \tag{2.66}$$

The equations (2.65) are supplemented with the boundary conditions:

$$\begin{aligned}
&P_r=0, \quad \frac{\partial P_\theta}{\partial r}=0, \quad \frac{\partial P_\varphi}{\partial r}=0 && \text{at } r=r_0,r_1,\\
&T=T_0 && \text{at } r=r_0,\\
&T=T_1 && \text{at } r=r_1,
\end{aligned} \tag{2.67}$$

where T_0 and T_1 are approximatively taken as constants and satisfy the physical condition

$$T_0>T_1.$$

A few remarks are now in order:

Remark 2.7. First, there are three important parameters: the Rayleigh number Re, the Prandtl number Pr and the δ-factor δ, which play an important role in astrophysical fluid dynamics:

$$\mathrm{Re}=\frac{mGr_0r_1\beta}{\kappa\nu}\frac{T_0-T_1}{h}, \qquad \mathrm{Pr}=\frac{\nu}{\kappa}, \qquad \delta=\frac{2mG}{c^2r_0}. \tag{2.68}$$

The δ-factor δ reflects the relativistic effect contained the Laplacian operator. □

Remark 2.8. Astronomic observations show that the Sun has three layers of atmospheres: the photosphere, the chromosphere, and the solar corona, where the solar atmospheric convections occur. It manifests that the thermal convection is a universal phenomenon for stellar atmospheres. In the classical fluid dynamics, the Rayleigh number dictates the Rayleign-Bénard convection. Here, however, both the Rayleigh number Re and the δ-factor defined by (2.68) play an important role in stellar atmospheric convections. □

Remark 2.9. For rotating stars with angular velocity $\vec{\Omega}$, we need add to the right hand side of (2.65) the Coriolis term:

$$-2\vec{\Omega}\times P=2\Omega(\sin\theta P_r-\cos\theta P_\theta,\cos\theta P_\varphi,-\sin\theta P_\varphi),$$

where Ω is the magnitude of $\vec{\Omega}$. □

Fluid dynamical equations inside open balls

As the fluid density in a stellar atmosphere is small, the equations (2.65) can be regarded as a precise model governing the stellar atmospheric motion. However, for a fluid sphere with high density, the fluid dynamic equations have to couple the gravitational field equations.

The Universe and all stars are in the momentum-flow state, i.e. they are fluid spheres. To investigate the interiors of the Universe, galaxies and stars, we need to develop dynamic models for fluid spheres.

The precise equations of fluid sphere should be defined in the Riemannian metric space as follows:

$$ds^2=g_{00}(x,t)c^2dt^2+g_{ij}(x,t)dx^idx^j \qquad \text{for } x\in\mathcal{M}^3, \tag{2.69}$$

where $\mathcal{M}^3$ is the spherical space. The gravitational field equations are expressed as

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-\frac{8\pi G}{c^4}T_{\mu\nu}-D_\mu\Phi_\mu, \quad (g_{j0}=g_{0j}=0, 1\le j\le 3), \tag{2.70}$$

where

$$T_{\mu\nu}=g_{\mu\alpha}g_{\nu\beta}\left[\varepsilon^\alpha\varepsilon^\beta+pg^{\alpha\beta}\right],$$

and ε^μ is the 4D energy-momentum vector.

For the fluid component of the system, it is necessary to simplify the model by making some physically sound approximations.

Hypothesis 2.10. *The metric (2.69) and the stationary solutions of the fluid dynamical equations are radially symmetric.*

Under Hypothesis 2.10, the metric (2.69) is as in (2.19), or is written in the following form

$$ds^2 = -\psi(r)c^2dt^2 + \alpha(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2, \tag{2.71}$$

and the fluid dynamic equations are rewritten as

$$\begin{aligned}
&\frac{\partial P}{\partial \tau} + \frac{1}{\rho}(P\cdot\nabla)P = \nu\Delta P + \mu\nabla(\mathrm{div}P) - \nabla p - \frac{c^2\rho}{2\alpha}\frac{d\psi}{dr}(1-\beta(T-T_0))\vec{k},\\
&\frac{\partial T}{\partial \tau} + \frac{1}{\rho}(P\cdot\nabla)T = \kappa\widetilde{\Delta}T + Q(r),\\
&\frac{\partial \rho}{\partial \tau} + \mathrm{div}P = 0,
\end{aligned} \tag{2.72}$$

where $P=(P_r,P_\theta,P_\varphi),\nabla P,\widetilde{\Delta}T,(P\cdot\nabla)P$, div P are as in (2.51)-(2.54), ∇p is as in (2.50), $\vec{k}=(1,0,0)$, and

$$(P\cdot\nabla)T = P_r\frac{\partial T}{\partial r} + \frac{P_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{P_\varphi}{r\sin\theta}\frac{\partial T}{\partial \varphi}.$$

The gravitational field equation (2.70) for the metric (2.71) is radially symmetric, therefore

$$\Phi_\nu = D_\nu\phi, \quad \phi = \phi(r).$$

Thus we have

$$\begin{aligned}
&D_0D^0\phi = \frac{1}{2\alpha\psi}\psi'\phi', && D_1D^1\phi = \frac{1}{\alpha}\phi'' - \frac{1}{2\alpha^2}\alpha'\phi',\\
&D_2D^2\phi = D_3D^3\phi = \frac{1}{r\alpha}\phi', && D_\mu D^\nu\phi = 0 \qquad \text{for } \mu\neq\nu.
\end{aligned}$$

Then, in view of (2.22)-(2.25), the equation (2.70) can be expressed by

$$\begin{aligned}
&\frac{1}{\alpha}\left(\frac{1}{r^2} - \frac{\alpha'}{r\alpha}\right) - \frac{1}{r^2} = -\frac{8\pi G}{c^2}\rho_0 + \frac{1}{2\alpha\psi}\psi'\phi',\\
&\frac{1}{\alpha}\left(\frac{1}{r^2} + \frac{\psi'}{r\psi}\right) - \frac{1}{r^2} = \frac{8\pi G}{c^2}p + \frac{1}{\alpha}\phi'' - \frac{1}{2\alpha^2}\alpha'\phi',\\
&\frac{1}{\alpha}\left[\frac{\psi''}{\psi} - \frac{1}{2}\left(\frac{\psi'}{\psi}\right)^2 - \frac{\alpha'\psi'}{2\alpha\psi} + \frac{1}{r}\left(\frac{\psi'}{\psi} - \frac{\alpha'}{\alpha}\right)\right] = \frac{16\pi G}{c^2}p + \frac{2}{r\alpha}\phi',
\end{aligned} \tag{2.73}$$

where the pressure p satisfies the stationary equations of (2.72) with $P=0$ as follows

$$\begin{aligned}
&p' = -\frac{c^2}{2}\psi'\rho[1-\beta(T-T_0)],\\
&\frac{\kappa}{\alpha r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -Q(r).
\end{aligned} \tag{2.74}$$

The functions ψ and α satisfy the boundary conditions (2.27), i.e.

$$\psi(r_0)=1-\frac{2Gm}{c^2r_0}, \quad \alpha(r_0)=\left(1-\frac{2Gm}{c^2r_0}\right)^{-1}. \tag{2.75}$$

In addition, for the ordinary differential equations (2.73)-(2.75), we also need the boundary conditions for ψ',ϕ' and T. Since $-\frac{1}{2}c^2\psi'$ represents the gravitational force, the condition of ψ' at $r=r_0$ is given by

$$\psi'(r_0)=\frac{2mG}{c^2r_0^2}. \tag{2.76}$$

Based on the Newton gravitational law, ϕ' is very small in the external sphere; also see [5]. Hence we can approximatively take that

$$\phi'(r_0)=0. \tag{2.77}$$

Finally, it is rational to take the temperature gradient in the boundary condition as follows

$$\frac{\partial T}{\partial r}(r_0)=-A \qquad (A>0). \tag{2.78}$$

Let the stationary solution of the problem (2.73)-(2.75) be given by $\widetilde{p},\widetilde{T},\psi,\alpha,\phi'$. Make the translation transformation

$$P\to P, \quad p\to p+\widetilde{p}, \quad T\to T+\widetilde{T}.$$

Then equations (2.72) are rewritten in the form

$$\begin{aligned}
&\frac{\partial P}{\partial \tau}+\frac{1}{\rho}(P\cdot\nabla)P=\nu\Delta P-\nabla p+\frac{c^2\rho}{2\alpha}\frac{d\psi}{dr}\beta\vec{k}T,\\
&\frac{\partial T}{\partial \tau}+\frac{1}{\rho}(P\cdot\nabla)T=\kappa\widetilde{\Delta}T-\frac{1}{\rho}\frac{d\widetilde{T}}{dr}P_r,\\
&\text{div } P=0,
\end{aligned} \tag{2.79}$$

supplemented with the boundary conditions:

$$\frac{\partial T}{\partial r}=0, \quad P_r=0, \quad \frac{\partial P_\theta}{\partial r}=\frac{\partial P_\varphi}{\partial r}=0 \quad \text{at } r=r_0. \tag{2.80}$$

The model (2.79)-(2.80), we just derived describes interior dynamics of the Universe, galaxies and stars.

3 Stars

3.1 Main driving force for stellar dynamics

Stars can be regarded as fluid balls. To investigate the stellar interior dynamic behavior, we need to use the fluid spherical models coupling the heat conductivity equation. There are two types of starts: stable and unstable. The sizes of stable stars do not change. The main-sequence stars, white dwarfs and neutron stars are stable stars. The radii of unstable stars may change; variable stars and expanding red giants are unstable stars. The dynamic equations governing the two types of stars are different, and will be addressed hereafter separately.

We note that the fluid dynamic equations (2.72) represent the Newton's second law, and their left-hand sides are the acceleration and their right-hand sides are the total force. The total force consists of four parts: the viscous friction, the pressure gradient, the relativistic effect, and the thermal expansion force, which are given as follows:

- The *viscous friction force* is caused by the electromagnetic, the weak and the strong interactions between the particles and the pressure, and is given by

$$F_\nu P=\nu\triangle P=\nu(\Delta P_\theta,\Delta P_\varphi,\Delta P_r), \tag{3.1}$$

  as defined by (2.51).

- The *pressure gradient* is defined by:

$$-\nabla p=-\left(\frac{1}{r}\frac{\partial p}{\partial\theta},\frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi},\frac{1}{\alpha}\frac{\partial p}{\partial r}\right). \tag{3.2}$$

- The *relativistic effect* is reflected in the following terms:

$$F_G P=\left(-\frac{\nu}{2\alpha^2 r}\frac{d\alpha}{dr}\frac{\partial}{\partial r}(rP_\theta),-\frac{\nu}{2\alpha^2 r}\frac{d\alpha}{dr}\frac{\partial}{\partial r}(rP_\varphi),\frac{\nu}{2\alpha}\frac{\partial}{\partial r}\left(\frac{1}{\alpha}\frac{d\alpha}{dr}P_r\right)\right), \tag{3.3}$$

  which can be regarded as the coupling interaction between the gravitational potential α and the electromagnetic, the weak and the strong potentials represented by the viscous coefficient ν.

  We shall see that the force (3.3) is responsible to the supernova's huge explosion.

- The *thermal expansion force* is due to the coupling between the gravity $\nabla\psi$ and the heat Q:

$$F_T=\left(0,0,\frac{c^2}{2\alpha}\frac{d\psi}{dr}\beta T\right), \tag{3.4}$$

  which is the main driving force for generating stellar interior circulations and nebular matter spurts of red giants.

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The two forces (3.3) and (3.4) are the main driving forces for the stellar motion, and hereafter we derive their explicit formulas.

1) *Formula for thermal expansion force.* The thermal expansion force (3.4) is radially symmetric, which is simply written in the r-component form

$$f_T = \frac{c^2}{2\alpha}\frac{d\psi}{dr}\beta T.$$

In its nondimensional form, f_T is expressed as

$$f_T = \sigma(r) T, \tag{3.5}$$

and $\sigma(r)$ is called the thermal factor given by

$$\sigma(r) = -\frac{c^2 r_0^4 \beta}{2\kappa^2}\frac{1}{\alpha}\frac{dT}{dr}\frac{d\psi}{dr}. \tag{3.6}$$

Here α,ψ,T satisfy equations (2.73)-(2.74) with the boundary conditions (2.75)-(2.78). The detailed derivation of (3.5)-(3.6) will be given hereafter.

The σ-factor (3.6) can be expressed in the following form, to be deduced later:

$$\begin{aligned}\sigma(r) = & \frac{c^2 r_0^3 (1-\delta)\beta}{2\kappa^2 r^2}\frac{e^{\zeta(r)}}{e^{\zeta(1)}}\cdot(1-\delta r^2-\eta)\cdot\left(\frac{1}{r^2}\frac{\delta r^2+\eta}{1-\delta r^2-\eta}+r\xi\right) \\ & \cdot\left(A-\frac{1}{\kappa}\int_r^1 \frac{r^2 Q}{1-\delta r^2-\eta}dr\right) \qquad \text{for } 0\le r\le 1,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}&\eta = \frac{1}{2r}\int_0^r \frac{r^2\psi'\phi'}{\alpha\psi}dr, \\ &\zeta = \int_0^r \left(\frac{\alpha}{r}+r\xi\right)dr, \\ &\xi = \frac{8\pi G}{c^2}\alpha p+\phi''-\frac{\alpha'\phi'}{2\alpha} \qquad \text{for } 0\le r\le 1,\end{aligned} \tag{3.8}$$

δ is called the δ-factor given by

$$\delta = \frac{2mG}{c^2 r_0}, \tag{3.9}$$

and m,r_0 are the mass and radius of the star.

2) *Formula for the relativistic effect.* The term $F_G P$ in (3.3) can be expressed in the following form:

$$F_G P = \begin{pmatrix} -\nu\left(\delta+\frac{\eta'}{2r}\right)\frac{\partial}{\partial r}(rP_\theta) \\ -\nu(\delta+\frac{\eta'}{2r})\frac{\partial}{\partial r}(rP_\varphi) \\ \frac{\nu}{2}\left(\frac{(2\delta r+\eta')^2}{1-\delta r^2-\eta}+2\delta+\eta''\right)P_r+\frac{\nu}{2}(2\delta r+\eta')\frac{\partial}{\partial r}P_r \end{pmatrix}, \tag{3.10}$$

where η,δ are as in (3.8) and (3.9).

The force $F_G P$ is of special importance in studying supernovae, black holes, and the galaxy cores. In fact, by the boundary conditions (2.75)-(2.77), we can reduce that the radial component of the force (3.10) on the stellar shell is as

$$f_r=\left(\frac{2\nu\delta^2}{1-\delta}+\delta+\phi''(1)\right)P_r+\delta\nu\frac{\partial P_r}{\partial r},$$

which has

$$f_r\sim\frac{2\nu\delta^2}{1-\delta}P_r\to\infty \quad \text{as} \quad \delta\to 1 \ (\text{for } P_r>0). \tag{3.11}$$

The property (3.11) will lead to a huge supernovae explosions as they collapse to the radii $r_0\to 2mG/c^2$. It is the explosive force (3.11) that prevents the formation of black holes; see Sections 3.4 and 4.3.

3) *Derivation of formula (3.10).* To deduce (3.10) we have to derive the gravitational potential α. The first equation of (2.73) can be rewritten as

$$\frac{dM}{dr}=4\pi r^2\rho_0-\frac{c^2}{4G}\frac{r^2\psi'\phi'}{\alpha\psi}, \qquad\qquad M=\frac{c^2 r}{2G}(1-\frac{1}{\alpha}),$$

It gives the solution as

$$M=\frac{4}{3}\pi r^3\rho_0-\frac{c^2}{4G}\int_0^r\frac{r^2\psi'\phi'}{\alpha\psi}dr, \qquad\qquad \alpha=\left(1-\frac{2MG}{c^2 r}\right)^{-1}.$$

By $\rho_0=m/\frac{4}{3}\pi r_0^3$, and in the nondimensional form $(r\to r_0 r)$, we get

$$\alpha=(1-\delta r^2-\eta)^{-1} \qquad \text{for } 0\le r\le 1, \tag{3.12}$$

where η,δ are as in (3.8) and (3.9). By (2.75) we have

$$\eta(1)=0, \qquad (\text{i.e. } \eta(r_0)=0). \tag{3.13}$$

Then, the formula (3.10) follows from (3.12).

4) *Derivation of σ-factor (3.7).* By (3.6) we need to calculate T' and ψ'. By (2.74), T' can be expressed in the form

$$\frac{dT}{dr}=-\frac{1}{\kappa r^2}\int_0^r r^2\alpha Q dr+\frac{a}{r^2},$$

where a is a determined constant. By (2.78) we obtain

$$a=-Ar_0^2+\frac{1}{\kappa}\int_0^{r_0} r^2\alpha Q dr.$$

In the nondimensional form, we have

$$\frac{dT}{dr} = -\frac{A}{r^2} + \frac{1}{\kappa r^2}\int_r^1 r^2 \alpha Q dr \qquad \text{for } 0 \le r \le 1, \quad A > 0. \tag{3.14}$$

To consider ψ', by the second equation of (2.73) we obtain

$$\psi = \frac{k}{r} e^{\zeta(r)}, \tag{3.15}$$

where $\zeta(r)$ is as in (3.8), k is a to-be-determined constant. In view of (2.75), i.e. $\psi(1) = 1-\delta$, we have

$$k = (1-\delta)e^{-\zeta(1)}.$$

Then, it follows from (3.15) that

$$\frac{d\psi}{dr} = \frac{(1-\delta)e^{\zeta(r)}}{e^{\zeta(1)r_0}} \left(\frac{\alpha - 1}{r^2} + r\xi \right) \qquad \text{for } 0 \le r \le 1, \tag{3.16}$$

where ξ is as in (3.8). By (2.76) and (3.12)-(3.13), we can deduce that

$$\xi(1) = 0, \quad (\text{i.e. } \xi(r_0) = 0). \tag{3.17}$$

Thus, the σ-factor (3.7) follows from (3.12), (3.14) and (3.16).

5) *Thermal Force and (3.17).* The thermal expansion force acting on the stellar shell (i.e. at $r = r_0$) can be deduced from (3.5) and (3.7) in the following (nondimensional) form

$$f_T = \sigma_0 T, \quad \sigma_0 = \frac{c^2 r_0^3 \beta (1-\delta)\delta}{2\kappa^2} A \quad (A > 0). \tag{3.18}$$

By $0 < \delta < 1$, we have

$$\sigma_0 > 0 \quad (\sigma_0 = \sigma(1), \text{ i.e. } \sigma(r_0)).$$

Hence, it follows that there is an $\varepsilon \ge 0$ such that

$$\sigma(r) > 0, \quad \text{for } \varepsilon < r \le 1. \tag{3.19}$$

The positiveness of $\sigma(r)$ in (3.19) shows that the thermal force f_T of (3.18) is an outward expanding force. It is this power that causes the swell and the nebular matter spurt of a red giant. We also remark that the temperature gradient A on the boundary is maintained by the heat source Q.

3.2 Stellar interior circulation

Recapitulation of dynamic transition theory

First we briefly recall the dynamic transition theory developed by the authors in [3] and the references therein. Many dissipative systems, both finite and infinite dimensional, can be written in an abstract operator equation form as follows

$$\frac{du}{dt}=L_\lambda u+G(u,\lambda), \tag{3.20}$$

where L_λ is a linear operator, G is a nonlinear operator, and λ is the control parameter.

It is clear that $u=0$ is a stationary solution of (3.20). We say that (3.20) undergoes a dynamic transition from $u=0$ at $\lambda=\lambda_1$ if $u=0$ is stable for $\lambda<\lambda_1$, and unstable for $\lambda>\lambda_1$. The dynamic transition of (3.20) depends on the linear eigenvalue problem:

$$L_\lambda\varphi=\beta(\lambda)\varphi.$$

Mathematically this eigenvalue problem has eigenvalues $\beta_k(\lambda)\in\mathbb{C}$ such that

$$\mathrm{Re}\beta_1(\lambda)>\mathrm{Re}\beta_2(\lambda)>\cdots.$$

The following are the main conclusions for the dynamic transition theory; see [3] for details:

- Dynamic transitions of (3.20) take place at $(u,\lambda)=(0,\lambda_1)$ provided that λ_1 satisfies the following principle of exchange of stability (PES):

$$\begin{aligned}
&\mathrm{Re}\beta_1\begin{cases}<0 & \text{for } \lambda<\lambda_1 \ (\text{or } \lambda>\lambda_1),\\ =0 & \text{for } \lambda=\lambda_1,\\ >0 & \text{for } \lambda>\lambda_1 \ (\text{or } \lambda<\lambda_1),\end{cases}\\
&\mathrm{Re}\beta_k(\lambda_1)<0 \qquad\qquad\qquad \forall k\geq 2.
\end{aligned} \tag{3.21}$$

- Dynamic transitions of all dissipative systems described by (3.20) can be classified into three categories: continuous, catastrophic, and random. Thanks to the universality, this classification is postulated in citeptd as a general principle called principle of dynamic transitions.

- Let u_λ be the first transition state. Then we can also use the same stratege outlined above to study the second transition by considering PES for the following linearized eigenvalue problem
$$L_\lambda\varphi+DG(u_\lambda,\lambda)\varphi=\beta^{(2)}(\lambda)\varphi.$$
Also we know that successive transitions can lead to chaos.

Stellar interior circulation

The governing fluid component equations are (2.79). We first make the nondimensional. Let

$$
\begin{aligned}
&(r,\tau)=(r_0r',r_0^2\tau'/\kappa),\\
&(P,T,p)=\left(\kappa P'/r_0,-\frac{d\widetilde{T}}{dr}r_0T',\rho_0\kappa^2p'/r_0^2\right).
\end{aligned}
$$

Then the equations (2.79) are rewritten as (drop the primes):

$$
\begin{aligned}
&\frac{\partial P}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)P=\Pr\Delta P+\frac{1}{\kappa}F_GP+\sigma(r)T\vec{k}-\nabla p,\\
&\frac{\partial T}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)T=\widetilde{\Delta}T+P_r,\\
&\operatorname{div}P=0,
\end{aligned}
\tag{3.22}
$$

where $P=(P_\theta,P_\varphi,P_r),\vec{k}=(0,0,1)$, $\sigma(r)$ and F_GP are as in (3.7) and (3.10), $\Pr=\nu/\kappa$ is the Prandtl number, and the Δ is given by

$$
\begin{aligned}
&\Delta P_\theta=\widetilde{\Delta}P_\theta+\frac{2}{r^2}\frac{\partial P_r}{\partial\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial P_\varphi}{\partial\varphi}-\frac{P_\theta}{r^2\sin^2\theta},\\
&\Delta P_\varphi=\widetilde{\Delta}P_\varphi+\frac{2}{r^2\sin\theta}\frac{\partial P_r}{\partial\varphi}+\frac{2\cos\theta}{r^2\sin\theta}\frac{\partial P_\theta}{\partial\varphi}-\frac{P_\varphi}{r^2\sin^2\theta},\\
&\Delta P_r=\widetilde{\Delta}P_r-\frac{2}{\alpha r^2}\left(P_r+\frac{\partial P_\theta}{\partial\theta}+\frac{\cos\theta}{\sin\theta}P_\theta+\frac{1}{\sin\theta}\frac{\partial P_\varphi}{\partial\varphi}\right),
\end{aligned}
\tag{3.23}
$$

Based on the dynamic transition theory introduced early in this section, the stellar circulation depends on the following three forces:

$$
\Pr\Delta P,\quad \frac{1}{\kappa}F_GP,\quad \sigma T\vec{k},
\tag{3.24}
$$

where in general $\Pr\Delta P$ prevents/slows-down the circulation.

By (3.10) we see that F_GP depends on the δ-factor. The Sun's δ-factor is $\delta_\odot=10^{-5}/2$, and in general the δ-factors for stars are of the order:

$$
\begin{aligned}
&\delta\sim10^{-8} && \text{for red giants,}\\
&\delta\sim10^{-5} && \text{for main-sequence stars,}\\
&\delta\sim10^{-3} && \text{for white dwarfs,}\\
&\delta\sim10^{-1} && \text{for neutron stars,}\\
&\delta=1 && \text{for black holes.}
\end{aligned}
\tag{3.25}
$$

Hence it is clear that all stars, except black holes, have small δ-factors.

On the other hand, for a small δ-factor, it follows from equations (2.73) and (2.74) that α,ψ,ϕ has the order

$$\begin{aligned} &\alpha\sim 1+\delta+\eta, && \eta,\eta'\sim\delta^2, && \alpha'\sim\delta,\\ &\psi\sim r^{\delta}, && \psi'/\psi\sim\delta, && \phi',\phi''\sim\delta. \end{aligned}$$

Hence, we deduce from (3.10) that

$$F_G P\sim\delta P \quad \text{for} \quad \delta\ll 1.$$

Thus, in view of (3.25) we conclude that the relativistic effect $F_G P$ is negligible on the stellar interior motion for all stars except supernovae.

Hence the main driving force for stellar circulations is the thermal expansion force characterized by the σ-factor σ_0 in (3.18). Due to $\delta\ll 1,\sigma_0$ can be approximately given by

$$\sigma_0=\frac{r_0^2 mG\beta}{\kappa^2}A, \tag{3.26}$$

which plays the similar role as the Rayleigh number Re in the earth's atmospheric circulation. The value σ_0 of (3.26) is large enough to generate thermal convections for main-sequence stars and red giants.

We remark that the heat source Q is caused not only by nuclear reactions, but also by the pressure gradient, the density and the gravitational potential energy. Based on the σ-factor in (3.26), we obtain the following physical conclusions:

1) *Main-sequence stars.* Based on the dynamic transition theory, by (3.19), we deduce that there is a critical number $\sigma_c>0$, independent of the parameter σ_0 in (3.26), such that equations (3.22) undergo no dynamic transition if $\sigma_0<\sigma_c$, and a dynamic transition if $\sigma_0>\sigma_c$:

$$\sigma_0-\sigma_c\begin{cases} <0 & \text{there is no stellar circulation,}\\ >0 & \text{there exists stellar circulation.}\end{cases} \tag{3.27}$$

In particular, σ_c has the same order of magnitude as the first eigenvalue λ_1 of the the following equations in the unit ball B_1:

$$\begin{aligned} &-\mathrm{Pr}\Delta P+\nabla p=\lambda_1 P && \text{for } x\in B_1\subset\mathbb{R}^3,\\ &\left(P_r,\frac{\partial P_\theta}{\partial r},\frac{\partial P_\varphi}{\partial r}\right)=0 && \text{at } r=1, \end{aligned}$$

where ΔP is as in (3.23).

For the main-sequence stars, the σ-factors are much larger than the first eigenvalue λ_1 of (3.28). For example, the Sun consists of hydrogen, and

$$r_0=7\times 10^8\text{m}, \quad m=2\times 10^{30}\text{kg}, \quad G=6.7\times 10^{-11}\text{m}^3/\text{kg}\cdot\text{s}^2.$$

Using the average temperature $T=10^6K$, the parameter κ is given by

$$\kappa=0.18\left(\frac{T}{190k}\right)^{1.72}10^{-4}\mathrm{m}^2/\mathrm{s}\simeq 50m^2/s.$$

With thermal expansion coefficient β in the order $\beta\sim 10^{-4}/K$, the σ-factor of (3.26) for the Sun is about

$$\sigma_{\odot}\sim 10^{30}A \quad [\mathrm{m/K}]. \tag{3.28}$$

Due to nuclear fusion, stars have a constant heat supply, which leads to a higher boundary temperature gradient A. Referring to (3.28), we conclude that there are always interior circulations and thermal motion in main-sequence stars and red giants, which has large σ-factors.

2) *Red giants.* The nuclear reaction of a red giant stops in its core, but does take place in the shell layer, which maintains a larger temperature gradient A on the boundary than the main-sequence phase. Therefore, the greater σ-factor makes the star to expand, and the increasing radius r_0 raises the σ-factor (3.26). The increasingly larger σ-factor provides a huge power to hurl large quantities of gases into space at very high speed.

3) *Neutron stars and pulsars.* Neutron stars are different from other stars, which have bigger δ-factors, higher rotation Ω and lower σ-factor (as the nuclear reaction stops). Instead of (3.22) the dynamic equations governing neutron stars are

$$\begin{aligned}&\frac{\partial P}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)P=\mathrm{Pr}\Delta P+\frac{1}{\kappa}F_G P-2\vec{\Omega}\times P-\nabla p+\sigma T\vec{k},\\&\frac{\partial T}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)T=\widetilde{\Delta}T+P_r,\\&\mathrm{div}P=0.\end{aligned} \tag{3.29}$$

As the nuclear reaction ceases, the temperature gradient A tends to zero as time $t\to\infty$, and consequently

$$\sigma\to 0 \quad \text{as} \quad t\to\infty. \tag{3.30}$$

Based on the dynamic transition theory briefly recalled earlier in this section, we derive from (3.29) and (3.30) the following assertions:

- By (3.30), neutron stars will eventually stop convection.
- Due to the high rotation Ω, the convection of (3.29) for the early neutron star is time periodic, and its period $\mathcal{T}$ is inversely proportional to Ω, and its convection intensity B is proportional to $\sqrt{\sigma-\sigma_c}$, i.e.

$$\begin{aligned}&\text{period } \mathcal{T}\simeq\frac{C_1}{\Omega},\\&\text{intensity } B\simeq C_2\sqrt{\sigma-\sigma_c},\end{aligned} \tag{3.31}$$

where σ_c is the critical value of the transition, and c_1,c_2 are constants. The properties of (3.31) explain that the early neutron stars are pulsars, and by (3.30) their pulsing radiation intensities decay at the rate of $\sqrt{\sigma-\sigma_c}$ or $\sqrt{kA-\sigma_c}$ (k is a constant).

3.3 Dynamics of stars with variable radii

For stars with varying sizes and for supernovae, their radii expand and shrink periodically. Therefore, the metric in the interior of such stars is as follows:

$$ds^2=-\psi c^2dt^2+R^2(t)\left[\alpha dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)\right],$$

where $\psi=\psi(r,t),\alpha=\alpha(r,t)$, and $R(t)$ is the scalar factor representing the star radius. For convenience, we denote

$$\psi=e^{u(r,t)},\quad \alpha=e^{v(r,t)},\quad R^2(t)=e^{k(t)},\quad 0\le r\le 1.$$

Then the metric is rewritten as

$$ds^2=-e^uc^2dt^2+e^k\left[e^vdr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)\right]. \tag{3.32}$$

The stars with variable radii are essentially in radial motion. Hence, the horizontal momentum (P_θ,P_φ) is assumed to be zero:

$$(P_\theta,P_\varphi)=0. \tag{3.33}$$

In the following we develop dynamic models for astronomical objects with variable sizes.

1). *Gravitational field equations.* We recall the gravitational field equations [5]:

$$R_{\mu\nu}=-\frac{8\pi G}{c^4}(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T)-(D_{\mu\nu}\phi-\frac{1}{2}g_{\mu\nu}\Phi). \tag{3.34}$$

The nonzero components of the metric (3.32) are

$$g_{00}=-e^u,\quad g_{11}=e^{k+v},\quad g_{22}=e^kr^2,\quad g_{33}=e^kr^2\sin^2\theta,$$

the nonzero components of the Levi-civita connections are

$$\begin{aligned}
&\Gamma^0_{00}=\frac{1}{2c}u_t, && \Gamma^0_{11}=\frac{1}{2c}e^{v-u}(k_t+v_t), && \Gamma^0_{10}=\frac{1}{2}u_r,\\
&\Gamma^0_{22}=\frac{r^2}{2c}e^{-u}k_t, && \Gamma^0_{33}=\frac{r^2}{2c}e^{-u}k_t\sin^2\theta, && \Gamma^1_{00}=\frac{1}{2}e^{u-v}u_r\\
&\Gamma^1_{11}=\frac{1}{2}v_r, && \Gamma^1_{10}=\frac{1}{2c}(k_t+v_t), && \Gamma^1_{22}=-re^{-v},\\
&\Gamma^1_{33}=-re^{-v}\sin^2\theta, && \Gamma^2_{21}=\frac{1}{r}, && \Gamma^2_{33}=\sin\theta\cos\theta,\\
&\Gamma^3_{31}=\frac{1}{r}, && \Gamma^3_{32}=\frac{\cos\theta}{\sin\theta},
\end{aligned}$$

and the nonzero components of the Ricci curvature tensor read

$$
\begin{aligned}
R_{00}=&\frac{1}{2c^2}\left[3k_{tt}+\frac{3}{2}k_t^2+v_{tt}+\frac{1}{2}v_t^2+k_tv_t-u_t(k_t+v_t)\right]-\frac{1}{2}e^{u-k-v}\left[u_{rr}+\frac{1}{2}u_r^2-\frac{1}{2}u_rv_r+\frac{2}{r}u_r\right],\\
R_{11}=&-\frac{e^{k+v-u}}{2c^2}\left[k_{tt}+\frac{3}{2}k_t^2+v_{tt}+v_t^2+3k_tv_t-\frac{1}{2}u_t(k_t+v_t)\right]+\frac{1}{2}\left[u_{rr}+\frac{1}{2}u_r^2-\frac{1}{2}u_rv_r-\frac{2}{r}v_r\right],\\
R_{22}=&-\frac{r^2e^{k-u}}{2c^2}\left[k_{tt}+\frac{3}{2}k_t^2+\frac{1}{2}k_t(v_t-u_t)\right]-e^{-v}\left[e^v+\frac{r}{2}(k_r+v_r-u_r)-1\right],\\
R_{33}=&R_{22}\sin^2\theta,\\
R_{10}=&-\frac{1}{cr}\left[(1+\frac{r}{2}u_r)k_t+v_r\right].
\end{aligned}
$$

The energy-momentum tensor is in the form

$$
T_{\mu\nu}=\begin{pmatrix}
\rho & g_{00}g_{11}P_rc & 0 & 0\\
g_{00}g_{11}P_rc & g_{11}p & 0 & 0\\
0 & 0 & g_{22}p & 0\\
0 & 0 & 0 & g_{33}p
\end{pmatrix},
$$

where ρ is the energy density, P_r is the radial component of the momentum density. Then direct computations imply that

$$
\begin{aligned}
&T=g^{\mu\nu}T_{\mu\nu}=-\rho+3p, && T_{00}-\frac{1}{2}g_{00}T=\frac{1}{2}(\rho+3p),\\
&T_{11}-\frac{1}{2}g_{11}T=\frac{1}{2}e^{k+v}(\rho-p), && T_{22}-\frac{1}{2}g_{22}T=\frac{1}{2}e^kr^2(\rho-p),\\
&T_{33}-\frac{1}{2}g_{33}T=(T_{22}-\frac{1}{2}g_{33}T)\sin^2\theta, && T_{10}-\frac{1}{2}g_{10}T=g_{00}g_{11}P_rc.
\end{aligned}
$$

To derive an explicit expression of (3.34), we need to compute the covariant derivatives of the dual gravitational field ϕ:

$$
D_{\mu\nu}\phi=\frac{\partial^2\phi}{\partial x^\mu\partial x^\nu}-\Gamma^\lambda_{\mu\nu}\frac{\partial\phi}{\partial x^\lambda}.
$$

Let $\phi=\phi(r,t)$. Then we have

$$
\begin{aligned}
D_{00}\phi=&\frac{1}{c^2}\phi_{tt}-\frac{1}{2c^2}u_t\phi_t-\frac{1}{2}e^{u-v}u_r\phi_r,\\
D_{11}\phi=&\phi_{rr}-\frac{1}{2c^2}e^{v-u}(k_t+v_t)\phi_t-\frac{1}{2}v_r\phi_r,\\
D_{22}\phi=&-\frac{r^2}{2c^2}e^{-u}k_t\phi_t+re^{-v}\phi_r,\\
D_{33}\phi=&D_{22}\phi\sin^2\theta,\\
D_{10}\phi=&\phi_{rt}-\frac{1}{2c}(u_r\phi_t+\phi_rk_t+\phi_rv_t).
\end{aligned}
$$

Thus, the field equations (3.34) are written as

$$R_{10} = -D_1 D_0 \phi,$$
$$R_{kk} = -\frac{8\pi G}{c^4}\left(T_{kk} - \frac{1}{2}g_{kk}T\right) - \left(D_{kk}\phi - \frac{1}{2}g_{kk}\Phi\right) \qquad \text{for } k = 0,1,2,$$

which are expressed as

$$\left(1+\frac{ru_r}{2}\right)k_t + v_t = \frac{8\pi Gr}{c^2}e^{u+k+v}P_r + cr\phi_{rt} - \frac{r}{2}(u_r\phi_t + \phi_r k_z + \phi_r v_t), \tag{3.35}$$

$$\begin{aligned}&3k_{tt} + \frac{3}{2}k_t^2 + v_{tt} + \frac{1}{2}v_t^2 + k_t v_t - u_t(k_t+v_t) - c^2 e^{u-k-v}\left[u_{rr} + \frac{1}{2}u_r^2 - \frac{1}{2}u_r v_r + \frac{2}{r}u_r\right]\\ &\quad = -\frac{8\pi G}{c^2}(\rho+3p) - c^2\left(D_{00}\phi + e^{u-k-v}D_{11}\phi + \frac{2e^{u-k}}{r^2}D_{22}\phi\right),\end{aligned} \tag{3.36}$$

$$\begin{aligned}&k_{tt} + \frac{3}{2}k_t^2 + v_{tt} + v_t^2 + 3k_t v_t - \frac{1}{2}u_t(k_t+v_t) - c^2 e^{u-k-v}\left[u_{rr} + \frac{1}{2}u_r^2 - \frac{1}{2}u_r v_r - \frac{2}{r}v_r\right]\\ &\quad = \frac{8\pi G}{c^2}e^u(\rho-p) + c^2\left(e^{u-k-v}D_{11}\phi - D_{00}\phi - \frac{2e^{u-k}}{r^2}D_{22}\phi\right),\end{aligned} \tag{3.37}$$

$$\begin{aligned}&k_{tt} + \frac{3}{2}k_t^2 + \frac{1}{2}k_t(v_t-u_t) + \frac{2c^2e^{u-k-v}}{r^2}\left[e^v + \frac{r}{2}(k_r+v_r-u_r) - 1\right]\\ &\quad = \frac{8\pi G}{c^2}e^u(\rho-p) + c^2(D_{00}\phi - e^{u-k-v}D_{11}\phi),\end{aligned} \tag{3.38}$$

The equations (3.35)-(3.38) have seven unknown functions u,v,k,ϕ,P_r,ρ,p, in which P_r,ρ,p satisfy the fluid dynamic equations and the equation of state introduced hereafter.

2) *Fluid dynamic model.* The fluid dynamic model takes the momentum representation equations coupling the heat equation. Under the condition (3.33) and the radial symmetry, they are given as follows:

$$\begin{aligned}\frac{\partial P_r}{\partial \tau} + \frac{1}{\rho}P_r\frac{\partial P_r}{\partial r} + \frac{1}{2}\frac{\partial v}{\partial r}P_r^2 &= \nu e^{-v}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P_r}{\partial r}\right) - \frac{2}{r^2}P_r + \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}P_r\right)\right]\\ &\quad + \gamma e^{-v}\frac{\partial}{\partial r}\left[\frac{e^{-v/2}}{r^2}\frac{\partial}{\partial r}(r^2e^{v/2}P_r)\right] - e^{-v}\left[\frac{\partial p}{\partial r} - \frac{\rho}{2}(1-\beta T)\frac{\partial e^u}{\partial r}\right],\end{aligned} \tag{3.39}$$

$$\frac{\partial T}{\partial \tau} + \frac{1}{\rho}P_r\frac{\partial T}{\partial r} = \frac{\kappa e^{-v}}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + Q(r), \tag{3.40}$$

$$\frac{\partial \rho}{\partial \tau} + \frac{e^{-v/2}}{r^2}\frac{\partial}{\partial r}(r^2e^{v/2}P_r) = 0. \tag{3.41}$$

3) *Equation of state.* We know that the gravitational field equations represent the law of gravity, which essentially dictates the gravity related unknowns: $e^u, e^v, R = e^{k/2}, \phi$.

The laws for describing the matter field are the motion equation (3.39), the heat equation (3.40), and the energy conservation equation (3.41). To close the system, one needs to supplement an equation of state given by thermal dynamics, which provides a relation between temperature T, pressure p, and energy density ρ:

$$f(T,p,\rho)=0, \tag{3.42}$$

which depends on the underlying physical system.

In summary, we have derived a consistent model coupling the gravitational field equations, the fluid dynamic equations and the equation of state consists of eight equations solving for eight unknowns: $\psi=e^u$, $\alpha=e^v$, $R=e^{k/2}$, ϕ, P_r, T, p and ρ.

4) *Energy conservation formula.* From the energy conservation equation (3.41), we can deduce energy conservation in the following form

$$R^3r^2e^{v/2}P_r+\frac{1}{4\pi}\frac{d}{dt}E_r=0 \qquad \text{for } 0<r<1, \tag{3.43}$$

where $r=1$ stands for the boundary $R=e^{k/2}$ of the star, E_r is the total energy in the ball B_r with radius r.

To see this, we first note that the volume differential element of the Riemannian manifold is given by

$$dV=\sqrt{g_{11}g_{22}g_{33}}drd\theta d\varphi=e^{3k/2}r^2e^{v/2}\sin\theta drd\theta d\varphi.$$

Taking volume integral for (3.41) on B_r implies that

$$\frac{d}{dt}\int_{B_r}\rho dV+R^3\int_0^r\frac{\partial}{\partial r}\left(r^2e^{v/2}P_r\right)\int_0^\pi\int_0^{2\pi}\sin\theta d\theta d\varphi=0,$$

which leads to

$$\frac{dM_r}{dt}+4\pi R^3r^2e^{v/2}P_r=0,$$

and (3.43) follows.

5) *Shock wave.* As the total energy E_R of the star is invariant, we have

$$\frac{d}{dt}E_R=0.$$

It follows from (3.43) that

$$P_r=0 \quad \text{on } r=1 \quad (\text{i.e. on the boundary } R). \tag{3.44}$$

On the other hand, the physically sound boundary condition for the star with variable radius is

$$\frac{\partial P_r}{\partial r}=0 \quad \text{on } r=1, \tag{3.45}$$

which means that there is no energy exchange between the star and its exterior. Thus, (3.44) and (3.45) imply that there is a shock wave outside the star near the boundary.

Remark 3.1. Formula (3.43) is very important. In fact, due to the boundary condition (2.75) and $e^{v/2} \simeq 1/\sqrt{1-\delta}$, in the star shell layer, (3.43) can be approximately written as

$$\rho P_r = -\frac{\sqrt{1-\delta}}{4\pi R^2}\frac{d}{dt}M_r \quad \text{for } R-r>0 \text{ small}, \tag{3.46}$$

where $\delta = 2M_{r_0}G/c^2R$. This shows that a collapsing supernova is prohibited to shrink into a black hole ($\delta = 1$). In fact, the strongest evidence for showing that black holes cannot be created comes from the relativistic effect of (3.10), which provides a huge explosive power in the star shell layer given by

$$\frac{\nu\delta^2}{1-\delta}P_r \to \infty \quad \text{as } \delta \to 1 \quad (P_r \neq 0). \tag{3.47}$$

Here P_r is the convective momentum different from the contracting momentum P_r in (3.46); see Section 4.3 for details. □

Remark 3.2. One difficulty encountered in the classical Einstein field equations is that the number of unknowns is less than the number of equations, and consequently the coupling between the field equations and fluid dynamic and heat equations become troublesome. □

3.4 Mechanism of supernova explosion

In its late stage of life, a massive red giant collapses, leading to a supernova's huge explosion. It was still a mystery where does the main source of driving force for the explosion come from, and the current viewpoint, that the blast is caused by the large amount of neutrinos erupted from the core, is not very convincing.

The stellar dynamic model (2.72)-(2.78) provides an alternative explanation for supernova explosions, and we proceed in a few steps as follows:

1). When a very massive red giant completely consumes its central supply of nuclear fuels, its core collapses. Its radius r_0 begins to decrease, and consequently the δ-factor increases:

$$r_0 \text{ decreases} \Rightarrow \delta = \frac{2mG}{c^2 r_0} \text{ increases.}$$

2). The huge mass m and the rapidly reduced radius r_0 make the δ-factor approaching one:

$$\delta \to 1 \quad \text{as } r_0 \to R_s$$

where $R_s = 2mG/c^2$ is the Schwarzschild radius.

3). By (3.46), the shrinking of the star slows down:

$$P_r \sim \sqrt{1-\delta},$$

and nearly stops as $\delta \to 1$.

4). Then the model (3.22) is valid, and the eigenvalue equations of (3.22) are given by

$$\begin{aligned}&\Pr\Delta P+\frac{1}{\kappa}F_G P+\sigma T\vec{k}-\nabla p=\beta P,\\&\widetilde{\Delta}T+P_r=\beta T,\\&\mathrm{div}P=0.\end{aligned}\tag{3.48}$$

The first eigenvalue β depends on the δ-factor, and by (3.10)

$$\beta_1\sim\left(\frac{\Pr\delta^2}{1-\delta}\right)^{1/2}\quad\text{as }\delta\to 1.\tag{3.49}$$

Based on the transition criterion (3.21), the property (3.49) implies that the star has convection in the shell layer, i.e., the radial circulation momentum flux P_r satisfies

$$P_r>0\quad\text{in certain regions of the shell layer.}$$

5). The radial force (3.11) in the shell layer is

$$f_r\simeq\frac{2P_r\delta^2}{1-\delta}P_r\to\infty\quad\text{as }\delta\to 1\text{ and }P_r>0,$$

which provides a very riving force, resulting in the supernova explosion, as shown in Figure 3.1.

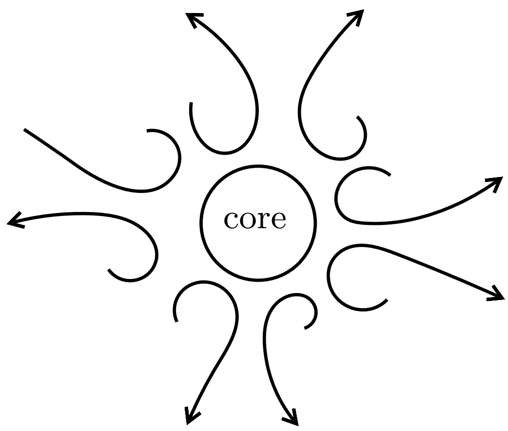


Figure 3.1: Circulation in a shell layer causing blast.

6). Since $P_r=0$ at $r=r_0$, the radial force of (3.10) is zero:

$$f_r=0\quad\text{at}\quad r=r_0.$$

Here r_0 is the radius of the blackhole core. Hence, the supernova's huge explosion preserves an interior core of smaller radius containing the blackhole core, which yields a neutron star. In particular, the huge explosion has no imploding force, and will not generate a new black hole.

The analysis in the above steps 1)-6) provides the supernova exploding mechanics, and clearly provide the power resource of the explosion.

In addition, by (3.8) and (3.12) we have

$$\alpha=\frac{1}{1-\delta r^2/r_0^2-\eta(r)}, \qquad \eta(r)=\frac{1}{2r}\int_0^r\frac{r^2\psi'\phi'}{\alpha\psi}dr \qquad \text{for } 0\le r\le r_0.$$

We can verify that

$$\eta(r)>0 \qquad \text{for } 0<r<r_0. \tag{3.50}$$

In fact, by (3.13) and (3.8) we have

$$\eta(0)=0, \quad \eta(r_0)=0. \tag{3.51}$$

Therefore, η has an extremum $\bar{r}$ $(0<\bar{r}<r)$ satisfying

$$\eta'(\bar{r})=0.$$

Let $\eta=\frac{1}{r}f$. Then

$$\eta'(r)=0 \Rightarrow f(r)=e^a r \quad (a=\text{constant}).$$

Hence, at the extremum $\bar{r},\eta$ takes a positive value

$$\eta(\bar{r})=\frac{1}{\bar{r}}f(\bar{r})=e^a>0 \qquad \text{for } 0<\bar{r}<r_0. \tag{3.52}$$

Thus, (3.50) follows from (3.51) and (3.52).

The fact (3.50) implies that the critical δ-factor δ_c for the supernova explosion is less than one, i.e. $\delta_c<1$.

4 Black Holes

4.1 Geometrical realization of black holes

The concept of black holes was originated from the Einstein general theory of relativity. Based on the Einstein gravitational field equations, K. Schwarzschild derived in 1916 an exact exterior solution for a spherically symmetrical matter field, and Tolman-Oppenheimer-Volkoff derived in 1939 an interior solution; see Section 2.3. In both solutions if the radius R of the matter field with mass M is less than or equal to a critical radius R_s, called the Schwarzschild radius:

$$R\le R_s=\frac{2MG}{c^2}, \tag{4.1}$$

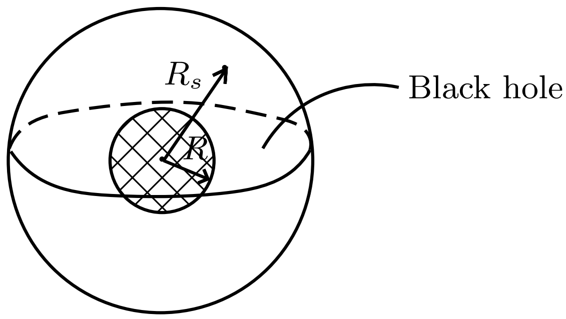


Figure 4.1: The spherical region enclosing a matter field with mass M and radius R satisfying (4.1) is called black hole.

then the matter field generates a singular spherical surface with radius R_s, where time stops and the spatial metric blows-up; see Figure 4.1. The spherical region with radius R_s is called the black hole.

We recall again the Schwarzschild metric in the exterior of a black hole written as

$$\begin{aligned} & ds^2 = g_{00}c^2dt^2 + g_{11}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \\ & g_{00} = -\left(1 - \frac{2MG}{c^2 r}\right), \quad g_{11} = \left(1 - \frac{2MG}{c^2 r}\right)^{-1}, \end{aligned} \tag{4.2}$$

where $r > R_s$ when the condition (4.1) is satisfied.

In (4.2) we see that at $r = R_s$, the time interval is zero, and the spatial metric blows up:

$$\sqrt{-g_{00}}dt = \left(1 - \frac{R_s}{r}\right)^{1/2} dt = 0 \qquad \text{at } r = R_s, \tag{4.3}$$

$$\sqrt{g_{11}}dr = \left(1 - \frac{R_s}{r}\right)^{-1/2} dr = \infty \qquad \text{at } r = R_s. \tag{4.4}$$

Physically, the proper time and distance for (4.2) are

$$\begin{aligned} & \text{proper time} = \sqrt{-g_{00}}\, t, \\ & \text{proper distance} = \sqrt{g_{11}dr^2 + r^2 d\theta^2 + r^2 \sin\theta d\varphi^2}. \end{aligned}$$

The coordinate system (t,x) with $x = (r,\theta,\varphi)$ represents the projection of the real world to the coordinate space. Therefore the radial motion speed dr/dt in the projected world differs from the proper speed v_r by a factor $\sqrt{-g_{00}/g_{11}}$, i.e.

$$\frac{dr}{dt} = \sqrt{-g_{00}/g_{11}}\, v_r.$$

Hence, the singularity (4.3) and (4.4) means that for an object moving toward to the boundary of a black hole, its projection speed vanishes:

$$\frac{dr}{dt} = 0 \qquad \text{at } r = R_s.$$

This implies that any object in the exterior of the black hole cannot pass through its boundary and enter into the interior. This result can also be best illustrated in the following geometric realization of black holes. Also, in the next subsection we shall give a proof that a black hole is a closed and innate system.

Mathematically, a Riemannian manifold $(\mathcal{M},g_{ij})$ is called a geometric realization (i.e. isometric embedding) in $\mathbb{R}^N$, if there exists a one to one mapping

$$\vec{r}:\ \mathcal{M}\to\mathbb{R}^N,$$

such that

$$g_{ij}=\frac{d\vec{r}}{dx^i}\cdot\frac{d\vec{r}}{dx^j}.$$

The geometric realization provides a "visual" diagram of $\mathcal{M}$, the real world of our Universe.

In the following we present the geometric realization of a 3D metric space of a black hole near its boundary. By (4.2), the space metric of a black hole is given by

$$ds^2=\left(1-\frac{R_s}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)\qquad \text{for } r>R_s=\frac{2MG}{c^2}. \tag{4.5}$$

It is easy to check that a geometric realization of (4.5) is given by $\vec{r}:\mathcal{M}\to\mathbb{R}^4$:

$$\vec{r}_{\text{ext}}=\left\{r\sin\theta\cos\varphi,r\sin\theta\sin\varphi,r\cos\theta,2\sqrt{R_s(r-R_s)}\right\}\qquad \text{for } r>R_s. \tag{4.6}$$

In the interior of a black hole, the Riemannian metric near the boundary is given by the TOV solution (2.36), and its space metric is in the form

$$ds^2=\left(1-\frac{r^2}{R_s^2}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)\qquad \text{for } r<R_s, \tag{4.7}$$

A geometrical realization of (4.7) is

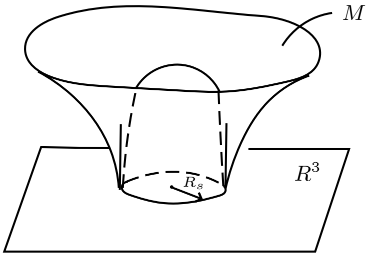
$$\vec{r}^{\pm}_{\text{int}}=\left\{r\sin\theta\cos\varphi,r\sin\theta\sin\varphi,r\cos\theta,\pm\sqrt{R_s^2-r^2}\right\}. \tag{4.8}$$

The diagrams of (4.6) and (4.8) are as shown in Figure 4.2, where case (a) is the embedding

$$\vec{r}^{+}=\begin{cases}\vec{r}_{\text{ext}} & \text{for } r>R_s,\\ \vec{r}^{+}_{\text{int}} & \text{for } r<R_s,\end{cases}$$

and case (b) is the embedding

$$\vec{r}^{-}=\begin{cases}\vec{r}_{\text{ext}} & \text{for } r>R_s,\\ \vec{r}^{-}_{\text{int}} & \text{for } r<R_s.\end{cases}$$



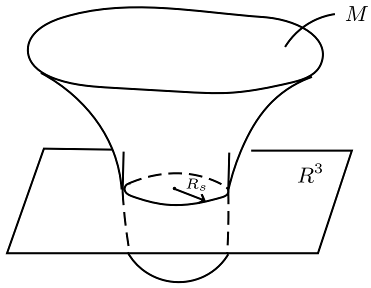


Figure 4.2: $\mathcal{M}$ is the real world with the metric (4.5) for $r > R_s$ and the metric (4.7) for $r < R_s$, and in the base space $\mathbb{R}^3$ the coordinate system is taken as spherical coordinates (r,θ,φ).

The base space marked as $\mathbb{R}^3$ in (a) and (b) are taken as the coordinate space (i.e. the projective space), and the surfaces marked by M represent the real world which are separated into two closed parts by the spherical surface of radius R_s: the black hole $(r < R_s)$ and the exterior world $(r > R_s)$.

In particular, the geometric realization of (4.7) for a black hole clearly manifests that the real world in the black hole is a hemisphere with radius R_s embedded in $\mathbb{R}^4$; see Figure 4.2(a):

$$x_1^2+x_2^2+x_3^2+x_4^2=R_s^2 \qquad \text{for } 0\le|x_4|\le R_s,$$

where

$$(x_1,x_2,x_3,x_4)=\left(r\sin\theta\cos\varphi,\, r\sin\theta\sin\varphi,\, r\cos\theta,\, \pm\sqrt{R_s^2-r^2}\right).$$

We remark that the singularity of $\mathcal{M}$ at $r=R_s$, where the tangent space of $\mathcal{M}$ is perpendicular to the coordinate space $\mathbb{R}^3$, is essential, and cannot be removed by any coordinate transformations. The coordinate transformations such as those given by Eddington and Kruskal possess the singularity as well, and, consequently, cannot be used as the coordinate systems for the metrics (4.5) and (4.7).

4.2 Blackhole theorem

The main objective of this section is to prove the following blackhole theorem.

Theorem 4.1 (Blackhole Theorem)**.** *Assume the validity of the Einstein theory of general relativity, then the following assertions hold true:*

1) *black holes are closed: matters can neither enter nor leave their interiors,*

2) *black holes are innate: they are neither born to explosion of cosmic objects, nor born to gravitational collapsing,*

3) *black holes are filled and incompressible, and if the matter field is non-homogeneously distributed in a black hole, then there must be sub-blackholes in the interior of the black hole.*

We prove this theorem in three steps as follows.

Step 1. Closedness of black holes. First, it is classical that all matter, including photons, cannot escape from a black hole when they are within the Schwarzschild radius.

Step 2. We have demonstrated that any object in the exterior of the black hole cannot pass through its boundary and enter into the interior. Now we see this in a different view, and we show that all external energy cannot enter into the interior of a black hole. By the energy-momentum conservation, we have

$$\frac{\partial E}{\partial \tau} + \text{div } P = 0, \tag{4.9}$$

where E and P are the energy and momentum densities. Take the volume integral of (4.9) on $B = \{x \in \mathbb{R}^3 \mid R_s < |x| < R_1\}$:

$$\int_B \left[\frac{\partial E}{\partial \tau} + \text{div} P\right] d\Omega = 0, \qquad d\Omega = \sqrt{g}\, dr d\theta d\varphi, \tag{4.10}$$

where $\text{div} P$ is as in (2.54), and

$$g = \det(g_{ij}) = g_{11} g_{22} g_{33} = \alpha r^4 \sin^2\theta, \qquad \alpha = \left(1 - \frac{2MG}{c^2 r}\right)^{-1}.$$

By the Gauss formula, we have

$$\int_B \text{div} P d\Omega = \int_{S_{R_1}} \sqrt{\alpha(R_1)} P_r dS_{R_1} - \lim_{r \to R_s} \int_{S_r} \sqrt{\alpha} P_r dS_r.$$

Here $S_r = \{x \in \mathbb{R}^3 \mid |x| = r\}$. In view of (4.10) we deduce that the total energy change

$$\int_B \frac{\partial E}{\partial \tau} d\Omega = \lim_{r \to R_s} \int_{S_r} \sqrt{\alpha} P_r dS_r - \sqrt{\alpha(R_1)} \int_{S_{R_1}} P_r dS_{R_1}. \tag{4.11}$$

The equality (4.11) can be rewritten as

$$\lim_{r \to R_s} \int_{S_r} P_r dS_r = \lim_{r \to R_s} \frac{1}{\sqrt{\alpha(r)}} \left[\int_B \frac{\partial E}{\partial \tau} d\Omega + \sqrt{\alpha(R_1)} \int_{S_{R_1}} P_r dS_{R_1}\right] = 0. \tag{4.12}$$

This together with no escaping of particles from the interior of the black hole shows that

$$\lim_{r \to R_s^+} P_r = 0.$$

In other words, there is no energy flux P_r on the Schwarzschild surface, and we have shown that no external energy can enter into a black hole.

In conclusion, we have shown that black holes are closed: no energy can penetrate the Schwarzschild surface.

Step 3. Innateness of black holes. The explosion mechanism introduced in Subsection 3.4 clearly manifests that any massive object cannot generate a new black hole. In other words, we conclude that black holes can neither be created nor be annihilated, and the total number of black holes in the Universe is conserved.

Step 4. Assertion 3) follows by applying conclusion (6.40) and the fact that sub black-holes are incompressible. The theorem is therefore proved. □

We remark again that the singularity on the boundary of black holes is essential and cannot be removed by any differentiable coordinate transformation with differentiable inverse. The Eddington and Kruskal coordinate transformations are non-differentiable, and are not valid.

Remark 4.2. The gravitational force F generated by a black hole in its exterior is given by

$$F=\frac{mc^2}{2}\nabla g_{00}=-mg^{11}\frac{\partial\psi}{\partial r},$$

where ψ is the gravitational potential. By (4.2) we have the following gravitational force:

$$F=-\left(1-\frac{2MG}{c^2r}\right)\frac{mMG}{r^2}. \tag{4.13}$$

Consequently, on the boundary of a black hole, the gravitational force is zero:

$$F=0 \qquad \text{at } r=R_s.$$

4.3 Critical δ-factor

Black holes are a theoretical outcome. Although we cannot see them directly due to their invisibility, they are, however, strong evidences from many astronomical observations and theoretic studies.

In the following, we first briefly recall the Chandrasekhar limit of electron degeneracy pressure and the Oppenheimer limit of neutron degeneracy pressure; then we present new criterions to classify pure black holes, which do not contain other black holes in their interior, into two types: the quark and weakton black holes, by using the δ-factor.

1) *Electron and neutron degeneracy pressures.* Classically we know that there are two kinds of pressure to resist the gravitational pressure, called the electron degeneracy pressure and the neutron degeneracy pressure. These pressures prevent stars from gravitational collapsing with the following mass relation:

$$m<\begin{cases}1.4M_\odot & \text{for electron pressure,}\\ 3M_\odot & \text{for neutron pressure.}\end{cases} \tag{4.14}$$

Hence, by (4.14), we usually think that a dead star is a white dwarf if its mass $m<1.4M_\odot$, and is a neutron star if its mass $m<3M_\odot$. However, if the dead star has mass $m>3M_\odot$, then it is regarded as a black hole. Hence the neutron pressure gradient is thought to be a final defense to prevent a star from collapsing into a black hole. Thus, $3M_\odot$ becomes a critical mass to determine the possible formation of a black hole.

2) *Interaction potential pressure.* However, thanks to the strong and weak interaction potentials established in [4,6], there still exist three kinds of potential pressures given by

$$\text{neutron potential, quark potential, weakton potential.} \tag{4.15}$$

These three potential pressures maintain three types of astronomical bodies:

$$\begin{aligned}&\text{neutron stars,}\\ &\text{quark black holes if they exist,}\\ &\text{weakton black holes if they exist.}\end{aligned} \tag{4.16}$$

We are now in position to discuss these potential pressures. By the theory of elementary particles, a neutron is made up of three quarks $n=uud$, and u,d quarks are made up of three weaktons as $u=w^*w_1\bar{w}_1, d=w^*w_1w_2$. The three levels of particles possess different potentials distinguished by their interaction charges:

$$\begin{aligned}&\text{neutron charge} && g_n=3\left(\frac{\rho_w}{\rho_n}\right)^3 g_s,\\ &\text{quark charge} && g_q=\left(\frac{\rho_w}{\rho_q}\right)^3 g_s,\\ &\text{weakton weak charge} && g_w,\end{aligned} \tag{4.17}$$

where ρ_n,ρ_q,ρ_w are the radii of neutron, quark and weakton.

Let g be a specific charge in (4.17). Then by the interaction potentials obtained in [6], the particle with charge g has a repulsive force:

$$f=\frac{g^2}{r^2}.$$

The force acts on particle's cross section with area $S=\pi r^2$, which yields the interaction potential pressure as

$$P=\frac{f}{S}=\frac{g^2}{\pi r^4}. \tag{4.18}$$

Let each ball B_r with radius r contain only one particle. Then the mass density ρ is given by

$$\rho=\frac{3m_0}{4\pi r^3}, \tag{4.19}$$

where m_0 is the particle mass. By the uncertainty relation, in B_r the particle energy ε_0 is

$$\varepsilon_0 = \frac{\hbar}{2t},$$

and $t = r/v$, where v is the particle velocity. Replacing v by the speed of light c, we have

$$\varepsilon_0 = \frac{\hbar c}{2r}.$$

By $m_0 = \varepsilon_0 / c^2$, the density ρ of (4.19) is written as

$$\rho = \frac{3\hbar}{8\pi c r^4} \qquad \text{or equivalently} \qquad r^4 = \frac{3\hbar}{8\pi c \rho}. \tag{4.20}$$

Inserting r^4 of (4.20) into (4.18), we derive the interaction potential pressure P in the form

$$P = \frac{8c\rho g^2}{3\hbar}. \tag{4.21}$$

3) *Critical δ-factors.* It is known that the central pressure of a star with mass m and radius r_0 can be expressed as

$$P_M = \frac{Gm^2}{r_0^4} = \frac{2\pi c^2}{3}\rho\delta, \qquad \delta = \frac{2mG}{c^2 r_0}, \tag{4.22}$$

where δ is the δ-factor.

We infer from (4.21) and (4.22) the critical δ-factor as

$$\delta_c = \frac{4}{\pi}\frac{g^2}{\hbar c}, \tag{4.23}$$

where g is one of the interaction charges in (4.17).

The critical δ-factor in (4.23) provides criterions for the three types of astronomical bodies of (4.16).

4) *Physical significance of δ_c.* It is clear that for a star with $m > 1.4 M_\odot$ if

$$\delta < \frac{4}{\pi}\frac{g_n^2}{\hbar c}, \tag{4.24}$$

then the neutron potential pressure P_n in (4.21) is greater than the star pressure P_M in (4.22):

$$P_n > P_M.$$

In this case, neutrons in the star cannot be crushed into quarks. Hence, (4.24) should be a criterion to determine if the body is a neutron star. It is known that

$$g_n^2 \sim \hbar c.$$

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Thus, we take

$$g_n^2 = \frac{\pi}{4}\hbar c, \tag{4.25}$$

and (4.24) is just the black hole criterion.

If the δ-factor satisfies that

$$\frac{4}{\pi}\frac{g_n^2}{\hbar c} < \delta < \frac{4}{\pi}\frac{g_q^2}{\hbar c}, \tag{4.26}$$

then the neutrons will be crushed to become quarks and gluons. The equality (4.25) shows that the star satisfying (4.26) must be a black hole which is composed of quarks and gluons, and is called quark black hole.

If δ satisfies

$$\frac{4}{\pi}\frac{g_q^2}{\hbar c} < \delta < \frac{4}{\pi}\frac{g_w^2}{\hbar c}, \tag{4.27}$$

then the quarks are crushed into weaktons, and the body is called weakton black hole.

In summary, we infer from (4.24), (4.26) and (4.27) the following conclusions:

$$\text{a body} = \begin{cases} \text{a neutron star} & \text{if } \delta < \delta_n^c \text{ and } m > 1.4M_\odot, \\ \text{a quark black hole} & \text{if } \delta_n^c < \delta < \delta_q^c, \\ \text{a weakton black hole} & \text{if } \delta_q^c < \delta < \delta_w^c, \end{cases} \tag{4.28}$$

where

$$\delta_j^c = \frac{4}{\pi}\frac{g_j^2}{\hbar c} \qquad \text{for } j = n,q,w.$$

5) *Upper limit of the radius.* Weaktons are elementary particles, which cannot be crushed. Therefore, there is no star with δ-factor greater than δ_w^c. Thus there exists an upper limit for the radius r_c for astronomical bodies with mass m, determined by

$$\frac{2MG}{c^2 r_c} = \delta_w^c.$$

Namely, the upper limit of the radius r_c reads

$$r_c = \frac{\pi m G}{2c^2}\frac{\hbar c}{g_w^2}. \tag{4.29}$$

4.4 Origin of stars and galaxies

The closeness and innateness of black holes provide an excellent explanation for the origin of planets, stars and galaxies.

In fact, all black holes are inherent. Namely, black holes exist at the very beginning of the Universe. During the evolution of the Universe, each black hole forms a core and adsorbs a ball of gases around it. The globes of gases eventually evolve into planets, stars and galaxy nuclei, according to the radii or masses of the inner cores of black holes. Of course, it is possible that several black holes can bound together to form a core of a bulk of gases.

Due to the closedness of black holes, planets, stars and galaxy nuclei are stable, which cannot be absorbed into the inner cores of black holes and vanish.

1) *Jeans theory on the origin of stars and galaxies.* In the beginning of the twentieth century, J. Jeans presented a general theory for the formation of galaxies and stars. He thought that the Universe in the beginning was filled with chaotic gas, and various astronomical objects were formed in succession by a process of gas decomposition into bulks of clouds, consequently forming galaxies, stars, and planets.

According to the Jeans theory, a ball of clouds with homogeneous density ρ can be held together only if

$$V+K\leq 0, \tag{4.30}$$

where V is the total gravitational potential energy, and K is the total kinetic energy of all particles. The potential energy V is

$$V=-\int_0^R \frac{GM_r}{r}\times 4\pi r^2\rho dr=-\frac{3GM^2}{5R}, \tag{4.31}$$

where M is the mass of the cloud, $M_r=4\pi r^3\rho/3$, and R is the radius. The kinetic energy K is expressed as the sum of thermal kinetic energies of all particles:

$$K=\frac{3}{2}NkT,$$

where N is the particle number, T is the temperature, and k is the Boltzmann constant. Assume that all particles have the same mass m, then $N=M/m$, and we have

$$K=\frac{3M}{2m}kT. \tag{4.32}$$

Thus, by (4.30)-(4.32) we obtain that

$$\frac{GM}{R}\geq\frac{5}{2m}kT. \tag{4.33}$$

The inequality (4.33) is called the Jeans condition.

2) *Masses of astronomical objects.* The Jeans condition (4.33) guarantees only the gaseous clouds being held together, and does not imply that the gas clouds can contract to form an astronomical object. However, a black hole must attract the nebulae around it to form a compact body.

We consider the mass relation between an astronomical object and its black hole core. The mass M of the object is

$$M=M_b+M_1, \tag{4.34}$$

where M_b is the mass of the black hole, and M_1 is the mass of the material attracted by this black hole. The total binding potential energy V of this object is given by

$$V = -\int_{R_s}^{R} \frac{GM_r}{r} \times 4\pi r^2 \rho dr, \tag{4.35}$$

where R is the object radius, R_s is the radius of the black hole, ρ is the mass density outside the core, and

$$M_r = M_b + \int_{R_s}^{r} 4\pi r^2 \rho dr. \tag{4.36}$$

Since $R_s \ll R$, we take $R_s = 0$ in the integrals (4.35) and (4.36). We assume that the density ρ is a constant. Then, it follows from (4.35) and (4.36) that

$$V = -4\pi G\rho \int_0^R \left[M_b r + \frac{4\pi}{3}\rho r^4 \right] dr = -4\pi G\rho \left(\frac{M_b R^2}{2} + \frac{4\pi}{3\times 5}\rho R^5 \right).$$

By $\rho = M_1 / \frac{4}{3}\pi R^3$ and $M_b = M - M_1$, we have

$$V = -\frac{G}{R}\left(\frac{3}{2} M M_1 - \frac{9}{10} M_1^2 \right). \tag{4.37}$$

The stability of the object requires that $-V$ takes its maximum at some M_1 such that $dV/dM_1 = 0$. Hence we derive from (4.37) that

$$M_1 = \frac{5}{6}M, \qquad M_b = \frac{1}{6}M. \tag{4.38}$$

The relation (4.38) means that a black hole with mass M_b can form an astronomical object with mass $M = 6M_b$.

3) *Relation between radius and temperature.* A black hole with mass M_b determines the mass M of the corresponding astronomical system: $M = 6M_b$. Then, by the Jeans relation (4.33), the radius R and average temperature T satisfy

$$T = \frac{2\times 6GmM_b}{5kR} \tag{4.39}$$

where T is expressed as

$$T = \frac{3}{4\pi R^3} \int_{B_R} \tau(x) dx,$$

where B_R is the ball of this system, and $\tau(x)$ is the temperature distribution. Let $\tau = \tau(r)$ depend only on r, then we have

$$T = \frac{3}{R^3} \int_0^R r^2 \tau(r) dr. \tag{4.40}$$

4) *Solar system.* For the Sun, $M=2\times 10^{30}$kg and $R=7\times 10^{8}$m. Hence the mass of the solar black hole core is about

$$M_{\odot b}=\frac{1}{3}\times 10^{30}\text{kg},$$

and the average temperature has an upper limit:

$$T=\frac{4}{5}\times\frac{6.7\times 10^{-11}\text{m}^3/\text{kg}\cdot\text{s}^2\times 10^{30}\text{kg}\times 1.7\times 10^{-27}\text{kg}}{1.4\times 10^{-24}\text{kg}\cdot\text{m}^2/\text{s}^2\cdot\text{K}\times 7\times 10^{8}\text{m}}\simeq 10^{8}\text{K}.$$

For the earth, $M=6\times 10^{24}\text{kg}, R=6.4\times 10^{6}$m. Thus we have

$$M_{eb}=10^{24}\text{kg},\qquad T=3.3\times 10^{4}.$$

5) *The radii of the solar and earth's black hole cores.* The radius of solar black hole is given by

$$R_s^{\odot}=500\text{ m},$$

and the radius of black hole of the earth is as

$$R_s^{e}=\frac{3}{2}\text{ cm}.$$

5 Galaxies

5.1 Galaxy dynamics

Galaxies are mainly either spiral or elliptical. Each galaxy possesses a compact core, known as galactic nucleus, which is supermassive and spherical-shaped. Thus, the galactic dynamic model is defined in an annular domain:

$$r_0<r<r_1,$$

where r_0 is the radius of galaxy nucleus and r_1 the galaxy radius. In the following we develop models for spiral and elliptical galaxies, and provide their basic consequences on galactic dynamics.

1) *Spiral galaxies.* Spiral galaxies are disc-shaped, as shown in Figure 5.1. We model the galaxy in a disc domain as

$$D=\{x\in\mathbb{R}^2 \mid r_0<|x|<r_1\}, \tag{5.1}$$

for which the spherical coordinates reduce to the polar coordinate system (φ,r):

$$(\theta,\varphi,r)=\left(\frac{\pi}{2},\varphi,r\right)\qquad \text{for } 0\leq\varphi\leq 2\pi,\quad r_0<r<r_1. \tag{5.2}$$

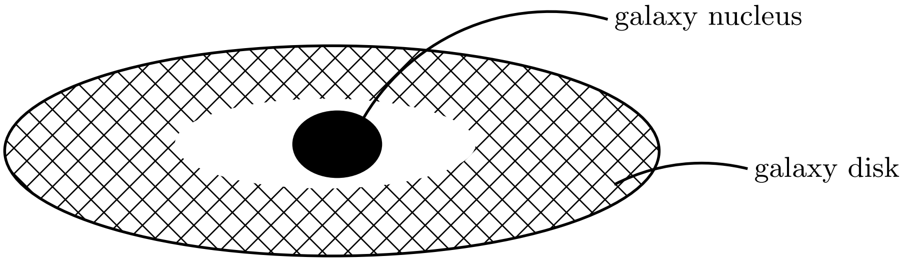

Figure 5.1: A schematic diagram of spiral galaxy.

The metric satisfying the gravitational field equations (2.56) of the galaxy nucleus is the Schwarzschild solution:

$$\begin{aligned} & g_{00}=-\left(1+\frac{2}{c^2}\psi\right), && \psi=-\frac{M_0G}{r}, \\ & g_{11}=\alpha(r)=\left(1-\frac{\delta r_0}{r}\right)^{-1}, && \delta=\frac{2M_0G}{c^2r_0}, \end{aligned} \tag{5.3}$$

where $r_0<r<r_1$ and M_0 is the mass of galactic nucleus.

With (5.2) and (5.3), the 2D fluid equations (2.59)-(2.62) are written as

$$\begin{aligned} & \frac{\partial P_\varphi}{\partial \tau}+\frac{1}{\rho}(P\cdot\nabla)P_\varphi=\nu\Delta P_\varphi-\frac{1}{r}\frac{\partial p}{\partial \varphi}, \\ & \frac{\partial P_r}{\partial \tau}+\frac{1}{\rho}(P\cdot\nabla)P_r=\nu\Delta P_r-\frac{1}{\alpha}\frac{\partial p}{\partial r}-\rho(1-\beta T)\frac{M_rG}{\alpha r^2}, \\ & \frac{\partial T}{\partial \tau}+\frac{1}{\rho}(P\cdot\nabla)T=\kappa\widetilde{\Delta}T+Q, \\ & \frac{\partial \rho}{\partial \tau}+\text{div}P=0, \end{aligned} \tag{5.4}$$

supplemented with boundary conditions:

$$\begin{aligned} & P_\varphi(r_0)=\zeta_0, \quad P_r(r_0)=0, \quad T(r_0)=T_0, \\ & P_\varphi(r_1)=\zeta_1, \quad P_r(r_1)=0, \quad T(r_1)=T_1. \end{aligned} \tag{5.5}$$

Here α is as in (5.3), and M_r is the total mass in the ball B_r.

2) *Elliptical galaxies.* Elliptical galaxies are spherically-shaped, defined in a spherical-annular domain, as shown in Figure 5.2:

$$\Omega=\{x\in\mathbb{R}^3 \mid r_0<|x|<r_1\} \tag{5.6}$$

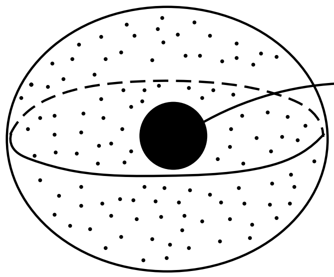

Figure 5.2: A schematic diagram of elliptical galaxy.

The metric is as in (5.3), and the corresponding fluid equations (2.59)-(2.62) are in the form:

$$\begin{aligned}&\frac{\partial P}{\partial \tau}+\frac{1}{\rho}(P\cdot\nabla)P=\nu\Delta P-\nabla p-\rho(1-\beta T)\frac{M_0G}{\alpha r^2}\vec{k},\\&\frac{\partial T}{\partial \tau}+\frac{1}{\rho}(P\cdot\nabla)T=\kappa\widetilde{\Delta}T+Q,\\&\frac{\partial \rho}{\partial \tau}+\text{div }P=0,\end{aligned}\tag{5.7}$$

supplemented with the physically sound conditions:

$$\begin{aligned}&P_r=0,\quad \frac{\partial P_\theta}{\partial r}=0,\quad \frac{\partial P_\varphi}{\partial r}=0\quad \text{at } r=r_0,r_1,\\&T(r_0)=T_0,\quad T(r_1)=T_1.\end{aligned}\tag{5.8}$$

3) *Galaxy dynamics.* Based on both models (5.4)-(5.5) and (5.7)-(5.8), we outline below the large scale dynamics of both spiral and elliptical galaxies.

Let the models be abstractly written in the following form

$$\frac{du}{dt}=F(u,\rho),\tag{5.9}$$

where $u=(P,T,p)$ is the unknown function, and ρ is the initial density distribution, which is used as a control parameter representing different physical conditions.

First, we consider the stationary equation of (5.9) given by

$$F(u,\rho)=0.\tag{5.10}$$

Let u_0 be a solution of (5.10), and consider the deviation from u_0 as

$$u=v+u_0.$$

Thus, (5.9) becomes the following form

$$\frac{dv}{dt}=L_\lambda v+G(v,\lambda,\rho),\tag{5.11}$$

where $\lambda=(\delta,\mathrm{Re})$, and the δ-factor and the Rayleigh number are defined by

$$\delta=\frac{2M_0G}{c^2r_0},\qquad \mathrm{Re}=\frac{M_0Gr_0r_1\beta}{\kappa\nu}\frac{T_0-T_1}{r_1-r_0}. \tag{5.12}$$

The L_λ is the derivative operator (i.e. the linearized operator) of $F(u,p)$ at u_0:

$$L_\lambda=DF(u_0,\rho),$$

and G is the higher order operator.

Then, we consider the dynamic transition of (5.11). Let $\widetilde{v}_\lambda$ be a stable transition solution of (5.11). Then the function

$$\widetilde{u}=u_0+\widetilde{v}_\lambda \tag{5.13}$$

provides the physical information of the galaxy.

5.2 Spiral galaxies

Spiral galaxies are divided into two types: normal spirals (S-type) and barred spirals (SB-type). We are now ready to discuss these two sequences of galaxies by using the spiral galaxy model (5.4)-(5.5).

Let the stationary solutions of (5.4)-(5.5) be independent of φ, given by

$$P_r=0,\quad P_\varphi=\widetilde{P}_\varphi(r),\quad T=\widetilde{T}(r),\quad p=\widetilde{p}(r).$$

The heat source is approximatively taken as $Q=0$. Then the stationary equations of (5.4)-(5.5) are

$$\begin{aligned}
&r\widetilde{P}''_\varphi+2\widetilde{P}'_\varphi-\frac{1}{r}\widetilde{P}_\varphi-\frac{\delta r_0}{r}\left(r\widetilde{P}''_\varphi+\frac{3}{2}\widetilde{P}'_\varphi-\frac{\widetilde{P}_\varphi}{2r}\right)=0,\\
&\frac{\partial\widetilde{p}}{\partial r}=\frac{1}{r\rho}\widetilde{P}^2_\varphi-\frac{1}{r^2}\rho(1-\beta\widetilde{T})M_rG,\\
&\frac{d}{dr}(r^2\frac{d\widetilde{T}}{dr})=0,\\
&\widetilde{P}_\varphi(r_0)=\zeta_0,\quad \widetilde{P}_\varphi(r_1)=\zeta_1,\quad \widetilde{T}(r_0)=T_0,\quad \widetilde{T}(r_1)=T_1.
\end{aligned} \tag{5.14}$$

The first equation of (5.14) is an elliptic boundary value problem, which has a unique solution P_φ. Since $\delta r_0/r$ is small in the domain (5.1), the first equation of (5.14) can be approximated by

$$\widetilde{P}''_\varphi+\frac{2}{r}P'_\varphi-\frac{1}{r^2}\widetilde{P}_\varphi=0,$$

which has an analytic solution as

$$\widetilde{P}_\varphi=\beta_1r^{k_1}+\beta_2r^{k_2},\quad k_1=\frac{\sqrt{5}-1}{2},\quad k_2=-\frac{\sqrt{5}+1}{2}. \tag{5.15}$$

By the boundary conditions in (5.14), we obtain that

$$\beta_1=\frac{r_0^{k_2}\zeta_1-r_1^{k_2}\zeta_0}{r_1^{k_1}r_0^{k_2}-r_0^{k_1}r_1^{k_2}},\qquad \beta_2=\frac{r_1^{k_1}\zeta_0-r_0^{k_1}\zeta_1}{r_1^{k_1}r_0^{k_2}-r_0^{k_1}r_1^{k_2}}. \tag{5.16}$$

Thus we derive the solution of (5.14) as

$$\widetilde{P}_\varphi,\quad \widetilde{T}=T_0+\frac{T_0-T_1}{r_1-r_0}r_1(\frac{r_0}{r}-1),\quad \widetilde{p}=\int\left[\frac{\widetilde{P}_\varphi^2}{r\rho}-\frac{\rho G}{r^2}(1-\beta\widetilde{T})M_r\right]dr.$$

Make the translation

$$P_r\to P_r,\quad P_\varphi\to P_\varphi+\widetilde{P}_\varphi,\quad T\to T+\widetilde{T},\quad p\to p+\widetilde{p};$$

then the equations (5.4) and boundary conditions (5.5) become

$$\begin{aligned}
&\frac{\partial P_\varphi}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)P_\varphi=\nu\Delta P_\varphi-(\frac{\widetilde{P}_\varphi}{r}+\frac{d\widetilde{P}_\varphi}{dr})P_r-\frac{1}{r}\frac{\partial p}{\partial\varphi},\\
&\frac{\partial P_r}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)P_r=\nu\Delta P_r+\frac{2\widetilde{P}_\varphi}{\alpha r}P_\varphi+\frac{\rho\beta M_rG}{\alpha r^2}T-\frac{1}{\alpha}\frac{\partial p}{\partial r},\\
&\frac{\partial T}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)T=\kappa\widetilde{\Delta}T+\frac{r_0r_1}{\rho r^2}\gamma P_r-\frac{1}{\rho r}\widetilde{P}_\varphi\frac{\partial T}{\partial\varphi},\\
&\text{div}\,P=0,\\
&P=0,\quad T=0\qquad \text{at } r=r_0,r_1,
\end{aligned} \tag{5.17}$$

where $r=(T_0-T_1)/(r_1-r_0)$.

The eigenvalue equations of (5.17) are given by

$$\begin{aligned}
&-\Delta P_\varphi+\frac{1}{\nu}(\frac{\widetilde{P}_\varphi}{r}+\frac{d\widetilde{P}_\varphi}{dr})P_r+\frac{1}{r\nu}\frac{\partial p}{\partial\varphi}=\lambda P_\varphi,\\
&-\Delta P_r-\frac{2\widetilde{P}_\varphi}{\alpha\nu r}P_\varphi-\frac{\rho\beta M_rG}{\alpha\nu r^2}T+\frac{1}{\alpha\nu}\frac{\partial p}{\partial r}=\lambda P_r,\\
&-\Delta T+\frac{1}{\rho r}\widetilde{P}_\varphi\frac{\partial T}{\partial\varphi}-\frac{r_0r_1\gamma}{\kappa\rho r^2}P_r=\lambda T,\\
&\text{div}\,P=0,\\
&P=0,\quad T=0\qquad \text{at } r=r_0,r_1.
\end{aligned} \tag{5.18}$$

The eigenvalues λ of (5.18) are discrete (not counting multiplicity):

$$\lambda_1>\lambda_2>\cdots>\lambda_k>\cdots,\quad \lambda_k\to-\infty \text{ as } k\to\infty.$$

The first eigenvalue λ_1 and first eigenfunctions

$$\Phi=(P_\varphi^0,P_r^0,T^0) \tag{5.19}$$

dictate the dynamic behaviors of spiral galaxies, which are determined by the physical parameters:

$$\zeta_0,\zeta_1,r_0,r_1,\kappa,\nu,\beta,\gamma=\frac{T_0-T_1}{r_1-r_0},\ \delta=\frac{2M_0G}{c^2r_0},\ M_r=M_0+4\pi\int_{r_0}^{r_1}r^2\rho dr. \tag{5.20}$$

Based on the dynamic transition theory in [3], we have the following physical conclusions:

- If the parameters in (5.20) make the first eigenvalue $\lambda_1<0$, then the spiral galaxy is of S0-type.
- If $\lambda_1>0$, then the galaxy is one of the types $S_a,S_b,S_c,SB_a,SB_b,SB_c$, depending on the structure of (P^0_φ,P^0_r) in (5.19).
- Let $\lambda_1>0$ and the first eigenvector (P^0_φ,P^0_r) of (5.19) have the vortex structure as shown in Figure 5.3. Then the number of spiral arms of the galaxy is $k/2$, where k is the vortex number of (P^0_φ,P^0_r). Hence, if $k=2$, the galaxy is of the SB_c-type.

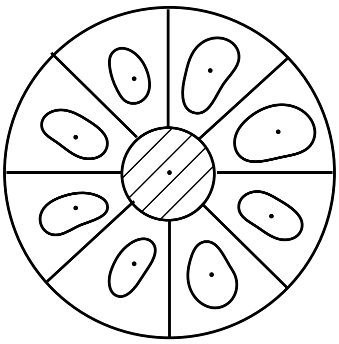

Figure 5.3: The vortex structure of the first eigenvector (P^0_φ,P^0_r).

The reason behind the number of spiral arms being $k/2$ is as follows. First the number of vortices in Figure 5.3 is even, and each pair of vortices have reversed orientations. Second, if the orientation of a vortex matches that of the stationary solution $P_\varphi(r)$ of (5.14), then the superposition of $P_\varphi(r)$ and P^0_φ of (5.19) gives rise to an arm; otherwise, the counteraction of $P_\varphi(r)$ and P^0_φ with reversed orientations reduces the energy momentum density, and the region becomes nearly void.

Remark 5.1. There are three terms in (5.18), which may generate the transition of (5.17):

$$
\begin{aligned}
&F_1=\left(0,-\frac{k_1T}{r^2},-\frac{k_2P_r}{r^2}\right), && k_2=\frac{\rho\beta M_rG}{\alpha\nu}, \quad k_2=\frac{r_0r_1\gamma}{\kappa\rho},\\
&F_2=\left(\frac{1}{\nu}\left(\frac{\widetilde{P}_\varphi}{r}+\frac{d\widetilde{P}_\varphi}{dr}\right)P_r,-\frac{2\widetilde{P}_\varphi}{\alpha\nu r}P_\varphi,0\right),\\
&F_3=\left(-\frac{1}{2\alpha^2r}\frac{d\alpha}{dr}\frac{\partial}{\partial r}(rP_\varphi),\frac{1}{2\alpha}\frac{\partial}{\partial r}(\frac{1}{\alpha}\frac{d\alpha}{dr}P_r),0\right).
\end{aligned}
$$

The term F_1 corresponds to the Rayleigh-Bénard convection with the Rayleigh number

$$R=k_1k_2=\frac{\beta M_rGr_0r_1\gamma}{\alpha\nu\kappa},$$

the term F_2 corresponds to the Taylor rotation which causes the instability of the basic flow $(P_\varphi,P_r)=(\widetilde{P}_\varphi,0)$, and F_3 is the relativistic effect which only plays a role in the case where $\delta\simeq 1$. □

5.3 Active galactic nuclei (AGN) and jets

The black hole core of a galaxy attracts a large amounts of gases around it, forming a galactic nucleus. The mass of a galactic nucleus is usually in the range

$$10^5M_\odot\sim 10^9M_\odot. \tag{5.21}$$

Galactic nuclei are divided into two types: normal and active. In particular, an active galactic nucleus emits huge quantities of energy, called jets. We focus in this section the mechanism of AGN jets.

1) *Model for AGN.* The domain of an galactic nucleus is a spherical annulus:

$$B=\left\{x\in\mathbb{R}^3 \mid R_s<|x|<R_1\right\}, \tag{5.22}$$

where R_s is the Schwarzschild radius of the black hole core, and R_1 is the radius of the galaxy nucleus.

The model governing the galaxy nucleus is given by (5.7)-(5.8), defined in the domain (5.22) with boundary conditions:

$$
\begin{aligned}
&P_r=0,\ \frac{\partial P_\theta}{\partial r}=0,\ P_\varphi=P_0,\ T=T_0 \quad \text{for } r=R_s,\\
&P_r=0,\ \frac{\partial P_\theta}{\partial r}=0,\ P_\varphi=P_1,\ T=T_1 \quad \text{for } r=R_1.
\end{aligned}
\tag{5.23}
$$

Let the stationary solution of the model be as

$$P_\theta=0,\quad P_r=0,\quad P_\varphi=P_\varphi(r,\theta),$$

and p,ρ,T be independent of φ. Then the stationary equations for the four unknown functions P_φ,T,p,ρ are in the form

$$\begin{aligned}
&\frac{\partial p}{\partial\theta}=-\frac{1}{\rho}\frac{\cos\theta}{\sin\theta}P_\varphi^2,\\
&\frac{\partial p}{\partial r}=\frac{1}{\rho r}P_\varphi^2-\rho(1-\beta T)\frac{M_bG}{r^2},\\
&-\nu\widetilde{\Delta}P_\varphi+\frac{P_\varphi}{r^2\sin\theta}+\frac{1}{2\alpha^2 r}\frac{d\alpha}{dr}\frac{\partial}{\partial r}(rP_\varphi)=0,\\
&-\frac{\kappa}{\alpha r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})T=Q(r),
\end{aligned}\tag{5.24}$$

where M_b is the mass of the black hole core, Q is the heat source generated by the nuclear burning, and

$$\begin{aligned}
&\widetilde{\Delta}=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{\alpha r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}),\\
&\alpha=\left(1-\frac{2M_bG}{c^2r}\right)^{-1}.
\end{aligned}$$

The boundary conditions of (5.23) become

$$P_\varphi(R_s)=R_s\Omega_0,\quad P_\varphi(R_1)=R_1\Omega_1,\quad T(R_s)=T_0,\quad T(R_1)=T_1,\tag{5.25}$$

where Ω_0,Ω_1 only depend on θ,T_0,T_1 are constants.

Make the translation

$$P_r\to P_r,\quad P_\theta\to P_\theta,\quad P_\varphi\to P_\varphi+\widetilde{P}_\varphi,\quad T\to T+\widetilde{T},\quad p\to p+\widetilde{p},$$

where $(\widetilde{P}_\varphi,\widetilde{T},\widetilde{p},\rho)$ is the solution of (5.24)-(5.25). Then the equations (5.7) are rewritten as

$$\begin{aligned}
&\frac{\partial P_\theta}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)P_\theta=\nu\Delta P_\theta-\frac{\widetilde{P}_\varphi}{\rho r\sin\theta}\frac{\partial P_\theta}{\partial\varphi}+\frac{2\cos\theta\widetilde{P}_\varphi}{\rho r\sin\theta}P_\varphi-\frac{1}{r}\frac{\partial p}{\partial\theta},\\
&\frac{\partial P_\varphi}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)P_\varphi=\nu\Delta P_\varphi-\frac{1}{\rho r}\frac{\partial\widetilde{P}_\varphi}{\partial\theta}P_\theta-\frac{\widetilde{P}_\varphi}{\rho r\sin\theta}\frac{\partial P_\varphi}{\partial\varphi}-\frac{1}{\rho}\frac{\partial\widetilde{P}_\varphi}{\partial r}P_r\\
&\qquad\qquad-\frac{\widetilde{P}_\varphi}{\rho r}P_r-\frac{\cos\theta\widetilde{P}_\varphi}{\rho r\sin\theta}P_\theta-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi},\\
&\frac{\partial P_r}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)P_r=\nu\Delta P_r-\frac{\widetilde{P}_\varphi}{\rho r\sin\theta}\frac{\partial P_r}{\partial\varphi}+\frac{2\widetilde{P}_\varphi}{\rho\alpha r}P_\varphi-\frac{1}{\alpha}\frac{\partial p}{\partial r}+\beta\rho\frac{M_bG}{\alpha r^2}T,\\
&\frac{\partial T}{\partial\tau}+\frac{1}{\rho}(P\cdot\nabla)T=\kappa\widetilde{\Delta}T-\frac{\widetilde{P}_\varphi}{\rho r\sin\theta}\frac{\partial T}{\partial\varphi}-\frac{1}{\rho}\frac{d\widetilde{T}}{dr}P_r,\\
&\mathrm{div}P=0,
\end{aligned}\tag{5.26}$$

with the boundary conditions

$$P_r=0, \quad P_\varphi=0, \quad \frac{\partial P_\theta}{\partial r}=0, \quad T=0 \quad \text{at } r=R_s,R_1. \tag{5.27}$$

2) *Taylor instability.* By the conservation of angular momentum and $R_1 \gg R_s$, the angular momentums Ω_0 and Ω_1 in (5.25) satisfy that

$$\Omega_0 \gg \Omega_1, \tag{5.28}$$

This property leads to the instability of the rotating flow represented by the stationary solution:

$$(P_r,P_\theta,P_\varphi)=(0,0,\widetilde{P}_\varphi), \tag{5.29}$$

which is similar to the Taylor-Couette flow in a rotating cylinder. The rotating instability can generate a circulation in the galactic nucleus, as the Taylor vortices in a rotating cylinder, as shown in Figure 5.4. The instability is caused by the force $F=(F_r,F_\theta,F_\varphi,T)$ in the equations of (5.26) given by

$$\begin{aligned}
&F_r=\frac{2\widetilde{P}_\varphi}{\rho\alpha r}P_\varphi-\frac{\widetilde{P}_\varphi}{\rho r\sin\theta}\frac{\partial P_r}{\partial\varphi},\\
&F_\theta=\frac{2\cos\theta\widetilde{P}_\varphi}{\rho r\sin\theta}P_\varphi-\frac{\widetilde{P}_\varphi}{\rho r\sin\theta}\frac{\partial P_\theta}{\partial\varphi},\\
&F_\varphi=-\frac{1}{\rho}\left(\frac{\widetilde{P}_\varphi}{r}+\frac{\partial\widetilde{P}_\varphi}{\partial r}\right)P_r-\frac{1}{\rho r}\left(\frac{\cos\theta}{\sin\theta}\widetilde{P}_\varphi+\frac{\partial\widetilde{P}_\varphi}{\partial\theta}\right)P_\theta-\frac{\widetilde{P}_\varphi}{\rho r\sin\theta}\frac{\partial P_\varphi}{\partial\varphi},\\
&T=-\frac{\widetilde{P}_\varphi}{\rho r\sin\theta}\frac{\partial T}{\partial\theta}.
\end{aligned} \tag{5.30}$$

3) *Rayleigh-Bénard instability.* Due to the nuclear reaction (fusion and fission) and the large pressure gradient, the galactic nucleus possesses a very large temperature gradient in (5.25) as

$$DT=T_0-T_1, \tag{5.31}$$

which yields the following thermal expansion force in (5.26), and gives rise to the Rayleigh-Bénard convection:

$$F_r=\beta\rho\frac{M_bG}{\alpha r^2}T, \qquad T=\frac{1}{\rho}\frac{d\widetilde{T}}{dr}P_r. \tag{5.32}$$

4) *Instability due to the gravitational effects.* Similar to (3.3), there is a radial force in the term $\nu\Delta u_r$ of the third equation of (5.26):

$$F_r=\frac{\nu}{2\alpha}\frac{\partial}{\partial r}\left(\frac{1}{\alpha}\frac{d\alpha}{dr}P_r\right), \tag{5.33}$$

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where

$$\alpha=(1-R_s/r)^{-1}, \qquad R_s<r<R_1. \tag{5.34}$$

In (5.33) and (5.34), we see the term

$$f_r=\frac{\nu}{1-R_s/r}\frac{R_s^2}{r^4}P_r, \tag{5.35}$$

which has the property that

$$f_r=\begin{cases} +\infty & \text{for } P_r>0 \quad \text{at } r=R_s, \\ -\infty & \text{for } P_r<0 \quad \text{at } r=R_s. \end{cases} \tag{5.36}$$

It is the force (5.36) that not only causes the instability of the basic flow (5.29), but also generates jets of the galaxy nucleus.

5) *Latitudinal circulation.* The above three types of forces: the rotating force (5.30), the thermal expansion force (5.32), and the gravitational effect (5.35), cause the instability of the basic flow (5.29) and lead to the latitudinal circulation of the galactic nucleus, as shown in Figure 5.4.

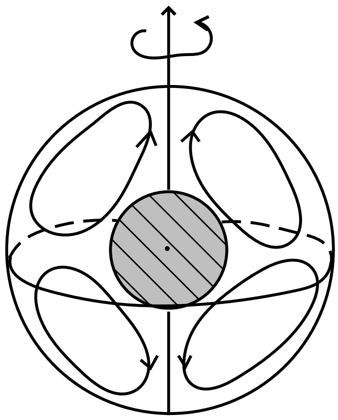

Figure 5.4: The latitudinal circulation with $k=2$ cells.

6) *Jets and accretions.* Each circulation cell has an exit as shown in Figure 5.5, where the circulating gas is pushed up by the radial force (5.35)-(5.36), and erupts leading to a jet. The cell has an entrance as shown in Figure 5.5, where the exterior gas is pulled into the nucleus, is cyclo-accelerated by the force (5.35), goes down to the inner boundary $r=R_s$, and then is pushed by F_θ of (5.30) toward to the exit. Thus the circulation cells form jets in their exits and accretions in their entrances. In Figure 5.6(a), we see that there is a jet in the latitudinal circulation with $k=1$ cell, and in Figure 5.6(b) there are two jets in the circulation with $k=2$ cells in its south and north poles, and an accretion disk near its equator.

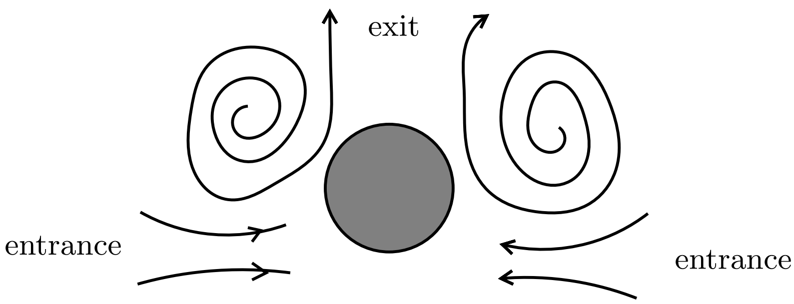

Figure 5.5:

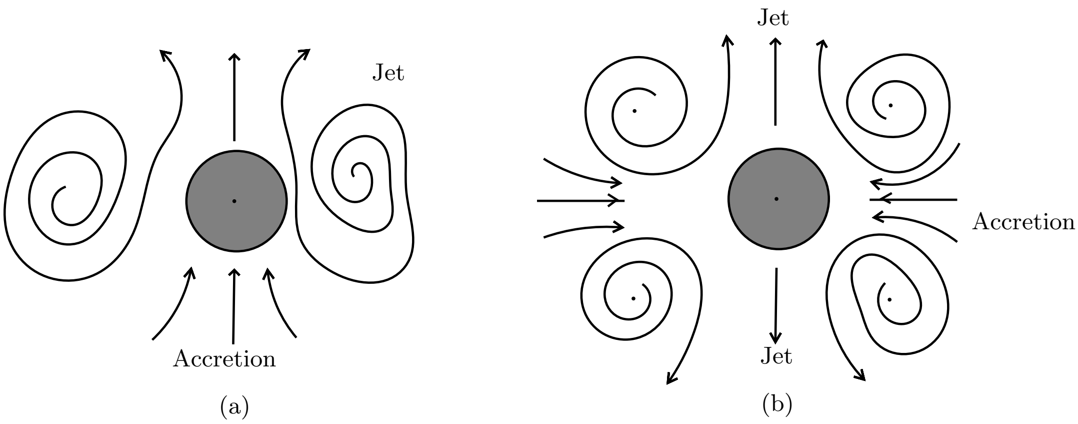

Figure 5.6: (a) A jet in the latitudinal circulation with $k=1$ cell, two jets in the latitudinal circulation with $k=2$ cells.

7) *Condition for jet generation.* The main power to generate jets comes from the gravitational effect of (5.35)-(5.36) by the black hole. The radial momentum P_r in (5.35) is the bifurcated solution of (5.26), which can be expressed as

$$P_r = R_s^2 Q_r,$$

where Q_r is independent of R_s. Thus, the radial force (5.35) near $r=R_s$ is approximatively written as

$$f_r = \frac{\nu}{1-R_s/r} Q_{R_s}, \quad r = R_s + \widetilde{r} \qquad \text{for } 0 < \widetilde{r} \ll R_s. \tag{5.37}$$

Let f_E be the lower limit of the effective force, which is defined as that the total radial force F_r in the third equation of (5.26) is positive provided $f_r > f_E$:

$$F_r > 0 \quad \text{if} \quad f_r > f_E.$$

Let R_E be the effective distance:

$$f_r > f_E \quad \text{if} \quad R_s < r < R_s + R_E.$$

Then, it follows from (5.37) that

$$R_E = kR_s \qquad (k = \nu Q_{R_s}/f_E). \tag{5.38}$$

It is clear that there is a critical distance R_c such that

$$\begin{aligned} &\text{a jet forms} && \text{if } R_E > R_c && \text{or } R_s > k^{-1}R_c, \\ &\text{no jet forms} && \text{if } R_E < R_c && \text{or } R_s < k^{-1}R_c. \end{aligned} \tag{5.39}$$

The criterion (5.39) is the condition for jet generation.

The condition (5.39) can be equivalently rewritten as that there is a critical mass M_c such that the galactic nucleus is active if its mass M is bigger than M_c, i.e. $M > M_c$. By (5.21), we have

$$10^5 M_\odot < M_c \qquad \text{or} \qquad 10^6 M_\odot < M_c.$$

Remark 5.2. The jets shown in Figures 5.4 and 5.5 are column-shaped. If the cell number $k \geq 3$ for the latitudinal circulation of galaxy nucleus, then there are jets which are disc-shaped. We don't know if there exist such galaxy nuclei which have the disc-shaped jets in the Universe. Theoretically, it appears to be possible. □

Remark 5.3. Galactic nucleus are made up of plasm. The precise description of AGN jets requires to take into consideration of the magnetic effect in the modeling. However the essential mechanism does not change and an explosive magnetic energy as in (5.37) will contribute to the supernovae explosion. □

6 The Universe

6.1 Classical theory of the Universe

In this section, we recall some basic aspects of modern cosmology, including the Hubble Law, the expanding universe, and the origin of our Universe, together with their experimental justifications.

The Hubble Law. In 1929, American astronomer Edwin Hubble discovered an approximatively linear relation between the recession velocity v and the distance R of remote galaxies, which is now called the Hubble Law:

$$v = HR, \tag{6.1}$$

where H is called the Hubble constant, depends on time, and its present-time value is

$$H = 70 \text{ km/s}\cdot\text{Mpc}, \qquad \text{Mpc} = 10^6\text{pc } (1\text{ pc} = 3.26\text{ ly}). \tag{6.2}$$

Formula (6.1) implies that the farther away the galaxy is from our galaxy, the greater its velocity is.

The Newton cosmology. The Newton cosmology is based on the Newton Gravitational Law. By Cosmological Principle [8], the universe is spherically symmetric. For any reference point $p\in\mathcal{M}$, the motion equation of an object with distance r from p is

$$\frac{d^2r}{dt^2}=-\frac{GM(r)}{r^2}, \tag{6.3}$$

where $M(r)=4\pi r^3\rho/3$, and ρ is the mass density. Thus, (6.3) can be rewritten as follows

$$r''=-\frac{4}{3}\pi Gr\rho. \tag{6.4}$$

Make the nondimensional

$$r=R(t)r_0,$$

where $R(t)$ is the scalar factor, which is the same as in the FLRW metric [5]. Let ρ_0 be the density at $R=1$. Then we have

$$\rho=\rho_0/R^3. \tag{6.5}$$

Thus, equation (6.4) is expressed as

$$R''=-\frac{4\pi G}{3}\frac{\rho_0}{R^2}, \tag{6.6}$$

which is the dynamic equation of Newtonian cosmology.

Multiplying both sides of (6.6) by R' we have

$$\frac{d}{dt}(\dot{R}^2-\frac{8\pi G}{3}\frac{\rho_0}{r})=0.$$

Hence, (6.6) is equivalent to the equation

$$\dot{R}^2=\frac{8\pi G}{3}\frac{\rho_0}{R}-\kappa, \tag{6.7}$$

where κ is a constant, and we shall see that $\kappa=kc^2$, and $k=-1,0$, or 1.

The Friedmann cosmology. The nonzero components of the Friedmann metric are

$$g_{00}=-1,\quad g_{11}=\frac{R^2}{1-kr^2},\quad g_{22}=R^2r^2,\quad g_{33}=R^2r^2\sin^2\theta.$$

Again by the Cosmological Principle [8], the energy-momentum tensor of the Universe is in the form

$$T_{\mu\nu}=\begin{pmatrix} pc^2 & 0 & 0 & 0\\ 0 & g_{11}p & 0 & 0\\ 0 & 0 & g_{22}p & 0\\ 0 & 0 & 0 & g_{33}p \end{pmatrix}.$$

By the Einstein gravitational field equations

$$\begin{aligned}&R_{\mu\nu}=-\frac{8\pi G}{c^4}(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T),\\&D^{\mu}T_{\mu\nu}=0,\end{aligned}$$

we derive three independent equations

$$\ddot{R}=-\frac{4\pi G}{3}\left(\rho+\frac{3p}{c^2}\right)R, \tag{6.8}$$

$$R\ddot{R}+2\dot{R}^2+2kc^2=4\pi G\left(\rho-\frac{p}{c^2}\right)R^2, \tag{6.9}$$

$$\dot{\rho}=-3\left(\frac{\dot{R}}{R}\right)\left(\rho+\frac{p}{c^2}\right), \tag{6.10}$$

where R,ρ,p are the unknown functions.

Equations (6.8)-(6.10) are called the Friedmann cosmological model, from which we can derive the Newtonian cosmology equations (6.7). To see this, by (6.8) and (6.9), we have

$$\left(\frac{\dot{R}}{R}\right)^2=-\frac{kc^2}{R^2}+\frac{8\pi G}{3}\rho. \tag{6.11}$$

By the approximate $p/c^2\simeq 0$, (6.5) follows from (6.10). Then we deduce (6.7) from (6.11) and (6.5).

From the equation (6.11), the density ρ_c corresponding to the case $k=0$ is

$$\rho_c=\frac{3}{8\pi G}\left(\frac{\dot{R}}{R}\right)^2=\frac{3}{8\pi G}H^2, \tag{6.12}$$

where $H=\dot{R}/R$ is the Hubble constant, and by (6.2) we have

$$\rho_c=10^{-26}\mathrm{kg/m^3}. \tag{6.13}$$

6.2 Globular universe with boundary

If the spatial geometry of a universe is open, then by our theory of black holes developed in Section 4, we have shown that the universe must be in a ball of a black hole with a fixed radius. In fact, according to the basic cosmological principle that the universe is homogeneous and isotropic [8], given the energy density $\rho_0>0$ of the universe, by Theorem 2.4, the universe will always be bounded in a black hole of open ball with the Schwarzschild radius:

$$R_s=\sqrt{\frac{3c^2}{8\pi G\rho_0}},$$

as the mass in the ball B_{R_s} is given by $M_{R_s}=4\pi R_s^3\rho_0/3$. This argument also clearly shows that

$$\boxed{\text{there is no unbounded universe.}}$$

In addition, since a black hole is unable to expand and shrink, by property (6.40) of black holes, all globular universes must be static.

Globular universe

We have shown that the universe is bounded, and suppose that the universe is open, i.e. its topological structure is homeomorphic to $\mathbb{R}^3$, and it begins with a ball. Let E be its total energy:

$$E=\text{mass}+\text{kinetic}+\text{thermal}+\Psi, \tag{6.14}$$

where Ψ is the energy of all interaction fields. Let

$$M=E/c^2. \tag{6.15}$$

At the initial stage, all energy is concentrated in a ball with radius R_0. By the theory of black holes, the energy contained in the ball generates a black hole in $\mathbb{R}^3$ with radius

$$R_s=\frac{2MG}{c^2}, \tag{6.16}$$

provided $R_s\geq R_0$; see (4.1) and Figure 4.1.

Thus, if the universe is born to a ball, then it is immediately trapped in its own black hole with the Schwarzschild radius R_s of (6.16). The 4D metric inside the black hole of the static universe is given by

$$ds^2=-\psi(r)c^2dt^2+\alpha(r)dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{6.17}$$

where ψ and α satisfy the equations (2.73) and (2.74) with boundary conditions:

$$\psi\to\widetilde{\psi},\quad \alpha\to\widetilde{\alpha}\quad \text{as}\quad r\to R_s,$$

where R_s is given by (6.16). Also, $\widetilde{\psi},\widetilde{\alpha}$ are given via the TOV metric (2.34)-(2.35):

$$\widetilde{\psi}=\frac{1}{4}\left(1-\frac{r^2}{R_s^2}\right),\quad \widetilde{\alpha}=\left(1-\frac{r^2}{R_s^2}\right)^{-1}\qquad \text{for } 0\leq r<R_s. \tag{6.18}$$

Basic problems

A static universe is confined in a ball with fixed radius R_s in (6.16), and the ball behaves like a black hole. We need to examine a few basic problems for a static universe, including the cosmic edge, the flatness, the horizon, the redshift, and the cosmic microwave background (CMB) radiation problems.

1) *The cosmic edge problem.* In the ancient Greece, the cosmic-edge riddle was proposed by the philosopher Archytas, a friend of Plato, who used “what happens when a spear is thrown across the outer boundary of the Universe?” The problem appears to be very difficult to answer. Hence, for a long time physicists always believe that the Universe is boundless.

Our theory of black holes presented in Section 4 shows that all objects in a globular universe cannot reach its boundary $r=R_s$. In particular, an observer in any position of the globular universe looking toward to the boundary will see no boundary due to the openness of the ball and the relativistic effect near the Schwarzschild surface. Hence the cosmic-edge riddle is no longer a problem.

2) *The flatness problem.* In modern cosmology, the flatness problem means that $k=0$ in the FLRW metric [5]. It is common to think that the flatness of the universe is equivalent to the fact that the present energy density ρ must be equal to the critical value given by (6.13). In fact, mathematically the flatness means that any geodesic triangle has the inner angular sum $\pi=180^\circ$.

Measurements by the WMAP (Wilson Microwave Anisotropy Probe) spacecraft in the last ten years indicated that the Universe is nearly flat. The present radius of the Universe is about

$$R=10^{26}\text{m}. \tag{6.19}$$

If the Universe is static, then (6.19) gives the Schwarzschild radius (6.16), from which it follows that the density ρ of our Universe is just the critical density of (6.13):

$$\rho=\rho_c=10^{-26}\text{kg}/\text{m}^3. \tag{6.20}$$

Thus, we deduce that if the universe is globular, then it is static. In addition, we have shown that any universe is bounded and confined in a 3D hemisphere of a black hole or in a 3D sphere as shown in Figure 6.2. Hence as the radius is sufficiently large, the universe is nearly flat.

3) *The horizon problem.* The cosmic horizon problem can be simply stated as that all places in a universe look as the same. It seems as if the static Universe with boundary violates the horizon problem. However, due to the gravitational lensing effect, the light bents around a massive object. Hence, the boundary of a globular niverse is like a concave spherical mirror, and all lights reaching close to it will be reflected back, as shown in Figure 6.1. It is this lensing effect that makes the globular universe looks as if everywhere is the same, and is horizontal. In Figure 6.1, if we are in position x, then we can also see a star as if it is in position $\widetilde{y}$, which is actually a virtual image of the star at y.

4) *The redshift problem.* Observations show that light coming from a remote galaxy is redshifted, and the farther away the galaxy is, the larger the redshift is. In astronomy, it is customary to characterize redshift by a dimensionless quantity z in the formula

$$1+z=\frac{\lambda_{\text{observ}}}{\lambda_{\text{emit}}},$$

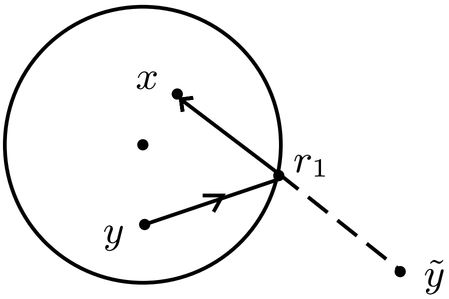

Figure 6.1: Due to the lensing effect, one at x can also see the star at y as if it is another star at $\tilde{y}$.

where λ_{observ} and λ_{emit} represent the observed and emitted wavelenths. There are three redshift types:

Doppler effect, cosmological redshift, gravitational redshift.

The gravitational redshift in a black hole are caused by both the gravitational fields of the emitting object and the black hole.

The first type of redshift, due to the gravitational field, is formulated as

$$1+z=\frac{\sqrt{1-\frac{2mG}{c^2r}}}{\sqrt{1-\frac{2mG}{c^2r_0}}}, \tag{6.21}$$

where m is the mass of the **emitting object**, r_0 is its radius, and r is the distance between the object and the observer.

The second type of redshifts, due to the cosmological effect or black hole effect, is

$$1+z=\frac{\sqrt{-g_{00}(r_0)}}{\sqrt{-g_{00}(r_1)}}, \tag{6.22}$$

where g_{00} is the time-component of the black hole gravitational metric, r_0 and r_1 are the positions of the observer and the emitting object (including virtual images).

If a universe is not considered as a black hole, then the gravitational redshift is simply given by (6.21) and is very small for remote objects. Likewise, the cosmological redshift is also too small to be significant. Hence, astronomers have to think the main portion of the redshift is due to the Doppler effect:

$$1+z=\frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}. \tag{6.23}$$

When v/c is small, (6.23) can be approximatively expressed as

$$z\simeq v/c. \tag{6.24}$$

In addition, Hubble discovered that the redshift has an approximatively linear relation with the distance:

$$z \simeq kR, \qquad k \text{ is a constant.} \tag{6.25}$$

Thus, the Hubble Law (6.1) follows from (6.24) and (6.25). It is the Hubble Law (6.1) that leads to the conclusion that our Universe is expanding.

However, if we adopt the view that the globular universe is in a black hole with the Schwarzschild radius R_s as in (6.16), the black hole redshift (6.22) cannot be ignored. By (6.17) and (6.18), the time-component g_{00} for the black hole can approximatively take the TOV solution as r near R_s:

$$g_{00} = -\frac{1}{4}\left(1 - \frac{r^2}{R_s^2}\right), \qquad \text{for } r \text{ near } R_s.$$

Hence, the redshift (6.22) is as

$$1+z = \frac{\sqrt{1 - r_0^2/R_s^2}}{\sqrt{1 - r_1^2/R_s^2}} \qquad \text{for } r_0, r_1 < R_s. \tag{6.26}$$

It is known that for a remote galaxy, r_1 is close to the boundary $r = R_s$. Therefore by (6.26) we have

$$z \to +\infty \quad \text{as} \quad r_1 \to R_s.$$

It reflects the redshifts observed from most remote objects. If the object is a virtual image as shown in Figure 6.1, then its position is the reflection point r_1. Thus, we see that even if the remote object is not moving, its redshift can still be very large.

5) *CMB problem.* In 1965, two physicists A. Penzias and R. Wilson discovered the low-temperature cosmic microwave background (CMB) radiation, which fills our Universe, and it is ever regarded as the Big-Bang product. However, for a static closed Universe, it is the most natural thing that there exists a CMB, because the Universe is a black-body and CMB is a result of black-body radiation.

6) *None expanding Universe.* As the energy of the Universe is given, the maximal radius, i.e. the Schwarzschild radius R_s, is determined, and the boundary is invariant. In fact, a globular universe must fill the ball with the Schwarzschild radius, although the distribution of the matter in this ball may be slightly non-homogenous. The main reason is that if the universe has a radius R smaller than R_s, then it must contain at least a sub-black hole with radius R_0 as follows

$$R_0 = \sqrt{\frac{R}{R_s}} R.$$

In Subsection 6.4 we shall discuss this topic.

6.3 Spherical Universe without boundary

Bounded universe has finite energy and space, and our Universe is bounded as we have demonstrated in the last section. Besides the globular universe, another type of bounded universe is the spherically-shaped corresponding to the $k=1$ case in the Friedmann model (6.8)-(6.10).

A globular universe must be static. With the same argument, a spherical closed universe have to be static as well. In this subsection, we are devoted to investigate the spherical cosmology.

1) *Cosmic radius.* For a static spherical universe, its radius R_c satisfies that

$$\dot{R}_c = 0, \qquad \ddot{R}_c = 0.$$

By the Friedmann equation (6.11), it leads to that

$$R_c^2 = \frac{3c^2}{8\pi G\rho}. \tag{6.27}$$

For a 3D sphere, its volume V is given by

$$V = 2\pi^2 R^3.$$

Thus, $\rho = M/2\pi^2 R_c^3$, and by (6.28) we get the radius R_c as

$$R_c = \frac{4MG}{3\pi c^2}. \tag{6.28}$$

This value (6.28) is also the maximal radius for a (possibly) oscillatory spherical universe.

2) *Negative pressure.* By (6.8) and $\ddot{R}_c = 0$, the pressure is negative:

$$p = -\frac{\rho c^2}{3}. \tag{6.29}$$

In order to resist the gravitational pulling, it is natural that there is a negative pressure in a static universe, which originates from three sources:

thermal expanding, radiation pressure, and dark energy.

These three types of forces are repulsive, and therefore yield the negative pressure as given by (6.29).

In fact, in our Universe both thermal and radiation (microwave radiation) pressures are very small. The main negative pressure is generated by the so called "dark energy". In [5], we have shown that the dark energy is the repulsive gravitational effect for a remote object of great distance. From the field theoretical point view, dark energy is an effect of the dual gravitational field Ψ_μ in the PID-induced gravitational field equations (1.6) discovered by the authors.

3) *Equivalence.* It seems that both spherical and globular geometries are very different. However, in the following we show that they are equivalent in cosmology. In fact, as the space-time curvature is caused by gravitation, a globular universe must be a 3D hemisphere as shown in Figure 6.2(a), and a spherical universe is as shown in Figure 6.2(b), which is a 3D sphere piecing the upper and lower hemispheres together.

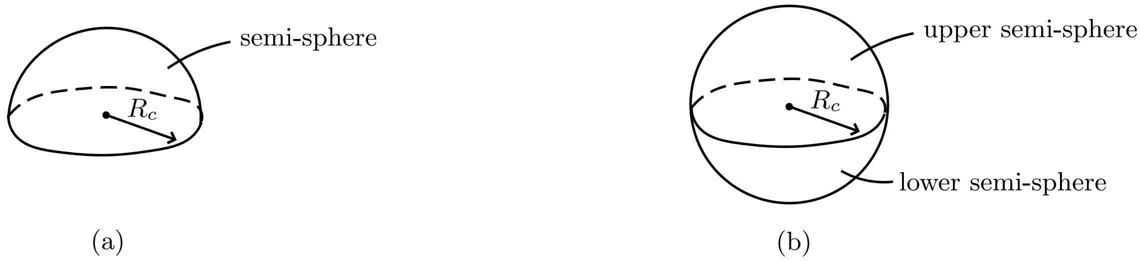

Figure 6.2: (a) A 3D hemisphere, and (b) a 3D sphere piecing the upper and lower hemispheres together.

In cosmology, the globular universe is a black hole, which likes as a 3D hemisphere, and the spherical universe can be regarded as if there were two hemispheres of black holes attached together.

We show this version from the cosmological dynamics.

First, by the Newtonian cosmological equation (6.7), i.e.

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2}. \tag{6.30}$$

For a static universe in a black hole with maximal radius R_c, the equation (6.30) becomes

$$\dot{R}_c = 0 \quad \Leftrightarrow \quad \frac{8\pi G}{3}\rho = \frac{kc^2}{R_c^2}. \tag{6.31}$$

The volume of the hemisphere is

$$V_0 = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{\frac{\pi}{2}} R_c^3 \sin\theta \sin^2\psi d\psi = \pi^2 R_c^3,$$

where $x \in \mathbb{R}^4$ takes the spherical coordinate:

$$(x_1,x_2,x_3,x_4) = (R_c\sin\psi\sin\theta\cos\varphi, R_c\sin\psi\sin\theta\sin\varphi, R_c\sin\psi\cos\theta, R_c\cos\psi). \tag{6.32}$$

Thus, the mass density is

$$\rho = M_{\text{total}}/\pi^2 R_c^3, \tag{6.33}$$

Then it follows from (6.31) that

$$R_c = \frac{8GM_{\text{total}}}{3\pi c^2} \quad \text{for } k=1. \tag{6.34}$$

Remark 6.1. The mass M_{total} in (6.33) contains the energy contributed by the space curvature, i.e.

$$M_{\text{total}} = M + \text{space curved energy},$$

where M is the mass of the flat space. By the invariance of density,

$$M / \frac{4\pi}{3} R_c^3 = M_{\text{total}} / \pi^2 R_c^3,$$

we get the relation

$$M_{\text{total}} = \frac{3\pi}{4} M. \tag{6.35}$$

With the flat space mass (6.35), from (6.34) we get the Schwarzschild radius $R_s = R_c$ for the cosmic black hole as follows

$$R_s = 2GM/c^2.$$

It means that the globular universe is essentially hemispherically-shaped. In particular the relation (6.35) can be generated to an arbitrary region $\Omega \subset \mathbb{R}^3$, i.e.

$$\widetilde{M}_\Omega = \frac{V_\Omega}{|\Omega|} M_\Omega, \tag{6.36}$$

where M_Ω is the flat space mass in $\Omega, \widetilde{M}_\Omega$ is the curved space mass, $|\Omega|$ is the normal volume of Ω,

$$V_\Omega = \int_\Omega \sqrt{g} dx, \quad g = \det(g_{ij}),$$

and g_{ij} $(1 \le i,j \le 3)$ is the spatial gravitational metric. □

Now, we return to the Friedmann model (6.11) with $k = 1$, which has the same form as that of the globular dynamic equation (6.30), and is of the same maximal radius R_c as that in (6.34). Hence, it is natural that a static spherical universe is considered as if there were two hemispherical black holes attached together. In fact, the static spherical universe forms an entire black hole as a closed space.

4) *Basic problems.* Since a static spherical universe is equivalent to two globular universes to be pieced together along with their boundary, an observer in its hemisphere is as if one is in a globular universe. Hence, the basic problems – the cosmic edge problem, flatness problem, horizon problem, and CMB problem– can be explained in the same fashion.

The redshift problem is slightly different, and the gravitational redshift is given by

$$1 + z = \frac{1}{\sqrt{-g_{00}(r)}}, \tag{6.37}$$

where r is the distance between the light source and the observer, and g_{00} is the time-component of the gravitational metric.

Due to the horizon of sphere, for an arbitrary point on a spherical universe, its opposite hemisphere relative to the point plays a similar role as a black hole. Hence, in the redshift formula (6.37), g_{00} can be approximatively taken as the Schwarzschild solution for distant objects as follows

$$-g_{00}=1-\frac{R_s}{\widetilde{r}}, \qquad R_s=\frac{2MG}{c^2}, \qquad \widetilde{r}=2R_s-r \qquad \text{for } 0\le r<R_s,$$

where M is the cosmic mass of hemisphere, and $\widetilde{r}$ is the distance from the light source to the opposite radial point, and r is from the light source to the point. Hence, formula (6.37) can be approximatively written as

$$1+z=\frac{1}{\sqrt{\alpha(r)(1-\frac{R_s}{\widetilde{r}})}}=\frac{\sqrt{2R_s-r}}{\sqrt{\alpha(r)(R_s-r)}} \qquad \text{for } 0<r<R_s. \tag{6.38}$$

where

$$\alpha(0)=2, \qquad \alpha(R_s)=1, \qquad \alpha'(r)<0.$$

We see from (6.38) that the redshift $z\to\infty$ as $r\to R_s$, and, consequently, we cannot see objects in the opposite hemisphere. Intuitively, $\alpha(r)$ represents the gravitational effect of the matter in the hemisphere of the observer.

5) *Physical conclusions.* In either case, globular or spherical, the universe is equivalent to globular universe(s). It is not originated from a Big-bang, is static, and confined in a black hole in the sense as addressed above. The observed mass M and the implicit mass M_{total} have the relation

$$M_{\text{total}}=2\times\frac{3\pi}{4}M=3\pi M/2, \tag{6.39}$$

which is derived by (6.35) adding the mass of another hemisphere.

The implicit mass M of (6.39) contains the dark matter. In [5], both the dark matter $M_{\text{total}}-M$ and the dark energy (i.e. the negative pressure (6.29)) just a property of gravity.

6.4 New cosmology

We start with two difficulties encountered in modern cosmology.

First, if the Universe were born to a Big-Bang and expanded continuously, then in the expansion process it would generate successively a large number of black holes, whose radii vary as follows:

$$\sqrt{\frac{R_0}{R_s}}R_0\le r\le\sqrt{\frac{R}{R_s}}R, \quad R_0<R\le R_s=\frac{2MG}{c^2}, \tag{6.40}$$

where M is the total mass in the universe, R_0 is the initial radius, R is the expanding radius, and r is the radius of sub-black holes.

To see this, we consider a homogeneous universe with radius $R<R_s$. Then the mass density ρ is given by

$$\rho=\frac{3M}{4\pi R^3}. \tag{6.41}$$

On the other hand, by Theorem 2.4, the condition for a ball B_r with radius r to form a black hole is that the mass M_r in B_r satisfies that

$$\frac{M_r}{r}=\frac{c^2}{2G}. \tag{6.42}$$

By (6.41), we have

$$M_r=\frac{4\pi}{3}r^3\rho=\frac{r^3}{R^3}M.$$

Then it follows from (6.42) that

$$r=\sqrt{\frac{R}{R_s}}R. \tag{6.43}$$

Actually, in general for a ball B_r in a universe with radius $R<R_s$, if its mass M_r satisfies (6.42) then it will form a black hole, and its radius r satisfies that

$$r\leq\sqrt{\frac{R}{R_s}}R.$$

In particular, there must exist a black hole whose radius r is as in (6.43). Thus, we derive the conclusion (6.40).

Based on (6.40) we can deduce that if the Universe were born to a Big-Bang and continuously expands, then it would contain many black holes with smaller ones being embedded in the larger ones. In particular, the Universe would contain a huge black hole whose radius r is almost equal to the cosmic radius R_s. This is not what we observed in our Universe.

The second difficulty of modern cosmology concerns with the Hubble Law (6.1), which is restated as $v=HR$, where $c/H=R_s$. Consider a remote object with mass M_0, then the observed mass M_{obser} is given by $M_{\text{obser}}=\frac{M_0}{\sqrt{1-\frac{v^2}{c^2}}}$. Consequently, the corresponding gravitational force F to the observer with mass m is

$$F=-\frac{mM_{\text{obser}}G}{r^2}=-\frac{mM_0G}{r^2\sqrt{1-\frac{v^2}{c^2}}}=-\frac{mM_0G}{r^2\sqrt{1-\frac{H^2}{c^2}r^2}}=-\frac{mM_0G}{r^2\sqrt{1-\frac{r^2}{R_s^2}}}.$$

It is clear then that as $r\to R_s$, $F\to-\infty$. This is clearly not what is observed.

In conclusion, we have rigorously derived the following new theory of cosmology:

Theorem 6.2. *Assume a) the Einstein theory of general relativity, and b) the principle of cosmological principle that the universe is homogeneous and isotropic. Then the following assertions hold true for our Universe:*

1) *All universes are bounded, are not originated from a Big-Bang, and are static.*

2) *The topological structure of our Universe can only be the 3D sphere such that to each observer, the corresponding equator with the observer at the center of the hemisphere can be viewed as the black hole horizon.*

Theorem 6.3. *If we assume only a) the Einstein theory of general relativity, and b′) the universe is homogeneous. Then all universes can only be either a 3D sphere as given in Theorem 6.2, or a globular universe, which is a 3D open ball B_{R_s} of radius R_s, forming the interior of a black hole with R_s as its Schwarzschild radius. In the later case, the Universe is also static, is not originated from a Big-Bang, and the matter fills the entire Universe. Also, the following assertions hold true:*

1) *The cosmic observable mass M and the total mass M_{total}, which includes both M and the non-observable mass due to the space curvature energy, satisfy the following relation*

$$M_{total} = \begin{cases} 3\pi M/2 & \text{for the spherical structure,} \\ 3\pi M/4 & \text{for the globular structure.} \end{cases} \tag{6.44}$$

The difference $M_{total} - M$ can be regarded as the dark matter.

2) *The static Universe has to possess a negative pressure to balance the gravitational attracting force. The negative pressure is actually the effect of the gravitational repelling force, also called dark energy.*

3) *Both dark matter and dark energy are a property of gravity, which is reflected in both spacetime curvature, and the attracting and repulsive gravitational forces in different scales of the Universe. This law of gravity is precisely described by the new gravitational field equations (1.8); see also [5].*

We end this section with three remarks and observations.

First, astronomical observations have shown that the measurable mass M is about one fifth of total mass M_{total}. By (6.44), for the spherical universe,

$$M_{\text{total}} = 4.7M.$$

This relation also suggest that the spherical universe case fits better the current understanding for our Universe.

Second, due to the horizon of sphere, for an arbitrary point in a spherical universe, its opposite hemisphere relative to the point is as if it is a black hole. Hence the main contribution to the redshifts is from the effect of the black hole, as explicitly given by (6.38).

Third, in modern cosmology, the view of expanding universe was based essentially on the Friedmann model and the Hubble Law. The observations can accurately measure the distances and redshifts for some galaxies, which allowed astronomers to get both measured and theoretical data, and their deviation led to the conclusion that the expanding universe is accelerating. The misunderstanding comes from the perception that the Doppler redshift is the main source of redshifts.

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