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Dynamics of a Deterministic and Stochastic Susceptible-exposed-infectious-recovered Epidemic Model*

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Abstract We investigate a susceptible-exposed-infectious-recovered (SEIR) epidemic model with asymptomatic infective individuals. First, we formulate a deterministic model, and give the basic reproduction number $\mathcal{R}_0$. We show that the disease is persistent, if $\mathcal{R}_0 > 1$, and it is extinct, if $\mathcal{R}_0 < 1$. Then, we formulate a stochastic version of the deterministic model. By constructing suitable stochastic Lyapunov functions, we establish sufficient criteria for the extinction and the existence of ergodic stationary distribution to the model. As a case, we study the COVID-19 transmission in Wuhan, China, and perform some sensitivity analysis. Our numerical simulations are carried out to illustrate the analytic results.

Keywords Asymptomatic infective individual, Extinction, Persistence, Stationary distribution.

MSC(2010) 92D30, 34D05, 60H10.

1. Introduction

It is well-known that mathematical modeling has become a powerful tool in studying dynamic behaviors and predicting the spreading trend of diseases [2,4,8,12,18]. The establishment of an appropriate epidemic model can clearly describe the transmission mechanism of infectious diseases, and then we analyze it and find effective measures for epidemic control. In 1927, Kermack and Mckendrick [14] first proposed the SIR epidemic model in which the population is separated into three mutually exclusive stages of infection: susceptible, infective and recovered individuals according to their status related to the disease with numbers at the time t denoted by $S(t)$, $I(t)$ and $R(t)$ respectively. Moreover, the basic reproduction number which determines the persistence or extinction of the disease is also described. The SIR model provides a sound theoretical basis for the use of mathematical models to study infectious diseases.

Following the idea of Kermack and Mckendrick, many realistic models have been proposed to investigate the transmission dynamics of infectious diseases (see,

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e.g., [3, 25, 34, 36]). For some diseases (e.g., tuberculosis, influenza, measles), on adequate contact with an infective individual, a susceptible individual becomes infected, but it has not been infective yet. This individual remains in the exposed class for a certain latent period before becoming infective [7]. Thus, in [19], the SEIR model was proposed to further consider the exposed individuals. As a basis, the SEIR model has multiple variants with different degrees of complexity, including those admitting controls, i.e., different kinds of incidence rates, constant and feedback vaccination and treatment controls or those involving several interacting patches associated with different towns or regions (see [3, 28, 32, 35, 39] and the references therein). In the reality, for some infectious diseases (e.g., COVID-19), since the strong concealment of the asymptomatic infective individuals is verified, the existence of these individuals has made the control of disease more difficult. Therefore, it is important to take into account the asymptomatic infective individuals in the SEIR model. We assume that any individual can move between the classes according to the following graph.

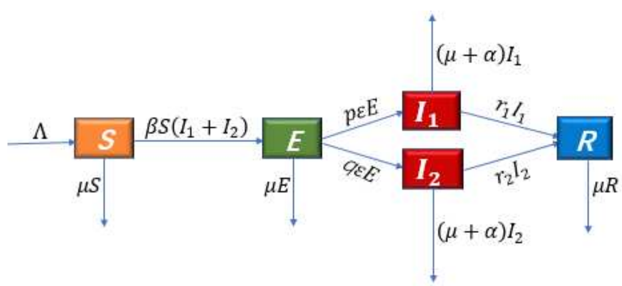


Figure 1. Transfer diagram of the model

In Figure 1, $I_1(t)$ is the number of the infective individuals, which are diagnosed and symptomatic, and $I_2(t)$ is the number of the infective individuals, which are diagnosed but asymptomatic. Thus, the model can be written as a system of differential equations with the form

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \beta S(I_1 + I_2) - \mu S, \\ \dfrac{dE}{dt} = \beta S(I_1 + I_2) - (\epsilon + \mu)E, \\ \dfrac{dI_1}{dt} = p\epsilon E - (\mu + \alpha + r_1)I_1, \\ \dfrac{dI_2}{dt} = q\epsilon E - (\mu + \alpha + r_2)I_2, \\ \dfrac{dR}{dt} = r_1 I_1 + r_2 I_2 - \mu R. \end{cases} \tag{1.1}$$

The biological interpretations of the parameters are shown as the table below.

Table 1. Biological interpretations for variables and parameters in system (1.1)

Parameters	Description
Λ	The number of recruitment of susceptible individuals per unit time
β	The contact transmission rate
μ	The natural mortality rate
ϵ	The rate of diagnosed individuals
p	The proportion of symptomatic individuals in all diagnosed individuals
q	The proportion of asymptomatic individuals in all diagnosed individuals
α	The disease-related mortality rate
r_1	The recovery rate of the symptomatic individuals
r_2	The recovery rate of the asymptomatic individuals

In Table 1, the parameters Λ and μ are assumed to be positive, the parameters β, ϵ, p, q, α, r_1 and r_2 are non-negative, and $p + q = 1$.

For disease-related epidemics, the nature of epidemic growth and spread is random due to the unpredictability of person-to-person contact [29]. Therefore, the variability and randomness of the environment are fed through the state of the epidemic [31]. Besides, in epidemic dynamics, the stochastic model may be a more appropriate way of modeling epidemics in many circumstances (see e.g., [1,5,11,16,17,20–22,37,38,40]). Following [24], we assume that the environmental influence on the individuals is proportional to the states $S(t)$, $E(t)$, $I_1(t)$, $I_2(t)$ and $R(t)$ to obtain a stochastic version of (1.1) as follows

$$\begin{cases} dS = [\Lambda - \beta S(I_1 + I_2) - \mu S]\, dt + \sigma_1 S dB_1(t), \\ dE = [\beta S(I_1 + I_2) - (\epsilon + \mu)E]\, dt + \sigma_2 E dB_2(t), \\ dI_1 = [p\epsilon E - (\mu + \alpha + r_1)I_1]\, dt + \sigma_3 I_1 dB_3(t), \\ dI_2 = [q\epsilon E - (\mu + \alpha + r_2)I_2]\, dt + \sigma_4 I_2 dB_4(t), \\ dR = [r_1 I_1 + r_2 I_2 - \mu R]\, dt + \sigma_5 R dB_5(t), \end{cases} \tag{1.2}$$

where $B_i(t)$ $(i = 1, 2, \cdots, 5)$ are independent standard Brownian motions, σ_i are the intensity of the standard Gaussian white noise, and $\sigma_1 S dB_1(t)$, $\sigma_2 E dB_2(t)$, $\sigma_3 I_1 dB_3(t)$, $\sigma_4 I_2 dB_4(t)$ and $\sigma_5 R dB_5(t)$ are used to model the interaction between the individuals and the environment.

When studying the transmission dynamics of the infection, it is important to know when the infection will extinct (prevail) in the population. For the deterministic model (1.1), this problem can be solved by showing that the disease-free equilibrium (endemic-equilibrium) is globally asymptotically stable. However, for model (1.2), there is no endemic-equilibrium. Then, Khasminskii [15] showed that the existence of an ergodic stationary distribution to model (1.2) can reveal the persistence of the infection.

The paper is to study the global dynamics of models (1.1) and (1.2). We will apply the basic reproduction number $\mathcal{R}_0$ to describe whether the disease will prevail to model (1.1) or not, and establish sufficient criteria for the extinction and existence of ergodic stationary distribution to model (1.2).

This paper is organized as follows. In Section 2, the existence and the global stability of disease-free equilibrium (endemic equilibrium) are investigated in model (1.1). In Section 3, we are devoted to establishing sufficient criteria for the extinction and existence of an ergodic stationary distribution. In Section 4, we present a case study on COVID-19 transmission in Wuhan, China, and carry out some sensitivity analysis to illustrate our results. Finally, we complete our paper with some concluding remarks.

2. The deterministic model (1.1)

First, we give some basic properties of the solution to model (1.1).

Lemma 2.1. *The solutions $S(t)$, $E(t)$, $I_1(t)$, $I_2(t)$, $R(t)$ of system (1.1) with initial values $S(0) \geq 0$, $E(0) \geq 0$, $I_1(0) \geq 0$, $I_2(0) \geq 0$, $R(0) \geq 0$ are positive for all $t \geq 0$.*

Proof. By system (1.1), we have

$$\frac{dS(t)}{dt}\Big|_{S(t)=0} = \Lambda \geq 0, \qquad \frac{dE(t)}{dt}\Big|_{E(t)=0} = \beta S(t)(I_1(t) + I_2(t) \geq 0,$$
$$\frac{dI_1(t)}{dt}\Big|_{I_1(t)=0} = p\epsilon E(t) \geq 0, \qquad \frac{dI_2(t)}{dt}\Big|_{I_2(t)=0} = q\epsilon E(t) \geq 0,$$
$$\frac{dR(t)}{dt}\Big|_{R(t)=0} = r_1 I_1(t) + r_2 I_2(t) \geq 0.$$

Since the solutions of system (1.1) are continuous, $S(t)$, $E(t)$, $I_1(t)$, $I_2(t)$, $R(t)$ remain positive for all $t \geq 0$. $\square$

Lemma 2.2. *The feasible region Ω is positive invariant for system (1.1) with initial conditions in $\mathbb{R}^5_+$ defined by*

$$\Omega = \left\{ (S(t), E(t), I_1(t), I_2(t), R(t)) \in \mathbb{R}^5_+ \mid 0 \leq S(t) + E(t) + I_1(t) + I_2(t) + R(t) \leq \frac{\Lambda}{\mu} \right\},$$

where $\mathbb{R}^5_+ = \{(x_1, x_2, x_3, x_4, x_5) | x_i \geq 0,\ i = 1, 2, \cdots, 5\}$.

Proof. Adding all the equations of system (1.1), we obtain

$$\frac{dN}{dt} = \Lambda - \mu N - \alpha(I_1 + I_2) \leq \Lambda - \mu N.$$

It follows that

$$0 \leq N(t) \leq \frac{\Lambda}{\mu} + N(0)e^{-\mu t},$$

where $N(0)$ represents the initial values of the total population.

Thus, $\lim_{t\to\infty} \sup N(t) = \frac{\Lambda}{\mu}$, which implies that the region Ω is a positive invariant set for system (1.1). $\square$

2.1. Disease-free equilibrium and the basic reproduction number

It is easy to see that model (1.1) has a disease-free equilibrium

$$\mathcal{E}_0 := (S_0, 0, 0, 0, 0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0 \right).$$

We rearrange (1.1) as follows

$$\begin{cases} \dfrac{dE}{dt} = \beta S(I_1 + I_2) - (\epsilon + \mu)E, \\ \dfrac{dI_1}{dt} = p\epsilon E - (\mu + \alpha + r_1)I_1, \\ \dfrac{dI_2}{dt} = q\epsilon E - (\mu + \alpha + r_2)I_2, \\ \dfrac{dS}{dt} = \Lambda - \beta S(I_1 + I_2) - \mu S, \\ \dfrac{dR}{dt} = r_1 I_1 + r_2 I_2 - \mu R. \end{cases}$$

Let $X := (E, I_1, I_2, S, R)^T$. Then system (1.1) can be written as

$$\frac{dX}{dt} = \mathcal{F}(x) - \mathcal{V}(x),$$

where

$$\mathcal{F}(x) = \begin{pmatrix} \beta S(I_1 + I_2) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{V}(x) = \begin{pmatrix} (\epsilon + \mu)E \\ (\mu + \alpha + r_1)I_1 - p\epsilon E \\ (\mu + \alpha + r_2)I_2 - q\epsilon E \\ \mu S + \beta S(I_1 + I_2) - \Lambda \\ \mu R - r_1 I_1 - r_2 I_2 \end{pmatrix}.$$

The Jacobian matrices of $\mathcal{F}(x)$ and $\mathcal{V}(x)$ at the disease-free equilibrium $\mathcal{E}_0$ are respectively

$$D\mathcal{F}(\mathcal{E}_0) = \begin{pmatrix} 0 & \beta S_0 & \beta S_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad D\mathcal{V}(\mathcal{E}_0) = \begin{pmatrix} \epsilon + \mu & 0 & 0 & 0 & 0 \\ -p\epsilon & \mu + \alpha + r_1 & 0 & 0 & 0 \\ -q\epsilon & 0 & \mu + \alpha + r_2 & 0 & 0 \\ 0 & \beta S_0 & \beta S_0 & \mu & 0 \\ 0 & -r_1 & -r_2 & 0 & \mu \end{pmatrix}.$$

The reproduction number of model (1.1), denoted by $\mathcal{R}_0$, is given by [9]

$$\begin{aligned} \mathcal{R}_0 =& \rho(FV^{-1}) \\ =& \frac{\beta S_0}{\mu + \epsilon} \left(\frac{p\epsilon}{\mu + \alpha + r_1} + \frac{q\epsilon}{\mu + \alpha + r_2} \right) = \frac{\beta \Lambda}{(\mu + \epsilon)\mu} \left(\frac{p\epsilon}{\mu + \alpha + r_1} + \frac{q\epsilon}{\mu + \alpha + r_2} \right). \end{aligned}$$

By Theorem 2 of [9], on the local stability of $\mathcal{E}_0$, we have the following result.

Theorem 2.1. *The disease-free equilibrium $\mathcal{E}_0$ is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable otherwise.*

2.2. Global stability of disease-free equilibrium

Theorem 2.2. *For system (1.1), the disease-free equilibrium $\mathcal{E}_0$ is globally asymptotically stable, if $\mathcal{R}_0 < 1$.*

Proof. We construct a Lyapunov function V as follows

$$V(t) = E(t) + \frac{\beta S_0}{\mu+\alpha+r_1} I_1(t) + \frac{\beta S_0}{\mu+\alpha+r_2} I_2(t).$$

Then, we obtain

$$\frac{dV}{dt} = \frac{dE}{dt} + \frac{\beta S_0}{\mu+\alpha+r_1}\frac{dI_1}{dt} + \frac{\beta S_0}{\mu+\alpha+r_2}\frac{dI_2}{dt}.$$

By system (1.1), we have

$$\begin{aligned}\frac{dV}{dt} &= \left[-(\epsilon+\mu) + \frac{\beta S_0 p\epsilon}{\mu+\alpha+r_1} + \frac{\beta S_0 q\epsilon}{\mu+\alpha+r_2}\right]E + \beta S I_1 - \beta S_0 I_1 + \beta S I_2 - \beta S_0 I_2 \\ &\le \left[-(\epsilon+\mu) + \frac{\beta S_0 p\epsilon}{\mu+\alpha+r_1} + \frac{\beta S_0 q\epsilon}{\mu+\alpha+r_2}\right]E \\ &= (\epsilon+\mu)(\mathcal{R}_0 - 1)E \le 0.\end{aligned}$$

Setting $\frac{dV}{dt} = 0$, then $E(t) = 0$.

Plugging it into the equations of system (1.1), we get

$$\lim_{t\to\infty} S(t) = \frac{\Lambda}{\mu}, \lim_{t\to\infty} I_1(t) = \lim_{t\to\infty} I_2(t) = \lim_{t\to\infty} R(t) = 0.$$

By LaSalle invariance principle [27], the disease-free equilibrium $\mathcal{E}_0$ is globally asymptotically stable, if

$$\mathcal{R}_0 < 1.$$

□

2.3. Global stability of the endemic equilibrium

If $\mathcal{R}_0 > 1$, then system (1.1) has a unique endemic equilibrium

$$\mathcal{E}^* := (S^*, E^*, I_1^*, I_2^*, R^*),$$

where

$$\begin{aligned} S^* &= \frac{\Lambda}{\beta E^*\left(\frac{p\epsilon}{\mu+\alpha+r_1} + \frac{q\epsilon}{\mu+\alpha+r_2}\right) + \mu},\ E^* = \frac{\Lambda(\mathcal{R}_0 - 1)}{\mathcal{R}_0(\mu+\epsilon)}, \\ I_1^* &= \frac{p\epsilon}{\mu+\alpha+r_1}E^*,\ I_2^* = \frac{q\epsilon}{\mu+\alpha+r_2}E^*,\ R^* = \frac{1}{\mu}\left(\frac{r_1 p\epsilon}{\mu+\alpha+r_1} + \frac{r_2 q\epsilon}{\mu+\alpha+r_2}\right).\end{aligned}$$

Theorem 2.3. *If $\mathcal{R}_0 > 1$, the endemic equilibrium $\mathcal{E}^*$ of system (1.1) is globally asymptotically stable.*

Proof. If $\mathcal{R}_0 > 1$, there exists a unique endemic equilibrium $\mathcal{E}^*$. Motivated by [13, 26], we define the following Lyapunov function

$$V = V_1 + V_2 + V_3 + V_4,$$

where

$$\begin{aligned}
V_1 &= S - S^* - S^* \ln\frac{S}{S^*}, \\
V_2 &= A\left(E - E^* - E^* \ln\frac{E}{E^*}\right), \\
V_3 &= B\left(I_1 - I_1^* - I_1^* \ln\frac{I_1}{I_1^*}\right), \\
V_4 &= C\left(I_2 - I_2^* - I_2^* \ln\frac{I_2}{I_2^*}\right).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\frac{dV}{dt} =&\frac{dS}{dt}\left(1-\frac{S^*}{S}\right) + A\frac{dE}{dt}\left(1-\frac{E^*}{E}\right) + B\frac{dI_1}{dt}\left(1-\frac{I_1^*}{I_1}\right) + C\frac{dI_2}{dt}\left(1-\frac{I_2^*}{I_2}\right) \\
=&\left(1-\frac{S^*}{S}\right)(\Lambda - \mu S - \beta S(I_1+I_2)) + A\left(1-\frac{E^*}{E}\right)(\beta S(I_1+I_2) \\
&-(\epsilon+\mu)E) + B\left(1-\frac{I_1^*}{I_1}\right)(p\epsilon E - (\mu+\alpha+r_1)I_1) \\
&+ C\left(1-\frac{I_2^*}{I_2}\right)(q\epsilon E - (\mu+\alpha+r_2)I_2) \\
=&\left(1-\frac{S^*}{S}\right)(\beta S^*(I_1^*+I_2^*) + \mu S^* - \mu S - \beta S(I_1+I_2)) \\
&+ A\left(1-\frac{E^*}{E}\right)\left(\beta S(I_1+I_2) - \frac{\beta S^*(I_1^*+I_2^*)}{E^*}E\right) + B\left(1-\frac{I_1^*}{I_1}\right)\Bigg(p\epsilon E - \\
&\frac{p\epsilon E^*}{I_1^*}I_1\Bigg) + C\left(1-\frac{I_2^*}{I_2}\right)\left(q\epsilon E - \frac{q\epsilon E^*}{I_2^*}I_2\right).
\end{aligned}$$

Letting $x := \frac{S}{S^*},\ y := \frac{E}{E^*},\ z := \frac{I_1}{I_1^*},\ m := \frac{I_2}{I_2^*}$, then

$$\begin{aligned}
&\frac{dV}{dt} \\
=&\left(1-\frac{1}{x}\right)(\beta S^*(I_1^*+I_2^*) + \mu S^* - \mu S^* x - \beta S^* I_1^* xz - \beta S^* I_2^* xm) \\
&+ A\left(1-\frac{1}{y}\right)(\beta S^* I_1^* xz + \beta S^* I_2^* xm - \beta S^*(I_1^*+I_2^*)y) \\
&+ B\left(1-\frac{1}{z}\right)(p\epsilon E^* y - p\epsilon E^* z) + C\left(1-\frac{1}{m}\right)(q\epsilon E^* y - q\epsilon E^* m) \\
=&\beta S^*(I_1^*+I_2^*) + \mu S^* - \mu S^* x - \beta S^* I_1^* xz - \beta S^* I_2^* xm - [\beta S^*(I_1^*+I_2^*) + \mu S^*]\frac{1}{x} \\
&+ \mu S^* + \beta S^* I_1^* z + \beta S^* I_2^* m + A(\beta S^* I_1^* xz + \beta S^* I_2^* xm - \beta S^*(I_1^*+I_2^*)y \\
&- (\beta S^* I_1^* xz + \beta S^* I_2^* xm)\frac{1}{y} + \beta S^*(I_1^*+I_2^*) + aR^*) \\
&+ B\left(p\epsilon E^* y - p\epsilon E^* z - p\epsilon E^* \frac{y}{z} + p\epsilon E^*\right) \\
&+ C\left(q\epsilon E^* y - q\epsilon E^* m - q\epsilon E^* \frac{y}{m} + q\epsilon E^*\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\mu S^* \left(x + \frac{1}{x} - 2 \right) + \beta S^*(I_1^* + I_2^*) + A\beta S^*(I_1^* + I_2^*) + Bp\epsilon E^* + Cq\epsilon E^* \\
&\quad + D(r_1 I_1^* + r_2 I_2^*) + xz\beta S^* I_1^*(A-1) + xm\beta S^* I_2^*(A-1) + y(-A\beta S^*(I_1^* + I_2^*) \\
&\quad + Bp\epsilon E^* + Cq\epsilon E^*) + z(-Bp\epsilon E^* + \beta S^* I_1^*) + m(-Cq\epsilon E^* + \beta S^* I_2^*) \\
&\quad - \frac{1}{x}\beta S^*(I_1^* + I_2^*) - \frac{xz}{y} A\beta S^* I_1^* - \frac{xm}{y} A\beta S^* I_2^* - \frac{y}{z} Bp\epsilon E^* - \frac{y}{m} Cq\epsilon E^*.
\end{aligned}
$$

The variable terms that appear in $\frac{dV}{dt}$ with positive coefficients are $xz,\ xm,\ y,\ z,\ m$. If the total of these coefficients is positive, then there is a possibility that $\frac{dV}{dt}$ is positive.

Let the coefficients of $xz,\ xm,\ y,\ z$ and m be equal to zero, we have

$$
\begin{cases}
A - 1 = 0, \\
-A\beta S^*(I_1^* + I_2^*) + Bp\epsilon E^* + Cq\epsilon E^* = 0, \\
-Bp\epsilon E^* + \beta S^* I_1^* = 0, \\
-Cq\epsilon E^* + \beta S^* I_2^* = 0,
\end{cases}
$$

by which we get

$$
A = 1, B = \frac{\beta S^* I_1^*}{p\epsilon E^*}, C = \frac{\beta S^* I_2^*}{q\epsilon E^*}.
$$

Hence, we have

$$
\begin{aligned}
\frac{dV}{dt} &= -\mu S^* \frac{(x-1)^2}{x} + 3\beta S^*(I_1^* + I_2^*) - \frac{1}{x}\beta S^*(I_1^* + I_2^*) - \frac{xz}{y}\beta S^* I_1^* - \frac{xm}{y}\beta S^* I_2^* \\
&\quad - \frac{y}{z}\beta S^* I_1^* - \frac{y}{m}\beta S^* I_2^* \\
&= -\mu S^* \frac{(x-1)^2}{x} + \beta S^* I_1^* \left(3 - \frac{1}{x} - \frac{xz}{y} - \frac{y}{z} \right) + \beta S^* I_2^* \left(3 - \frac{1}{x} - \frac{xm}{y} - \frac{y}{m} \right).
\end{aligned}
$$

Since the arithmetical mean is greater than or equal to the geometrical mean,

$$
3 - \frac{1}{x} - \frac{xz}{y} - \frac{y}{z} \le 0, \ \text{for } x > 0, y > 0, z > 0 \text{ and } 3 - \frac{1}{x} - \frac{xz}{y} - \frac{y}{z} = 0
$$

if and only if

$$
x = 1, \quad y = z;
$$

$$
3 - \frac{1}{x} - \frac{xm}{y} - \frac{y}{m} \le 0, \ \text{for } x > 0,\ y > 0,\ m > 0 \text{ and } 3 - \frac{1}{x} - \frac{xm}{y} - \frac{y}{m} = 0
$$

if and only if

$$
x = 1, \quad y = m.
$$

Then,

$$
\dot{V} \le 0.
$$

Thus, for system (1.1), the endemic equilibrium is globally asymptotically stable, if $\mathcal{R}_0 > 1$ by LaSalle invariance principle [27]. □

3. The stochastic model (1.2)

First, we give some preliminaries in this section.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual condition (i.e., it is increasing and right continuous, while $\{\mathscr{F}_0\}$ contains all P-null sets), and we also let $\mathbb{R}_+^d = \left\{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\right\}$.

In general, consider the d-dimensional stochastic differential equation

$$dx(t) = f\left(t, x(t)\right) dt + g\left(t, x(t)\right) dB_t, \tag{3.1}$$

where $f\left(t, x(t)\right)$ is a function in $\mathbb{R}^d$ defined in $[t_0, \infty] \times \mathbb{R}^d$, $g\left(t, x(t)\right)$ is a $d \times m$ matrix, and $f,\ g$ are locally Lipschitz functions in x. B_t denotes an m-dimensional standard Brownian motion defined on the complete probability space $(\Omega,\ \mathscr{F},\ \{\mathscr{F}_t\}_{t\geq 0},$ $P)$. Denoted by $C^{2,1}\left(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+\right)$, the family of all nonnegative functions $V(x,t)$ defined on $\mathbb{R}^d \times [t_0, \infty]$ are continuously twice differentiable in x and once in t. The differential operator L of equation (3.1) is defined [23] by

$$L = \tfrac{\partial}{\partial t} + \textstyle\sum_{i=1}^d f_i(t) \tfrac{\partial}{\partial x_i} + \dfrac{1}{2} \textstyle\sum_{i,j=1}^d \left[g^T(x,t)\, g(x,t)\right]_{ij} \tfrac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on a function $V \in C^{2,1}\left(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+\right)$, then

$$LV(x,t) = V_t(x,t) + V_x(x,t) f(x,t) + \frac{1}{2} trace\left[g^T(x,t) V_{xx} g(x,t)\right],$$

where $V_t(x,t) = \frac{\partial V}{\partial t}$, $V_x(x,t) = \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_d}\right)$, $V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{d\times d}$.

By Itô's formula, if $x(t)$ is the solution of (3.1), then

$$dV(x,t) = LV(x,t)\, dt + V_x(x,t)\, g(x,t)\, dB_t.$$

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}_+^d$ described by

$$dX(t) = b(X)dt + \sum_{r=1}^k \sigma_r(X) dB_r(t),$$

and the diffusion matrix is defined by

$$A(X) = (a_{ij}(x)),\ a_{ij}(x) = \sum_{r=1}^k \sigma_r^i(x) \sigma_r^j(x).$$

3.1. The well-posedness of the solution

Lemma 3.1. *For any initial value $(S(0), E(0), I_1(0), I_2(0), R(0)) \in \mathbb{R}_+^5$, there is a unique solution $X(t) = (S(t), E(t), I_1(t), I_2(t), R(t))$ of system (1.2) on $t \geq 0$, and the solution will remain in $\mathbb{R}_+^5$ with probability one.*

Proof. Since the coefficients of system (1.2) satisfy the local Lipschitz condition, for any initial value $(S(0), E(0), I_1(0), I_2(0), R(0)) \in \mathbb{R}_+^5$, there is a unique local solution in $[0,\tau_e]$, where τ_e is the explosion time [23]. To show that this solution is global, we only need to prove that $\tau_e = \infty$ is almost sure (abbreviated as a.s.).

To this end, let k_0 be sufficiently large such that every component of X_0 lies in the interval $[\frac{1}{k_0}, k_0]$. For each integer $k > k_0$, we define the stopping time with the form

$$\tau_k = \inf\left\{ t \in [0, \tau_e) : \min X(t) \le \frac{1}{k} \ or \ \max X(t) \ge k \right\}.$$

Throughout this paper, we set $inf\emptyset = \infty$ (as usual, $\emptyset$ denotes the empty set). It is easy to see that τ_k is increasing as $k \to \infty$. Setting $\tau_\infty = \lim_{k\to\infty} \tau_k$, then $\tau_\infty \le \tau_e$ a.s. If we show $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $X(t) \in \mathbb{R}^5_+$ a.s., for all $t \ge 0$. In other words, to complete the proof, what we need to show is $\tau_\infty = \infty$ a.s. If this assertion is violated, then there are a pair of constants $T > 0$ and $\eta \in (0, 1)$ such as $P\{\tau_\infty \le T\} > \eta$. Consequently, there is an integer $k_1 \ge k_0$ such as $P\{\tau_k \le T\} > \eta$, for all $k \ge k_1$.

Define a C^2-function V: $\mathbb{R}^5_+ \to \mathbb{R}^1_+$ as

$$\begin{aligned} &V(S, E, I_1, I_2, R) \\ =&(S - c - \ln\frac{S}{c}) + (E - 1 - \ln E) + (I_1 - 1 - \ln I_1) + (I_2 - 1 - \ln I_2) + (R - 1 - \ln R). \end{aligned}$$

The non-negativity of this function can be obtained from

$$u - 1 - \ln u \ge 0 \text{ for any } u > 0.$$

Using It$\hat{o}$'s formula, we have

$$\begin{aligned} &dV(S, E, I_1, I_2, R) \\ =&LV dt + \sigma_1(S - 1)dB_1(t) + \sigma_2(E - 1)dB_2(t) + \sigma_3(I_1 - 1)dB_3(t) \\ &+ \sigma_4(I_2 - 1)dB_4(t) + \sigma_5(R - 1)dB_5(t), \end{aligned}$$

where

$$\begin{aligned} LV =& \left(1 - \frac{c}{S}\right)(\Lambda - \mu S - \beta S(I_1 + I_2)) + \left(1 - \frac{1}{E}\right)(\beta S(I_1 + I_2) - (\epsilon + \mu)E) \\ &+ \left(1 - \frac{1}{I_1}\right)(p\epsilon E - (\mu + \alpha + r_1)I_1) + \left(1 - \frac{1}{I_2}\right)(q\epsilon E - (\mu + \alpha + r_2)I_2) \\ &+ \left(1 - \frac{1}{R}\right)(r_1 I_1 + r_2 I_2 - \mu R) + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2) \\ =&\Lambda + 5\mu + \epsilon + 2\alpha + r_1 + r_2 + c\beta(I_1 + I_2) - \frac{c\Lambda}{S} - \frac{\beta S(I_1 + I_2)}{E} - \frac{p\epsilon E}{I_1} - \frac{q\epsilon E}{I_2} \\ &- \frac{r_1 I_1 + r_2 I_2}{R} - \mu(S + E + I_1 + I_2 + R) - \alpha(I_1 + I_2) \\ &+ \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2). \end{aligned}$$

Choosing $c = \frac{\mu}{\beta}$, we can get

$$LV \le \Lambda + 5\mu + \epsilon + 2\alpha + r_1 + r_2 + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2) \doteq F,$$

where F is a suitable positive constant, which is independent of $S(t)$, $E(t)$, $I_1(t)$, $I_2(t)$, $R(t)$ and t. Therefore, we have

$$\begin{aligned} dV \le& F dt + \sigma_1(S - 1)dB_1(t) + \sigma_2(E - 1)dB_2(t) + \sigma_3(I_1 - 1)dB_3(t) \\ &+ \sigma_4(I_2 - 1)dB_4(t) + \sigma_5(R - 1)dB_5(t). \end{aligned}$$

Integrating both sides from 0 to $\tau_k \wedge T$ and taking expectations, then we have

$$EV \leq V(S(0), E(0), I_1(0), I_2(0), R(0)) + FT.$$

For any positive $k \geq k_1$ we set $\Omega_k = \{\tau_k < T\}$, and then it leads to $P(\Omega_k) > \frac{\eta}{2}$. Note that for every $\upsilon \in \Omega_k$, there is at least one of $(S(\upsilon), E(\upsilon), I_1(\upsilon), I_2(\upsilon), R(\upsilon))$ equaling $\frac{1}{k}$ or k, then

$$V \geq (\frac{1}{k} - 1 - \ln\frac{1}{k}) \wedge (\frac{1}{k} - c - c\ln\frac{1}{ck}) \text{ or } V \geq (k - 1 - \ln k) \wedge (k - c - c\ln\frac{k}{c}).$$

Therefore, we obtain

$$\begin{aligned}
&V(S(0), E(0), I_1(0), I_2(0), R(0)) + FT \\
\geq &E[I_{\Omega_k} V(S(t), E(t), I_1(t), I_2(t), R(t))] \\
= &P(\Omega_k) V(S(t), E(t), I_1(t), I_2(t), R(t)) \\
> &\frac{\eta}{2}\left[\left(\frac{1}{k} - 1 - \ln\frac{1}{k}\right) \wedge (\frac{1}{k} - c - c\ln\frac{1}{ck})\right] \\
&\wedge \frac{\eta}{2}\left[(k - 1 - \ln k) \wedge (k - c - c\ln\frac{k}{c})\right],
\end{aligned}$$

where I_{Ω_k} is the indicator function of Ω_k.
Setting $k \to \infty$, we have

$$\infty > V(S(0), E(0), I_1(0), I_2(0), R(0)) + FT = \infty.$$

This completes the proof. □

3.2. Extinction of the stochastic model

First, we give a lemma that can be proved by using the same arguments as that in Lemma 3.1 of [41]. Thus, it is omitted here.

Lemma 3.2. *Let $(S(t), E(t), I_1(t), I_2(t), R(t))$ be any solution of system (1.2) with initial value. Assume $\mu > \frac{\sigma^2_{\max}}{2}$. Then,*

(i) $\lim\limits_{t\to\infty} \frac{S(t)}{t} = \lim\limits_{t\to\infty} \frac{E(t)}{t} = \lim\limits_{t\to\infty} \frac{I_1(t)}{t} = \lim\limits_{t\to\infty} \frac{I_2(t)}{t} = \lim\limits_{t\to\infty} \frac{R(t)}{t} = 0$ *a.s. Moreover,* $\lim\limits_{t\to\infty} \frac{\ln S(t)}{t} = \lim\limits_{t\to\infty} \frac{\ln E(t)}{t} = \lim\limits_{t\to\infty} \frac{\ln I_1(t)}{t} = \lim\limits_{t\to\infty} \frac{\ln I_2(t)}{t} = \lim\limits_{t\to\infty} \frac{\ln R(t)}{t} = 0$ *a.s.;*

(ii) $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t S(u)dB_1(u) = \lim\limits_{t\to\infty} \frac{1}{t}\int_0^t E(u)dB_2(u) = \lim\limits_{t\to\infty} \frac{1}{t}\int_0^t I_1(u)dB_3(u) = \lim\limits_{t\to\infty} \frac{1}{t}\int_0^t I_2(u)dB_4(u) = \lim\limits_{t\to\infty} \frac{1}{t}\int_0^t R(u)dB_5(u) = 0$ *a.s., where* $\sigma_{\max} = \sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2$.

Set

$$\mathcal{R}_0^s \doteq \frac{3\epsilon\beta\Lambda(\epsilon + \mu)}{\mu[(\epsilon + \mu)^2(\mu + \alpha + r_1 + \frac{1}{2}\sigma_3^2) \wedge (\epsilon + \mu)^2(\mu + \alpha + r_2 + \frac{1}{2}\sigma_4^2) \wedge \frac{1}{2}\epsilon^2\sigma_2^2]}.$$

Theorem 3.1. *Let $(S(t), E(t), I_1(t), I_2(t), R(t))$ be the solution of model (1.2) with any initial value $(S(0), E(0), I_1(0), I_2(0), R(0) \in \mathbb{R}^5_+$.*

If $\mathcal{R}_0^s < 1$, $\mu > \frac{\sigma_{\max}^2}{2}$, then the solution $(S(t), E(t), I_1(t), I_2(t), R(t))$ of model (1.2) satisfies

$$
\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t}\int_0^t S(u)du = \frac{\Lambda}{\mu},\\
&\lim_{t\to\infty} \frac{1}{t}\int_0^t E(u)du\\
=&\lim_{t\to\infty} \frac{1}{t}\int_0^t I_1(u)du = \lim_{t\to\infty} \frac{1}{t}\int_0^t I_2(u)du = \lim_{t\to\infty} \frac{1}{t}\int_0^t R(u)du = 0.
\end{aligned}
$$

Proof. Let $P(t) = \epsilon E(t) + (\epsilon+\mu)(I_1(t)+I_2(t))$. Applying Itô's formula, we have

$$
\begin{aligned}
d\ln P(t) =& \bigg(\frac{\epsilon\beta S(t)(I_1(t)+I_2(t)) - (\epsilon+\mu)(\mu+\alpha+r_1)I_1(t)}{\epsilon E(t) + (\epsilon+\mu)(I_1(t)+I_2(t)}\\
&\frac{-(\epsilon+\mu)(\mu+\alpha+r_2)I_2(t)}{\epsilon E(t) + (\epsilon+\mu)(I_1(t)+I_2(t)} + \frac{\epsilon^2\sigma_2^2E^2(t) + (\epsilon+\mu)^2\sigma_3^2I_1^2(t)}{2[\epsilon E(t) + (\epsilon+\mu)(I_1(t)+I_2(t))]^2}\\
&+\frac{(\epsilon+\mu)^2\sigma_4^2I_2^2(t)}{2[\epsilon E(t) + (\epsilon+\mu)(I_1(t)+I_2(t))]^2}\bigg)dt\\
&+\frac{\epsilon\sigma_2E(t)dB_2(t) + (\epsilon+\mu)\sigma_3I_1(t)dB_3(t) + (\epsilon+\mu)\sigma_4I_2(t)dB_4(t)}{\epsilon E(t) + (\epsilon+\mu)(I_1(t)+I_2(t))}\\
\le&\frac{\epsilon\beta S(t)}{\epsilon+\mu}dt - \frac{1}{[\epsilon E(t) + (\epsilon+\mu)(I_1(t)+I_2(t))]^2}\big[(\epsilon+\mu)^2(\mu+\alpha+r_1\\
&+\frac{1}{2}\sigma_3^2)I_1^2(t) + (\epsilon+\mu)^2(\mu+\alpha+r_2+\frac{1}{2}\sigma_4^2)I_2^2(t) + \frac{1}{2}\epsilon^2\sigma_2^2E^2(t)\bigg]dt\\
&+\frac{\epsilon\sigma_2E(t)dB_2(t) + (\epsilon+\mu)\sigma_3I_1(t)dB_3(t) + (\epsilon+\mu)\sigma_4I_2(t)dB_4(t)}{\epsilon E(t) + (\epsilon+\mu)(I_1(t)+I_2(t))}\\
\le&\frac{\epsilon\beta S(t)}{\epsilon+\mu}dt - \frac{1}{3(\epsilon+\mu)^2}\bigg[(\epsilon+\mu)^2(\mu+\alpha+r_1+\frac{1}{2}\sigma_3^2)\\
&\wedge(\epsilon+\mu)^2\left(\mu+\alpha+r_2+\frac{1}{2}\sigma_4^2\right)\wedge\frac{1}{2}\epsilon^2\sigma_2^2\bigg]\\
&+\frac{\epsilon\sigma_2E(t)dB_2(t) + (\epsilon+\mu)\sigma_3I_1(t)dB_3(t) + (\epsilon+\mu)\sigma_4I_2(t)dB_4(t)}{\epsilon E(t) + (\epsilon+\mu)(I_1(t)+I_2(t))}.
\end{aligned}
\tag{3.2}
$$

It follows from system (1.2) that

$$
\begin{aligned}
dN(t) =& [\Lambda - \mu N(t) - \alpha(I_1(t)+I_2(t))]dt + \sigma_1S(t)dB_1(t) + \sigma_2E(t)dB_2(t)\\
&+\sigma_3I_1(t)dB_3(t) + \sigma_4I_2(t)dB_4(t) + \sigma_5R(t)dB_5(t).
\end{aligned}
\tag{3.3}
$$

Integrating (3.3) from 0 to t, together with Lemma 3.2, we have

$$
\lim_{t\to\infty}\sup\left(\frac{1}{t}\int_0^t N(s)ds\right) \le \frac{\Lambda}{\mu} \quad a.s. \tag{3.4}
$$

Integrating (3.2) from 0 to t, together with (3.4), and noting $\mathcal{R}_0^s < 1$, we get

$$
\begin{aligned}
\lim_{t\to\infty}\sup\frac{\ln P(t)}{t} \le& \frac{\epsilon\beta\Lambda}{\mu(\epsilon+\mu)} - \frac{1}{3(\epsilon+\mu)^2}\bigg[(\epsilon+\mu)^2(\mu+\alpha+r_1+\frac{1}{2}\sigma_3^2)\\
&\wedge(\epsilon+\mu)^2(\mu+\alpha+r_2+\frac{1}{2}\sigma_4^2)\wedge\frac{1}{2}\epsilon^2\sigma_2^2\bigg] < 0 \qquad a.s.,
\end{aligned}
\tag{3.5}
$$

which implies $\lim_{t\to\infty} E(t) = \lim_{t\to\infty} I_1(t) = \lim_{t\to\infty} I_2(t) = 0$ a.s.

That is to say, the prepatent individual $E(t)$ and the infective individuals $I_1(t)$ and $I_2(t)$ will exponentially tend to zero with probability one.

By model (1.2), it is easy to see

$$\lim_{t\to\infty} R(t) = 0 \quad a.s. \tag{3.6}$$

It also means

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I_1(u)du = \lim_{t\to\infty} \frac{1}{t}\int_0^t I_2(u)du = \lim_{t\to\infty} \frac{1}{t}\int_0^t R(u)du = 0 \;\; a.s. \tag{3.7}$$

On the other hand, applying Itô's formula, we have

$$\begin{aligned} dN(t) =& [\Lambda - \mu N(t) - \alpha(I_1(t) + I_2(t)]dt + \sigma_1 S(t)dB_1(t) + \sigma_2 E(t)dB_2(t) \\ &+ \sigma_3 I_1(t)dB_2(t) + \sigma_4 I_2(t)dB_4(t) + \sigma_5 dB_5(t). \end{aligned} \tag{3.8}$$

Integrating (3.8) from 0 to t and then dividing by t on both sides, we can derive

$$\begin{aligned} \frac{N(t)-N(0)}{t} =& \Lambda - \frac{\mu}{t}\int_0^t N(u)du - \frac{\alpha}{t}\int_0^t I_1(u)du - \frac{\alpha}{t}\int_0^t I_2(u)du \\ &+ \frac{\sigma_1}{t}\int_0^t S(u)dB_1(u) + \frac{\sigma_2}{t}\int_0^t E(u)dB_2(u) + \frac{\sigma_3}{t}\int_0^t I_1(u)dB_3(u) \\ &+ \frac{\sigma_4}{t}\int_0^t I_2(u)dB_4(u) + \frac{\sigma_5}{t}\int_0^t R(u)dB_5(u). \end{aligned} \tag{3.9}$$

Taking the superior limit on the both sides of (3.9) and combining with Lemma 3.2 and (3.7), one can obtain

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(u)du = \frac{\Lambda}{\mu} \; a.s.$$

This completes the proof. □

3.3. Stationary distribution and ergodicity of the stochastic model

The following lemma in [15] is crucial to proving the existence of a stationary distribution.

Lemma 3.3. *The Markov process $X(t)$ has a unique ergodic stationary distribution $m(\cdot)$, if there exists a bounded domain $U \in \mathbb{R}^d$ with regular boundary such that its closure $\tilde{U} \subset \mathbb{R}^d$ has the following properties.*

(i) In the open domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(t)$ is bounded away from zero.

(ii) If $x \in \mathbb{R}^d \setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E^x\tau < \infty$ for every compact $K \subset \mathbb{R}^d$.

Moreover, if $f(\cdot)$ is a function integrable with respect to measure m, then

$$P\left(\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X^x(t))dt = \int_{\mathbb{R}^d} f(X^x(t))m(dx)\right) = 1,$$

for all $x \in \mathbb{R}^d$.

Remark 3.1. To verify condition (i), it is sufficient to prove that F is uniformly elliptical in U, where $F(u) = b(x)u_x + \frac{1}{2}trace(A(x)u_{xx})$. That is, there is a positive number M such that $\sum_{i,j=1}^{d} a_{ij}(x)\zeta_i\zeta_j \geq M|\zeta|^2,\ x \in U,\ \zeta \in \mathbb{R}_d$ [10,30]. To validate condition (ii), it is sufficient to show that there are a nonnegative C^2-function V and a neighborhood U such that for some $\mathcal{K} > 0$, LV¡$-\mathcal{K}$, $x \in \mathbb{R}^d \setminus U$ [42].

Set

$$\hat{\mathcal{R}_0^s} = \frac{3^6\Lambda^2\beta^2 pq\epsilon^2}{2^8\left(\mu + \frac{\sigma_1^2}{2}\right)^2\left(\epsilon + \mu + \frac{\sigma_2^2}{2}\right)^2\left(\mu + \alpha + r_1 + \frac{\sigma_3^2}{2}\right)\left(\mu + \alpha + r_2 + \frac{\sigma_4^2}{2}\right)}.$$

Theorem 3.2. *Assume that* $\hat{\mathcal{R}_0^s} > 1$. *Then, model (1.2) has a unique stationary distribution* $m(\cdot)$*, and it has the ergodic property.*

Proof. The diffusion matrix of system (1.2) is given by

$$A = \begin{pmatrix} \sigma_1^2 S^2(t) & 0 & 0 & 0 & 0 \\ 0 & \sigma_2^2 E^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3^2 I_1^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4^2 I_2^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5^2 R^2 \end{pmatrix}.$$

Choosing

$$m = \min_{X(t)\in\mathbb{R}_+^5}\{\sigma_1^2 S^2(t), \sigma_2^2 E^2(t), \sigma_3^2 I_1^2(t), \sigma_4^2 I_2^2(t), \sigma_5^2 R^2(t)\},$$

we can get

$$\begin{aligned} &\sum_{i,j=1}^{5} a_{ij}(S(t), E(t), I_1(t), I_2(t), R(t)\zeta_i\zeta_j \\ =&\sigma_1^2 S^2(t) + \sigma_2^2 E^2(t) + \sigma_3^2 I_1^2(t) + \sigma_4^2 I_2^2(t) \\ &+ \sigma_5^2 R^2(t) \geq m \mid \zeta \mid^2, \end{aligned} \tag{3.10}$$

where $(S(t), E(t), I_1(t), I_2(t), R(t)) \in D_\delta,\ \zeta \in \mathbb{R}_+^5$, and D_δ is defined as

$$D_\delta = \left\{(S, E, I_1, I_2, R) \in \mathbb{R}_+^5 : \delta \leq S, I_1, I_2 \leq \frac{1}{\delta}, \delta^2 \leq R \leq \frac{1}{\delta^2}, \delta^3 \leq E \leq \frac{1}{\delta^3}\right\}, \tag{3.11}$$

where $\delta > 0$ is a sufficiently small number.

Thus, condition (i) in Lemma 3.3 is satisfied.

By Lemma 3.1, we have obtained that for any initial value $(S(0), E(0), I_1(0), I_2(0), R(0)) \in \mathbb{R}_+^5$, and there is a unique global solution $X(t)$: $(S(t), E(t), I_1(t), I_2(t), R(t)) \in \mathbb{R}_+^5$. First, we define

$$V_1 = -C_1\ln S - C_2\ln E - C_3\ln I_1 - C_4\ln I_2.$$

Applying It$\hat{o}$'s formula, we have

$$
\begin{aligned}
LV_1 =& - C_1\left(\frac{\Lambda}{S} - \mu - \beta(I_1+I_2) - \frac{\sigma_1^2}{2}\right) - C_2\left(\frac{\beta S(I_1+I_2)}{E} - (\epsilon+\mu) - \frac{\sigma_2^2}{2}\right) \\
& - C_3\left(\frac{p\epsilon E}{I_1} - (\mu+\alpha+r_1) - \frac{\sigma_3^2}{2}\right) - C_4\left(\frac{q\epsilon E}{I_2} - (\mu+\alpha+r_2) - \frac{\sigma_4^2}{2}\right) \\
=& - \frac{\Lambda C_1}{S} - \frac{\beta C_2 S(I_1+I_2)}{E} - \frac{C_3 p\epsilon E}{I_1} - \frac{C_4 q\epsilon E}{I_2} + \beta C_1(I_1+I_2) \\
& + \left(\mu+\frac{\sigma_1^2}{2}\right)C_1 + \left(\epsilon+\mu+\frac{\sigma_2^2}{2}\right)C_2 + \left(\mu+\alpha+r_1+\frac{\sigma_3^2}{2}\right)C_3 \\
& + \left(\mu+\alpha+r_2+\frac{\sigma_4^2}{2}\right)C_4 \\
=& - \left(\frac{\Lambda C_1}{2S} + \frac{\beta C_2 S I_1}{E} + \frac{C_3 p\epsilon E}{I_1}\right) - \left(\frac{\Lambda C_1}{2S} + \frac{\beta C_2 S I_2}{E} + \frac{C_4 q\epsilon E}{I_2}\right) \\
& + \beta C_1(I_1+I_2) + \left(\mu+\frac{\sigma_1^2}{2}\right)C_1 + \left(\epsilon+\mu+\frac{\sigma_2^2}{2}\right)C_2 \\
& + \left(\mu+\alpha+r_1+\frac{\sigma_3^2}{2}\right)C_3 + \left(\mu+\alpha+r_2+\frac{\sigma_4^2}{2}\right)C_4 \\
\leq& - 6\left(\frac{\Lambda^2 C_1^2 \beta^2 C_2^2 C_3 C_4 pq\epsilon^2}{4}\right)^{\frac{1}{6}} + \beta C_1(I_1+I_2) + \left(\mu+\frac{\sigma_1^2}{2}\right)C_1 \\
& + \left(\epsilon+\mu+\frac{\sigma_2^2}{2}\right)C_2 + \left(\mu+\alpha+r_1+\frac{\sigma_3^2}{2}\right)C_3 \\
& + \left(\mu+\alpha+r_2+\frac{\sigma_4^2}{2}\right)C_4.
\end{aligned}
$$

Let

$$
\begin{gathered}
\left(\mu+\frac{\sigma_1^2}{2}\right)C_1 = \left(\epsilon+\mu+\frac{\sigma_2^2}{2}\right)C_2 = \Lambda, \\
\left(\mu+\alpha+r_1+\frac{\sigma_3^2}{2}\right)C_3 = \left(\mu+\alpha+r_2+\frac{\sigma_4^2}{2}\right)C_4 = \Lambda,
\end{gathered}
$$

then

$$
\begin{aligned}
C_1 &= \frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}, \quad C_2 = \frac{\Lambda}{\epsilon+\mu+\frac{\sigma_2^2}{2}}, \\
C_3 &= \frac{\Lambda}{(\mu+\alpha+r_1+\frac{\sigma_3^2}{2})}, \quad C_4 = \frac{\Lambda}{(\mu+\alpha+r_2+\frac{\sigma_4^2}{2})}.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
&LV_1 \\
&\leq -4\Lambda\left(\frac{3^6\Lambda^2\beta^2pq\epsilon^2}{2^8\left(\mu+\frac{\sigma_1^2}{2}\right)^2\left(\epsilon+\mu+\frac{\sigma_2^2}{2}\right)^2\left(\mu+\alpha+r_1+\frac{\sigma_3^2}{2}\right)\left(\mu+\alpha+r_2+\frac{\sigma_4^2}{2}\right)}\right)^{\frac{1}{6}} \\
&\quad +4\Lambda+\frac{\beta b(I_1+I_2)}{\mu+\frac{\sigma_1^2}{2}} \\
&=-4\Lambda[(\hat{\mathcal{R}_0^s})-1]+\frac{\beta\Lambda(I_1+I_2)}{\mu+\frac{\sigma_1^2}{2}} \\
&\doteq -\lambda+\frac{\beta\Lambda(I_1+I_2)}{\mu+\frac{\sigma_1^2}{2}}.
\end{aligned}
$$

Then, we define

$$V_2=\frac{1}{\theta+1}(S+E+I_1+I_2+R)^{\theta+1}.$$

By simple calculation, we have

$$
\begin{aligned}
LV_2 &= N^\theta[\Lambda-\mu N-\alpha(I_1+I_2)]+\frac{\theta}{2}N^{\theta-1}(\sigma_1^2S^2+\sigma_2^2E^2+\sigma_3^2I_1^2+\sigma_4^2I_2^2+\sigma_5^2R^2) \\
&\leq N^\theta(\Lambda-\mu N)+\frac{\theta}{2}N^{\theta+1}\sigma_{\max}^2 \\
&=\Lambda N^\theta-N^{\theta+1}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right) \\
&\leq P-\frac{1}{2}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)N^{\theta+1} \\
&\leq P-\frac{1}{2}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1}+E^{\theta+1}+I_1^{\theta+1}+I_2^{\theta+1}+R^{\theta+1}\right),
\end{aligned}
$$

where

$$P=\sup_{X(t)\in\mathbb{R}_+^5}\left\{\Lambda N^\theta-\frac{1}{2}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)N^{\theta+1}\right\}.$$

We define V_3 as follows

$$V_3=-\ln S-\ln E-\ln I_1-\ln R.$$

Then, we have

$$
\begin{aligned}
LV_3 =& -\frac{\Lambda}{S}+\beta(I_1+I_2)-\frac{\beta S(I_1+I_2)}{E}-\frac{p\epsilon E}{I_1}-\frac{r_1I_1+r_2I_2}{R}+4\mu+\epsilon+\alpha \\
&+r_1+\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2).
\end{aligned}
$$

After that, we construct a C^2-function $Q:\ \mathbb{R}_+^5\to R$ in the following form

$$Q(S(t),E(t),I_1(t),I_2(t),R(t))=MV_1+V_2+V_3,$$

where $M > 0$ satisfies $-M\lambda + W \leq -2$, and

$$\begin{aligned} & W \\ = & \sup_{X(t)\in\mathbb{R}_+^5} \{-\frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1} + E^{\theta+1} + I_1^{\theta+1} + I_2^{\theta+1} + R^{\theta+1}\right) + P + 4\mu + \epsilon \\ & + \alpha + r_1 + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2\right)\}. \end{aligned}$$

It is easy to check

$$\lim_{k\to\infty,(S(t),E(t),I_1(t),I_2(t),R(t)\in\mathbb{R}_+^5\setminus U_k} Q(S(t), E(t), I_1(t), I_2(t), R(t)) = +\infty,$$

where $U_k = \left(\frac{1}{k}, k\right) \times \left(\frac{1}{k}, k\right) \times \left(\frac{1}{k}, k\right) \times \left(\frac{1}{k}, k\right) \times \left(\frac{1}{k}, k\right)$.

In addition, $Q(S(t), E(t), I_1(t), I_2(t), R(t))$ is a continuous function. Thus, $Q(S(t), E(t), I_1(t), I_2(t), R(t))$ has a minimum point Q_{min} in the interior of $\mathbb{R}_+^5$.

Then, we define a nonnegative C^2-function $\mathcal{V} : \mathbb{R}_+^5 \to \mathbb{R}$ as

$$\mathcal{V}(S(t), E(t), I_1(t), I_2(t), R(t)) = Q(S(t), E(t), I_1(t), I_2(t), R(t)) - Q_{min}.$$

The differential operator L acting on the function $\mathcal{V}(S(t), E(t), I_1(t), I_2(t), R(t))$ yields

$$\begin{aligned} & L\mathcal{V} \\ \leq & - M\lambda + \frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + (I_1 + I_2) + P - \frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1} + E^{\theta+1} + I_1^{\theta+1} + I_2^{\theta+1}\right. \\ & \left. + R^{\theta+1}\right) - \frac{\Lambda}{S} + \beta(I_1 + I_2) - \frac{\beta S(I_1 + I_2)}{E} - \frac{r_1 I_1 + r_2 I_2}{R} - \frac{p\epsilon E}{I_1} + 4\mu + \epsilon + \alpha + r_1 \\ & + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2\right) \\ \leq & - M\lambda + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right)(I_1 + I_2) - \frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1} + E^{\theta+1} + I_1^{\theta+1} + I_2^{\theta+1}\right. \\ & \left. + R^{\theta+1}\right) - \frac{\Lambda}{S} - \frac{\beta S I_1}{E} - \frac{r_1 I_1}{R} + P + 4\mu + \epsilon + \alpha + r_1 + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2). \end{aligned}$$

In the set $\mathbb{R}_+^5 \setminus D_\delta$, D_δ is defined here in (3.11), we can choose the sufficiently small δ such that

$$-\frac{\Lambda}{\delta} + W_1 \leq -1, \tag{3.12}$$

$$-M\lambda + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right)\delta + W_2 \leq -1, \tag{3.13}$$

$$-M\lambda + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right)\delta + W_3 \leq -1, \tag{3.14}$$

$$-\frac{r_1}{\delta} + W_1 \leq -1, \tag{3.15}$$

$$-\frac{\beta}{\delta} + W_1 \leq -1, \tag{3.16}$$

$$-\frac{1}{4}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\frac{1}{\delta^{\theta+1}} + W_4 \leq 1, \tag{3.17}$$

$$-\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)\frac{1}{\delta^{\theta+1}}+W_5\leq 1, \tag{3.18}$$

$$-\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)\frac{1}{\delta^{\theta+1}}+W_6\leq 1, \tag{3.19}$$

$$-\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)\frac{1}{\delta^{2\theta+2}}+W_7\leq 1, \tag{3.20}$$

$$-\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)\frac{1}{\delta^{3\theta+3}}+W_8\leq 1, \tag{3.21}$$

where $W_i(i=1,...,8)$ are positive constants which can be found from the following inequations.

For the sake of convenience, we divide $\mathbb{R}_+^5\setminus D_\delta$ into the following domains

$$\begin{aligned}
&D_1=\{X(t)\in\mathbb{R}_+^5, 0<S<\delta\}; && D_2=\{X(t)\in\mathbb{R}_+^5, 0<I_1<\delta\};\\
&D_3=\{X(t)\in\mathbb{R}_+^5, 0<I_2<\delta\};\\
&D_4=\{X(t)\in\mathbb{R}_+^5, I_1\geq\delta, 0<R<\delta^2\};\\
&D_5=\{X(t)\in\mathbb{R}_+^5, S\geq\delta, I_1\geq\delta, 0<E<\delta^2\}; && D_6=\{X(t)\in\mathbb{R}_+^5, S>\frac{1}{\delta}\};\\
&D_7=\{X(t)\in\mathbb{R}_+^5, I_1>\frac{1}{\delta}\}; && D_8=\{X(t)\in\mathbb{R}_+^5, I_2>\frac{1}{\delta}\};\\
&D_9=\{X(t)\in\mathbb{R}_+^5, R>\frac{1}{\delta^2}\}; && D_{10}=\{X(t)\in\mathbb{R}_+^5, E>\frac{1}{\delta^3}\}.
\end{aligned}$$

Next, we will show $LV\leq -1$ on $\mathbb{R}_+^5\setminus D_\delta$, which is equivalent to proving it on the above 10 domains.

Case 1. If $X(t)\in D_1$, one can see

$$\begin{aligned}
&L\mathcal{V}\\
\leq&-M\lambda+\left(\frac{M\beta\Lambda}{\mu+\frac{1}{2}\sigma_1^2}+\beta\right)(I_1+I_2)-\frac{1}{2}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1}+E^{\theta+1}+I_1^{\theta+1}\right.\\
&\left.+I_2^{\theta+1}+R^{\theta+1}\right)-\frac{\Lambda}{S}-\frac{\beta SI_1}{E}-\frac{r_1I_1}{R}+P+4\mu+\epsilon+\alpha+r_1\\
&+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2\right)\\
\leq&-\frac{\Lambda}{S}+W_1\leq-\frac{\Lambda}{\delta}+W_1\leq -1,
\end{aligned}$$

where

$$\begin{aligned}
W_1=\sup_{X(t)\in\mathbb{R}_+^5}&\{-\frac{1}{2}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1}+E^{\theta+1}+I_1^{\theta+1}+I_2^{\theta+1}+R^{\theta+1}\right)\\
&+(\frac{M\beta\Lambda}{\mu+\frac{1}{2}\sigma_1^2}+\beta)(I_1+I_2)+P+4\mu+\epsilon+\alpha+r_1+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2\right)\}.
\end{aligned}$$

According to (3.12), we have $L\mathcal{V}\leq -1$, for all $X(t)\in D_1$.

Case 2. If $X(t) \in D_2$, we get

$$
\begin{aligned}
&L\mathcal{V} \\
\leq & -M\lambda + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right)(I_1 + I_2) - \frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1} + E^{\theta+1} + I_1^{\theta+1}\right. \\
&\left. + I_2^{\theta+1} + R^{\theta+1}\right) - \frac{\Lambda}{S} - \frac{\beta S I_1}{E} - \frac{r_1 I_1}{R} + P + 4\mu + \epsilon + \alpha + r_1 \\
&+ \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2\right) \\
\leq & -M\lambda + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right) I_1 + W_2 \leq -M\lambda + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right)\delta + W_2 \leq -1,
\end{aligned}
$$

where

$$
\begin{aligned}
W_2 = \sup_{X(t)\in\mathbb{R}_+^5} & \{-\frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1} + E^{\theta+1} + I_1^{\theta+1} + I_2^{\theta+1} + R^{\theta+1}\right) \\
& + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right) I_2 + P + 4\mu + \epsilon + \alpha + r_1 + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2\right)\}.
\end{aligned}
$$

According to (3.13), we have $L\mathcal{V} \leq -1$, for all $X(t) \in D_2$.

Case 3. If $X(t) \in D_3$, we derive

$$
\begin{aligned}
&L\mathcal{V} \\
\leq & -M\lambda + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right)(I_1 + I_2) - \frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1} + E^{\theta+1} + I_1^{\theta+1}\right. \\
&\left. + I_2^{\theta+1} + R^{\theta+1}\right) - \frac{\Lambda}{S} - \frac{\beta S I_1}{E} - \frac{r_1 I_1}{R} + P + 4\mu + \epsilon + \alpha + r_1 \\
&+ \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2\right) \\
\leq & -M\lambda + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right) I_2 + W_3 \leq -M\lambda + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right)\delta + W_3 \leq -1,
\end{aligned}
$$

where

$$
\begin{aligned}
W_3 = \sup_{X(t)\in\mathbb{R}_+^5} & \{-\frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1} + E^{\theta+1} + I_1^{\theta+1} + I_2^{\theta+1} + R^{\theta+1}\right) \\
& + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right) I_1 + P + 4\mu + \epsilon + \alpha + r_1 + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2\right)\}.
\end{aligned}
$$

According to (3.14), we have $L\mathcal{V} \leq -1$, for all $X(t) \in D_3$.

Case 4. If $X(t) \in D_4$, one can obtain

$$
\begin{aligned}
&L\mathcal{V} \\
\leq & -M\lambda + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right)(I_1 + I_2) - \frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1} + E^{\theta+1} + I_1^{\theta+1}\right. \\
&\left. + I_2^{\theta+1} + R^{\theta+1}\right) - \frac{\Lambda}{S} - \frac{\beta S I_1}{E} - \frac{r_1 I_1}{R} + P + 4\mu + \epsilon + \alpha + r_1 \\
&+ \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2\right) \\
\leq & -\frac{r_1 I_1}{R} + W_1 \leq -\frac{r_1}{\delta} + W_1 \leq -1.
\end{aligned}
$$

According to (3.15), we have $L\mathcal{V} \le -1$, for all $X(t) \in D_4$.

Case 5. If $X(t) \in D_5$, we can obtain

$$\begin{aligned}
&L\mathcal{V}\\
\le& -M\lambda + \left(\frac{M\beta\Lambda}{\mu+\frac{1}{2}\sigma_1^2}+\beta\right)(I_1+I_2) - \frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1}+E^{\theta+1}+I_1^{\theta+1}\right.\\
&\left.+I_2^{\theta+1}+R^{\theta+1}\right) - \frac{\Lambda}{S} - \frac{\beta SI_1}{E} - \frac{r_1 I_1}{R} + P + 4\mu + \epsilon + \alpha + r_1\\
&+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2\right)\\
\le& -\frac{\beta SI_1}{E} + W_1 \le -\frac{\beta}{\delta} + W_1 \le -1.
\end{aligned}$$

According to (3.16), we have $L\mathcal{V} \le -1$, for all $X(t) \in D_5$.

Case 6. If $X(t) \in D_6$, one can see

$$\begin{aligned}
&L\mathcal{V}\\
\le& -\frac{1}{4}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)S^{\theta+1} - \frac{1}{4}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)S^{\theta+1} - \frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(E^{\theta+1}\right.\\
&\left.+I_1^{\theta+1}+I_2^{\theta+1}+R^{\theta+1}\right) + \left(\frac{M\beta\Lambda}{\mu+\frac{1}{2}\sigma_1^2}+\beta\right)(I_1+I_2) + P + 4\mu + \epsilon + \alpha + r_1\\
&+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2\right)\\
\le& -\frac{1}{4}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)S^{\theta+1} + W_4 \le -\frac{1}{4}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\frac{1}{\delta^{\theta+1}} + W_4 \le -1,
\end{aligned}$$

where

$$\begin{aligned}
W_4 = \sup_{X(t)\in\mathbb{R}_+^5}&\{-\frac{1}{4}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)S^{\theta+1} - \frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(E^{\theta+1}+I_1^{\theta+1}+I_2^{\theta+1}\right.\\
&\left.+R^{\theta+1}\right) + \left(\frac{M\beta\Lambda}{\mu+\frac{1}{2}\sigma_1^2}+\beta\right)(I_1+I_2) + P + 4\mu + \epsilon + \alpha + r_1\\
&+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2\right)\}.
\end{aligned}$$

In view of (3.17), we can conclude $L\mathcal{V} \le -1$ on D_6.

Case 7. If $X(t) \in D_7$, we can see

$$\begin{aligned}
&L\mathcal{V}\\
\le& -\frac{1}{4}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)I_1^{\theta+1} - \frac{1}{4}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)I_1^{\theta+1} - \frac{1}{2}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1}\right.\\
&\left.+E^{\theta+1}+I_2^{\theta+1}+R^{\theta+1}\right) + \left(\frac{M\beta\Lambda}{\mu+\frac{1}{2}\sigma_1^2}+\beta\right)(I_1+I_2) + P + 4\mu + \epsilon + \alpha + r_1\\
&+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2\right)\\
\le& -\frac{1}{4}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)I_1^{\theta+1} + W_5 \le -\frac{1}{4}\left(\mu - \frac{\theta\sigma_{\max}^2}{2}\right)\frac{1}{\delta^{\theta+1}} + W_5 \le -1,
\end{aligned}$$

where

$$
\begin{aligned}
& W_5 \\
= & \sup_{X(t)\in\mathbb{R}_+^5} \{-\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right) I_1^{\theta+1}-\frac{1}{2}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1}+E^{\theta+1}+I_2^{\theta+1}\right. \\
& \left.+R^{\theta+1}\right)+\frac{M\beta\Lambda}{\mu+\frac{1}{2}\sigma_1^2}(I_1+I_2)+P+4\mu+\epsilon+\alpha+r_1+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2\right)\}.
\end{aligned}
$$

In view of (3.18), we can conclude $L\mathcal{V}\leq -1$ on D_7.

Case 8. If $X(t)\in D_8$, we can see

$$
\begin{aligned}
& L\mathcal{V} \\
\leq & -\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right) I_2^{\theta+1}-\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right) I_2^{\theta+1}-\frac{1}{2}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)(S^{\theta+1} \\
& +E^{\theta+1}+I_1^{\theta+1}+R^{\theta+1})+\left(\frac{M\beta\Lambda}{\mu+\frac{1}{2}\sigma_1^2}+\beta\right)(I_1+I_2)+P+4\mu+\epsilon+\alpha \\
& +r_1+\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2\right) \\
\leq & -\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right) I_2^{\theta+1}+W_6\leq -\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)\frac{1}{\delta^{\theta+1}}+W_6\leq -1,
\end{aligned}
$$

where

$$
\begin{aligned}
& W_6 \\
= & \sup_{X(t)\in\mathbb{R}_+^5} \{-\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right) I_2^{\theta+1}-\frac{1}{2}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)\left(S^{\theta+1}+E^{\theta+1}+I_1^{\theta+1}\right. \\
& \left.+R^{\theta+1}\right)+\left(\frac{M\beta\Lambda}{\mu+\frac{1}{2}\sigma_1^2}+\beta\right)(I_1+I_2)+P+4\mu+\epsilon+\alpha+r_1 \\
& +\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2\right)\}.
\end{aligned}
$$

In view of (3.19), we can conclude $L\mathcal{V}\leq -1$ on D_8.

Case 9. If $X(t)\in D_9$, one can obtain

$$
\begin{aligned}
& L\mathcal{V} \\
\leq & -\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right) R^{\theta+1}-\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right) R^{\theta+1}-\frac{1}{2}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)(S^{\theta+1} \\
& +E^{\theta+1}+I_1^{\theta+1}+I_2^{\theta+1})+\left(\frac{M\beta\Lambda}{\mu+\frac{1}{2}\sigma_1^2}+\beta\right)(I_1+I_2)+P+4\mu+\epsilon+\alpha+r_1 \\
& +\frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2\right) \\
\leq & -\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right) R^{\theta+1}+W_7\leq -\frac{1}{4}\left(\mu-\frac{\theta\sigma_{\max}^2}{2}\right)\frac{1}{\delta^{2\theta+2}}+W_7\leq -1,
\end{aligned}
$$

where

$$W_7 = \sup_{X(t)\in\mathbb{R}^5_+} \{-\frac{1}{4}\left(\mu - \frac{\theta\sigma^2_{\max}}{2}\right) R^{\theta+1} - \frac{1}{2}\left(\mu - \frac{\theta\sigma^2_{\max}}{2}\right)\left(S^{\theta+1} + E^{\theta+1} + I_1^{\theta+1}\right.$$
$$\left. + I_2^{\theta+1}\right) + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right)(I_1 + I_2) + P + 4\mu + \epsilon + \alpha + r_1$$
$$+ \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2\right)\}.$$

In view of (3.20), we can conclude $L\mathcal{V} \le -1$ on D_9.

Case 10. If $X(t) \in D_{10}$, we can see

$$\begin{aligned}
&L\mathcal{V}\\
\le& -\frac{1}{4}\left(\mu - \frac{\theta\sigma^2_{\max}}{2}\right) E^{\theta+1} - \frac{1}{4}\left(\mu - \frac{\theta\sigma^2_{\max}}{2}\right) E^{\theta+1} - \frac{1}{2}\left(\mu - \frac{\theta\sigma^2_{\max}}{2}\right)(S^{\theta+1}\\
&+ I_1^{\theta+1} + I_2^{\theta} + R^{\theta+1}) + \left(\frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2} + \beta\right)(I_1 + I_2) + P + 4\mu + \epsilon + \alpha + r_1\\
&+ \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2\right)\\
\le& -\frac{1}{4}\left(\mu - \frac{\theta\sigma^2_{\max}}{2}\right) E^{\theta+1} + W_8 \le -\frac{1}{4}\left(\mu - \frac{\theta\sigma^2_{\max}}{2}\right)\frac{1}{\delta^{3\theta+3}} + W_8 \le -1,
\end{aligned}$$

where

$$\begin{aligned}
&W_8\\
=& \sup_{X(t)\in\mathbb{R}^5_+} \{-\frac{1}{4}\left(\mu - \frac{\theta\sigma^2_{\max}}{2}\right) E^{\theta+1} - \frac{1}{2}\left(\mu - \frac{\theta\sigma^2_{\max}}{2}\right)\left(S^{\theta+1} + I_1^{\theta+1} + I_2^{\theta} + R^{\theta+1}\right)\\
&+ \frac{M\beta\Lambda}{\mu + \frac{1}{2}\sigma_1^2}(I_1 + I_2) + P + 4\mu + \epsilon + \alpha + r_1 + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2\right)\}.
\end{aligned}$$

In view of (3.21), we can conclude $L\mathcal{V} \le -1$ on D_{10}.

Clearly, from above, we can draw that $L\mathcal{V} \le -1$, for all $X(t) \in \mathbb{R}^5_+ \setminus D_\delta$.

Therefore, (ii) in Lemma 3.3 is satisfied.

This completes the proof. □

4. A case study

In this section, we study the COVID-19 transmission case in Wuhan, Hubei, China. In the following, we use numerical simulation to verify the correctness of the theoretical analysis, and give some suggestions on pandemic control according to the real data.

4.1. Model validation

Since the outbreak of COVID-19 in Wuhan, China, in late December, 2019, the Chinese government had quickly adopted a series of effective measures such as keeping social distancing, contact tracing and testing, self-quarantine or isolation, closing schools, and so on. Beside, up to now, the pandemic in China has been fundamentally controlled. Based on the official data and the study of some scholars, with the assistance of simple data analysis, we get some important parameters in the following table.

Table 2. Relevant variables and parameters values

Parameters	Values	References
β	$2.65\times10^{-}8$	[33]
Λ	100000	Estimate
μ	7.14×10^{-3}	[6]
ϵ	0.2	[33]
p	0.6834	[33]
q	0.3166	[33]
α	0.0009	[33]
r_1	0.1029	[33]
r_2	0.2978	[33]
σ_1	0.006	Estimate
σ_2	0.1	Estimate
σ_3	0.08	Estimate
σ_4	0.08	Estimate
σ_5	0.008	Estimate
$S(0)$	11081000	[33]
$E(0)$	600	[33]
$I_1(0)$	410	[33]
$I_2(0)$	30	[33]
$R(0)$	2	[33]

According to the data in Table 2 and our system (1.2), we simulate the spread of the COVID-19 in the early days in China. By numerically computing the threshold $\hat{\mathcal{R}}_0^s$ under the same set of parameter values as Table 2, we can obtain $\hat{\mathcal{R}}_0^s = 2.1279 > 1$, and the long-term behaviors of the individuals at different stages are shown in Figure 2. In this case, the pandemic will persist.

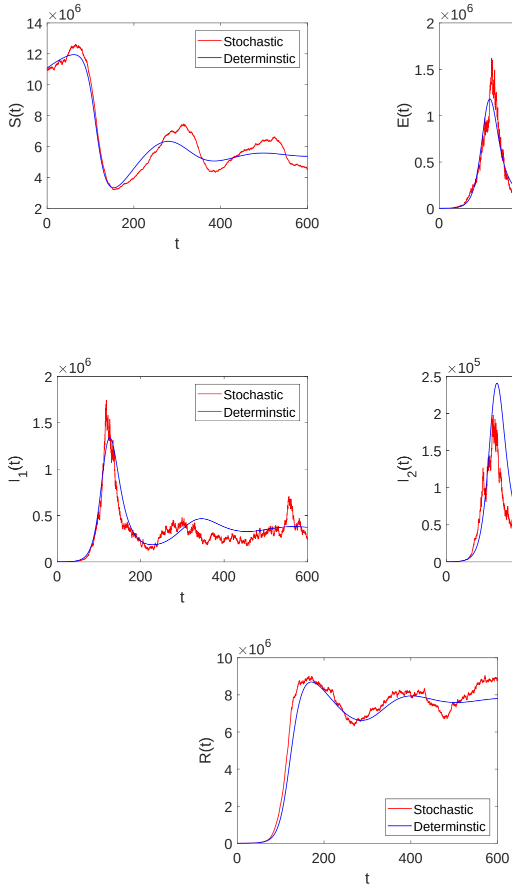
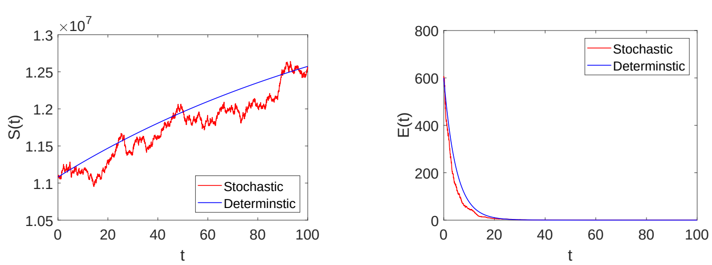


Figure 2. The spread of the COVID-19 in Wuhan

Since the outbreak of the pandemic, the Chinese government has tried to control the spread of the disease by sealing off cities and the traffic. The aim is to reduce population mobility and cut down the number of exposed individuals. Therefore, the contact transmission coefficient at the later stage of the pandemic is smaller. We set $\beta = 9.13 \times 10^{-11}$, and keep the other parameters unchanged in Table 2. Then, we can obtain the threshold $\mathcal{R}_0^s = 0.7950 < 1$. In this case, the long-term behaviors of the individuals at different stages are shown in Figure 3, which implies that the pandemic will die out.



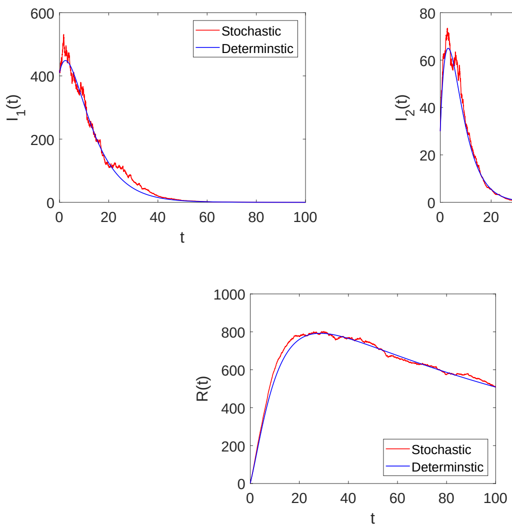
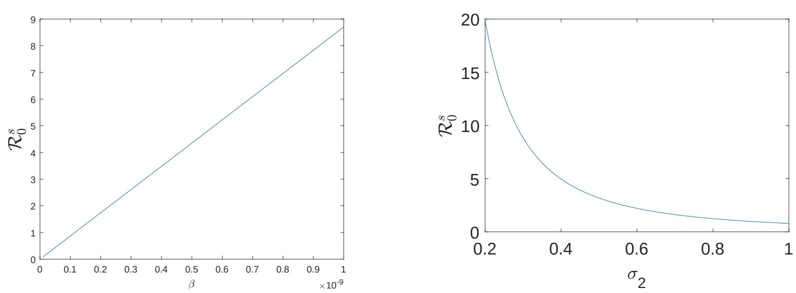


Figure 3. The spread of the COVID-19 in Wuhan

4.2. Sensitivity analysis of $\mathcal{R}_0^s$ and $\hat{\mathcal{R}}_0^s$

In order to control the pandemic, it is important to explore the effect of different factors for COVID-19 transmission. Thus, we study the relationship between some parameters, $\mathcal{R}_0^s$, $\hat{\mathcal{R}}_0^s$ and the possible measures to control the spread of pandemic. In what follows, except the parameter interval we set, all the other parameter values are the same as those in Table 2.

The influence of the contact transmission rate β and the intensity of the white noises σ_2 on $\mathcal{R}_0^s$ are shown (see Figure 4(a)-(b)).



(a) Relationship between $\mathcal{R}_0^s$ and β (b) Relationship between $\mathcal{R}_0^s$ and σ_2

Figure 4. The relationship between $\mathcal{R}_0^s$ and the related parameters

The influence of the contact transmission rate β, the intensity of the white noise $\sigma_i,\ i = 1, 2, \ldots, 4$, and the recovery rate of the infected individuals $r_1,\ r_2$ on $\hat{\mathcal{R}}_0^s$ are shown (see Figure 5(a)-(e)).

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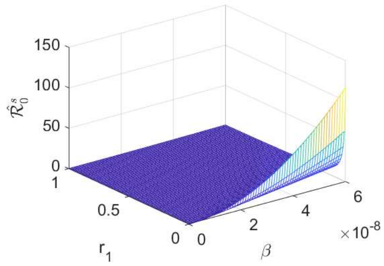
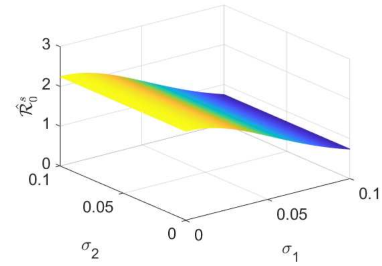
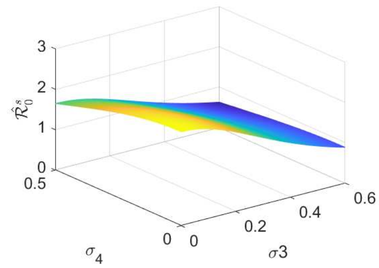


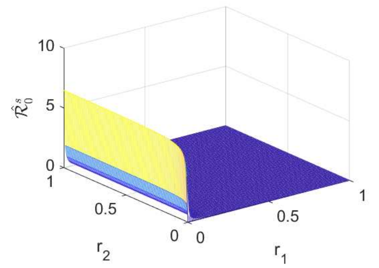
(a) Relationship between $\hat{\mathcal{R}}_0^s$ and β, r_1



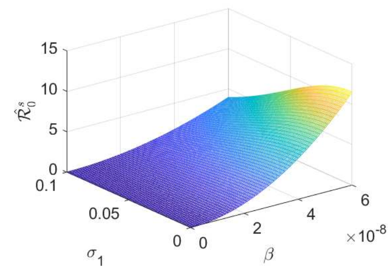
(b) Relationship between $\hat{\mathcal{R}}_0^s$ and σ_1, σ_2



(c) Relationship between $\hat{\mathcal{R}}_0^s$ and σ_3, σ_4



(d) Relationship between $\hat{\mathcal{R}}_0^s$ and r_1, r_2



(e) Relationship between $\hat{\mathcal{R}}_0^s$ and β, σ_1

Figure 5. The relationship between $\hat{\mathcal{R}}_0^s$ and the related parameters

The sensitivity analysis of $\mathcal{R}_0^s$ and $\hat{\mathcal{R}}_0^s$ above shows that reducing contact rate (see Figure 4(a), Figure 5(a) and Figure 5(e)), improving recovery rate (see Figure 5(d)) and increasing the intensity of the white noise (see Figure 5(b)-(c)) can inhibit the spread of disease. However, compared with the effect of reducing the contact rate, the effect of increasing the intensity of the white noise is not relatively noticeable (see Figure 5(e) and Figure 4(a)-(b)). Moreover, compared with the recovery rate, it is obvious that $\hat{\mathcal{R}}_0^s$ is more sensitive to variation in the value of β (see Figure 5(a)). In conclusion, reducing contact rate is the most effective measure in controlling the spread of the disease. This is also in line with the current response to the pandemic.

5. Concluding remarks

In this paper, we have proposed an SEIR epidemic model by considering asymptomatic infective individuals. First, we construct the deterministic model (1.1), and obtain the basic reproduction number $\mathcal{R}_0$. We show that the disease-free equilibrium is globally asymptotically stable, if $\mathcal{R}_0 < 1$, and the endemic equilibrium is globally asymptotically stable, if $\mathcal{R}_0 > 1$. Due to the stochastic perturbations in the environment, we further construct the stochastic model (1.2). By constructing suitable Lyapunov functions, we establish sufficient criteria for the existence of ergodic stationary distribution as well as the extinction of the pandemic. These results show whether the pandemic will be cleared or persist.

We obtain some feasible coefficients from some of the published works on COVID-19 transmission in Wuhan, Hubei, China as a case. Moreover, we measure the risk of the pandemic by $\mathcal{R}_0^s$ and $\hat{\mathcal{R}}_0^s$. Our numerical results show that the contact transmission rate β strongly affects the value of $\mathcal{R}_0^s$ and $\hat{\mathcal{R}}_0^s$ compared with the influence of recovery rates r_i $(i = 1, 2)$ and the intensities of the white noises σ_i $(i = 1, 2, \cdots, 5)$. This indicates that prevention is the most effective measure in controlling the spread of the pandemic.

As we know, quarantine and vaccination are important ways of controlling the spread of diseases. What effect these factors will make is worthy of further study. At the same time, it should be noted that we assume that the environmental influence on the individuals is proportional to each state in our paper. In fact, there are some different continuous and discontinuous stochastic perturbations like Lévy jumps noises, which can be considered in the model. We leave these investigations for our further work.

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