Journal of Nonlinear Modeling and Analysis
Volume 5, Number 1, March 2023, 54–72
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2023.54

An Accelerated Algorithm Involving Quasi-ϕ-Nonexpansive Operators for Solving Split Problems

Bashir Ali[1], Aisha Aminu Adam[2] and Abubakar Adamu[2,3,†]

Abstract In this paper, an algorithm of inertial type for approximating solutions of split equality fixed point problems involving quasi-ϕ-nonexpansive maps is proposed and studied in the setting of certain real Banach spaces. Weak and strong convergence theorems are proved under some conditions. Some applications of the theorems are presented. The results presented extend and improve some existing results. Finally, some numerical illustrations are presented to support our theorems and their applications.

Keywords Fixed point, Quasi-ϕ-nonexpansive, Inertia.

MSC(2010) 47H05, 47H09, 47H10, 47H20.

1. Introduction

Let E_1, E_2 and E_3 be real Hilbert spaces, and let D and Q be nonempty closed and convex subsets of E_1 and E_2 respectively. Let $S : E_1 \to E_3$, $T : E_2 \to E_3$ be bounded linear mappings, and let $B : E_1 \to E_1$ and $A : E_2 \to E_2$ be nonlinear mappings such that $F(B)$ and $F(A)$ are nonempty respectively. The split equality fixed point problem (SEFPP) is to find

$$u \in F(B) \quad \text{and} \quad v \in F(A) \quad \text{such that} \quad Su = Tv. \tag{1.1}$$

The problem was first introduced by Moudafi [29], and since then, it has been studied by many researchers (see, e.g. [14, 33, 36, 37] and the references therein). It allows asymmetric relations between the two variables u and v, and also covers many problems such as decomposition methods for partial differential equations (PDEs), and has applications in game theory and in intensity modulated radiation therapy (see, e.g. [9]).

Remark 1.1. If $E_2 = E_3$ and $T = I$, the SEFPP (1.1) reduces to the split common fixed point problem, which was first studied by Censor and Segal [10]. The problem is to find $u \in E_1$ with

$$u \in F(B) \quad \text{and} \quad Su \in F(A).$$

†The corresponding author.
Email address: bashiralik@yahoo.com (B. Ali), aaminu@aust.edu.ng (A. A. Aminu), aadamu@aust.edu.ng (A. Adamu)
[1]Department of Mathematics, Bayero University Kano, Kano, Nigeria
[2]Department of Pure and Applied Mathematics, African University of Science and Technology, Abuja, Nigeria
[3]Center of Excellence in Theoretical and Computational Science, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

Various algorithms for approximating solutions of the SEFPP (1.1) have been introduced and studied by numerous researchers in Hilbert spaces and Banach spaces more general than Hilbert spaces (see, for example, [1,9,12,13,21,24] and the references therein).

In 2019, Chidume, Romanus and Nyaba [22] considered the following algorithm in the setting of some Banach spaces:

Algorithm 1.

$$\begin{cases} x_1 \in X_1, y_1 \in X_2, z_n \in J_{X_3}(Sx_n - Ty_n); \\ x_{n+1} = J_{X_1}^{-1}(a_n J_{X_1} u_n + (1-a_n) J_{X_1} Bu_n),\ u_n = J_{X_1}^{-1}(J_{X_1} x_n - \gamma S^* z_n); \\ y_{n+1} = J_{X_2}^{-1}(a_n J_{X_2} v_n + (1-a_n) J_{X_2} Av_n),\ v_n = J_{X_2}^{-1}(J_{X_2} y_n + \gamma T^* z_n), \end{cases} \tag{1.2}$$

where X_1 and X_2 are Banach spaces that are uniformly smooth and 2-uniformly convex with weak continuous duality maps J_{X_1} and J_{X_2}, respectively, X_3 is a Banach space with duality map J_{X_3}, A and B are quasi-ϕ-nonexpansive mappings, T and S are bounded linear mappings, $\{a_n\}$ is a sequence in $(0,1)$ and γ is a constant that satisfies a certain condition. They proved that the sequence generated by Algorithm 1 converges weakly to a solution of the SEFPP (1.1).

Many efforts have been devoted to improving the convergence speed of the existing iterative algorithms (see, e.g. [2,8,15,16,18,26]). An inertial algorithm was introduced by Polyak [31] to accelerate the process of solving the convex minimization problem. Since then, various iterative algorithms involving inertial extrapolation term have been proposed by numerous authors (see [6–8,11,19,23,28,31]).

Motivated by the research on inertial acceleration technique, in this paper, we incorporate the inertial extrapolation term in Algorithm 1 of Chidume, Romanus and Nyaba [22] for approximating solution(s) of the SEFPP to get an algorithm which accelerates approximation of solution of the SEFPP in some Banach spaces. Unlike in the theorem Chidume, Romanus and Nyaba [22] where weak convergence was established under weak sequencial continuity of the duality mappings, we prove weak convergence of theorem in the setting of Opial spaces. In addition, we prove strong convergence under semi-compactness condition on the quasi-ϕ-nonexpansive maps. Furthermore, we give applications of our theorem to *split equality equilibrium problem, split equality variational inclusion problem* and *split equality problem.* Finally, some numerical examples are given to support our theorems.

2. Preliminaries

Let X be a real Banach space which is smooth and let $\phi : X \times X \to \mathbb{R}$ be a map given by

$$\phi(r,s) = \|r\|^2 - 2\langle r, Js\rangle + \|s\|^2, \quad \forall\, r,s \in X, \tag{2.1}$$

with J being the normalized duality map whose definition and properties on some Banach spaces can be found in, for example, [4]. Alber [4] first introduced this function, and since then numerous researchers have been studying it (see, for example, [3,17,20,27]). By the definition of ϕ, we can see that if X is a real Hilbert space, (2.1) reduces to $\phi(r,s) = \|r-s\|^2$, $\forall r,s \in X$. Furthermore, given $r,s,t,u \in X$, ϕ has the following properties

$$(\|r\| - \|s\|)^2 \le \phi(r,s) \le (\|r\| + \|s\|)^2,$$

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

$$\phi(r,s) = \phi(r,u) + \phi(u,s) + 2\langle u - r, Js - Ju\rangle$$

and

$$2\langle r - s, Ju - Jt\rangle = \phi(r,t) + \phi(s,u) - \phi(r,u) - \phi(s,t).$$

Defining a mapping $V : X \times X^* \to \mathbb{R}$ by

$$V(r, r^*) = \|r\|^2 - 2\langle r, r^*\rangle + \|r^*\|^2,$$

we can see by the definition of ϕ that

$$V(r, r^*) = \phi(r, J^{-1}r^*), \forall r \in X, r^* \in X^*.$$

Definition 2.1. A Banach space X is called Opial space (see, Opial, [30]) or satisfies an Opial condition, if given any sequence $\{y_n\}$ in X with weak convergent limit $y \in X$, the following holds for $x \neq y$:

$$\liminf_{n\to\infty} \|y_n - y\| < \liminf_{n\to\infty} \|y_n - x\|. \tag{2.2}$$

Every real Hilbert space is known to be an Opial space (see, for example, Opial, [30]). Furthermore, l_p spaces, $1 < p < \infty$, are Opial spaces, but L_p spaces $1 < p < \infty$, $p \neq 2$ are not.

Remark 2.1. Gosse and Lami-Dozo [25] have shown that when a norm space has a duality map which is weakly continuous, then it is an Opial space (i.e., it satisfies condition (2.2)), but the converse implication is not true.

Definition 2.2. Let X be a reflexive, strictly convex and smooth real Banach space. Let D be a nonempty convex and closed subset of X. The generalized projection $\Pi_D : X \to D$ is defined by $\tilde{u} = \Pi_D(u) \in D$ such that $\phi(\tilde{u}, u) = \inf_{v\in D} \phi(v, u)$. The metric projection P_D in a real Hilbert space coincides with the generalized projection Π_D .

Definition 2.3. Let X_1 and X_2 be smooth, strictly convex and reflexive real Banach spaces. The collection of linear and continuous maps $B : X_1 \to X_2$ is a normed linear space. The adjoint operator $B^* : X_2^* \longrightarrow X_1^*$ is defined by $\langle B^*u^*, v\rangle = \langle u^*, Bv\rangle$, $\forall\, v \in X_1$, $u^* \in X_2^*$, and $\|B^*\| = \|B\|$.

Definition 2.4. Let X be a real Banach space, and let $\emptyset \neq D \subset X$ be convex and closed. The mapping $S : D \to D$ is

- quasi-ϕ-nonexpansive, if $F(S) := \{q \in D : Sq = q\} \neq \emptyset$ and
$$\phi(p, Sq) \leq \phi(p, q)\ \forall\, p \in F(S),\ q \in D.$$
- semi-compact, if any bounded sequence $\{y_n\} \in D$ with $y_n - Sy_n \to 0$ is given, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ strongly converges to some $y \in C$.
- The mapping $(I - S) : D \to D$ is demiclosed at origin, when $\{y_n\}$ in D converges weakly to $y \in D$ and $\{(I - S)y_n\}$ strongly converges to 0, then $(I - S)y = 0$.

Lemma 2.1 ([3]). *Let X be a smooth, strictly convex and reflexive real Banach Space, and let X^* be its dual space. Then,*

$$V(r, r^*) + 2\langle J^{-1}r^* - r, s^*\rangle \leq V(r, r^* + s^*), \quad \forall\, r \in X,\ r^*, s^* \in X^*.$$

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11

OPEN ACCESS

Lemma 2.2 ([34]). *If X is a smooth and 2-uniformly convex real Banach space, then for all $r, s \in X^*$,*

$$\|J^{-1}r - J^{-1}s\| \leq \frac{1}{k}\|r - s\|, for\ some\ \ k > 0.$$

Lemma 2.3 ([4]). *Let $D \neq \emptyset$ be a convex and closed subset of a smooth, reflexive and smooth real Banach space X. Then,*

$$\phi(r, \Pi_D s) + \phi(\Pi_D s, s) \leq \phi(r, s),\ \forall\ r \in D,\ s \in X.$$

Lemma 2.4 ([32]). *Let X be a smooth, strictly convex and reflexive Banach space, and let $\emptyset \neq D \subset X$ be convex and closed. If $B : X \to 2^{X^*}$ is a maximal monotone map with $B^{-1}(0) \neq \emptyset$, then for $d > 0$, $s \in X$ and $r \in B^{-1}(0)$, we have*

$$\phi(r, Q_d^B s) + \phi(Q_d^B s, s) \leq \phi(r, s),$$

where $Q_d^B : X \to X$ is defined by $Q_d^B y := (J + dB)^{-1} Js$.

Lemma 2.5 ([5]). *Let the sequences $\{\Theta_n\}$,$\{\gamma_n\}$ and $\{\beta_n\}$ be in $[0, \infty)$ with*

$$\Theta_{n+1} \leq \Theta_n + \alpha_n(\Theta_n - \Theta_{n-1}) + \gamma_n,$$

for all $n \geq 1$, $\sum_{n=1}^{\infty} \gamma_n < +\infty$, and there exists $\beta \in \mathbb{R}$ with $0 \leq \beta_n \leq \beta < 1$, for all $n \in \mathbb{N}$. Then, the following holds:
(i) $\sum_{n \geq 1}[\Theta_n - \Theta_{n-1}]_+ < +\infty$, where $[r]_+ = \max\{r, 0\}$.
(ii) There exists $\Theta^* \in [0, \infty)$ such that $\lim_{n\to\infty} \Theta_n = \Theta^*$.

Lemma 2.6 ([34]). *Let $r > 0$. X is uniformly convex, if and only if there exists a continuous, strictly increasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ such that*

$$\|\gamma u + (1 - \gamma)s\|^2 \leq \gamma\|u\|^2 + (1 - \gamma)\|s\|^2 - \gamma(1 - \gamma)f(\|u - s\|), \tag{2.3}$$

for all $\gamma \in [0, 1]$, and $u, s \ \in\ B_r(0)$, where $B_r(0) = \{w \in X : \|w\| \leq r\}$.

3. Main results

Here, we present the main results of this paper, and start by presenting the following algorithm.

Algorithm 2.

Step 1: Choose the positive sequences $\{\epsilon_n\}$ and $\{a_n\}$ satisfying $\sum_{n=1}^{\infty} \epsilon_n < \infty$, $0 < a_n < 1$, $0 < \gamma < \frac{c}{\|S\|^2+\|T\|^2}$, $c = \min\{c_1, c_2\}$, where c_1, c_2 are constants as in Lemma 2.2.
Step 2: Select the arbitrary starting points $x_0, x_1 \in X_1$, $y_0, y_1 \in X_2$, $\alpha \in (0, 1)$ and choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n = \begin{cases} \min\Big\{\alpha, \epsilon_n\|J_{X_1}x_n - J_{X_1}x_{n-1}\|^{-2},\ \epsilon_n\phi(x_n, x_{n-1})^{-1}, \\ \ \epsilon_n\|J_{X_2}y_n - J_{X_2}y_{n-1}\|^{-2},\ \epsilon_n\phi(y_n, y_{n-1})^{-1}\Big\}, \quad x_n \neq x_{n-1}, y_n \neq y_{n-1}; \\ \alpha, \quad otherwise. \end{cases}$$

Step 3: Compute

$$w_n = J_{X_1}^{-1}(J_{X_1}x_n + \alpha_n(J_{X_1}x_n - J_{X_1}x_{n-1}))$$

and

$$x_{n+1} = J_{X_1}^{-1}(a_n J_{X_1}u_n + (1 - a_n)J_{X_1}Bu_n), u_n = J_{X_1}^{-1}(J_{X_1}w_n - \gamma S^* J_{X_3}(Sw_n - Tt_n).$$

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

Step 4: Compute

$$t_n = J_{X_2}^{-1}(J_{X_2}y_n + \alpha_n(J_{X_2}y_n - J_{X_2}y_{n-1}))$$

and

$$y_{n+1} = J_{X_2}^{-1}(a_n J_{X_2} v_n + (1-a_n)J_{X_2}Av_n), v_n = J_{X_2}^{-1}(J_{X_2}t_n + \gamma T^* J_{X_3}(Sw_n + Tt_n).$$

Step 5: Set $n = n+1$ and go to Step 2.

Remark 3.1. Step 2 of Algorithm 2 can be easily implemented, since it only involves the computation of two previous iterates x_{n-1} and x_n, and y_{n-1} and y_n.

3.1. Weak convergence

Theorem 3.1. *Let X_1 and X_2 be 2-uniformly convex and uniformly smooth real Banach spaces which satisfy the Opial condition, and let X_3 be a smooth real Banach space. Let $S : X_1 \to X_3$ and $T : X_2 \to X_3$ (with $S, T \neq 0$) be two bounded linear maps with adjoints S^* and T^* respectively. Let $B : X_1 \to X_1$ and $A : X_2 \to X_2$ be quasi-ϕ-nonexpansive mappings. Suppose $I - B$ and $I - A$ are demiclosed at origin, we set $\Omega = \{(x, y) \in F(B) \times F(A) : Sx = Ty\}$ and assume $\Omega \neq \emptyset$. Let the sequence $\{(x_n, y_n)\}$ be generated by Algorithm 2, and then $\{(x_n, y_n)\}$ converges weakly to a point (x^*, y^*) in Ω.*

Proof. Let $(x, y) \in \Omega$. Using Lemma 2.1 and fact that B is quasi-ϕ-nonexpansive, we get

$$\begin{aligned} \phi(x, x_{n+1}) &= \phi(x, J_{X_1}^{-1}(a_n J_{X_1} u_n + (1-a_n)J_{X_1}Bu_n)) \\ &= V(x, a_n J_{X_1} u_n + (1-a_n)J_{X_1}Bu_n) \\ &\leq a_n V(x, J_{X_1}u_n) + (1-a_n)V(x, J_{X_1}Bu_n) \\ &= a_n \phi(x, u_n) + (1-a_n)\phi(x, Bu_n) \\ &\leq \phi(x, u_n). \end{aligned} \tag{3.1}$$

Using Lemma 2.1, we obtain

$$\begin{aligned} \phi(x, u_n) &= \phi(x, J_{X_1}^{-1}(J_{X_1}w_n - \gamma S^* J_{X_3}(Sw_n - Tt_n)) \\ &= V(x, J_{X_1}w_n - \gamma S^* J_{X_3}(Sw_n - Tt_n) \\ &\leq V(x, J_{X_1}w_n) \\ &- 2\gamma\langle J_{X_1}^{-1}(J_{X_1}w_n - \gamma S^* J_{X_3}(Sw_n - Tt_n)) - x, S^* J_{X_3}(Sw_n - Tt_n)\rangle \\ &= \phi(x, w_n) - 2\gamma\langle Su_n - Sx, J_{X_3}(Sw_n - Tt_n)\rangle. \end{aligned} \tag{3.2}$$

Therefore,

$$\phi(x, x_{n+1}) \leq \phi(x, w_n) - 2\gamma\langle Su_n - Sx, J_{X_3}(Sw_n - Tt_n)\rangle. \tag{3.3}$$

Similarly,

$$\phi(y, y_{n+1}) \leq \phi(y, t_n) - 2\gamma\langle Tv_n - Ty, J_{X_3}(Sw_n - Tt_n)\rangle. \tag{3.4}$$

Since $Sx = Ty$, by adding (3.3) and (3.4), we obtain

$$\phi(x, x_{n+1}) + \phi(y, y_{n+1}) \leq \phi(x, w_n) + \phi(y, t_n) - 2\gamma\langle Su_n - Tv_n, J_{X_3}(Sw_n - Tt_n)\rangle. \tag{3.5}$$

Using the properties of ϕ, we have

$$\begin{aligned} \phi(x, w_n) &= \phi(x, x_n) + \phi(x_n, w_n) + 2\langle x_n - x, J_{X_1}w_n - J_{X_1}x_n\rangle \\ &= \phi(x, x_n) + \phi(x_n, w_n) + 2\alpha_n\langle x_n - x, J_{X_1}x_n - J_{X_1}x_{n-1}\rangle \\ &= \phi(x, x_n) + \phi(x_n, w_n) + \alpha_n\phi(x_n, x_{n-1}) + \alpha_n\phi(x, x_n) \end{aligned}$$

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

$$- \alpha_n \phi(x, x_{n-1}). \tag{3.6}$$

$$\begin{aligned}
\phi(x, w_n) &= \phi(x, J_{X_1}^{-1}(J_{X_1}x_n + \alpha_n(J_{X_1}x_n - J_{X_1}x_{n-1}))) \\
&= \|x\|^2 + \|J_{X_1}x_n + \alpha_n(J_{X_1}x_n - J_{X_1}x_{n-1})\|^2 \\
&- 2\langle x, J_{X_1}x_n + \alpha_n(J_{X_1}x_n - J_{X_1}x_{n-1})\rangle \\
&= \|x\|^2 + \|J_{X_1}x_n + \alpha_n(J_{X_1}x_n - J_{X_1}x_{n-1})\|^2 - 2\langle x, J_{X_1}x_n\rangle \\
&- 2\alpha_n\langle x, J_{X_1}x_n - J_{X_1}x_{n-1}\rangle \\
&\leq \|x\|^2 + \|x_n\|^2 + k_2{\alpha_n}^2\|J_{X_1}x_n - J_{X_1}x_{n-1}\|^2 \\
&+ 2\alpha_n\langle J_{X_1}x_n - J_{X_1}x_{n-1}, x_n\rangle - 2\langle x, J_{X_1}x_n\rangle - 2\alpha_n\langle x, J_{X_1}x_n - J_{X_1}x_{n-1}\rangle \\
&= \phi(x, x_n) + k_2{\alpha_n}^2\|J_{X_1}x_n - J_{X_1}x_{n-1}\|^2 \\
&+ 2\alpha_n\langle J_{X_1}x_n - J_{X_1}x_{n-1}, x_n\rangle - 2\alpha_n\langle x, J_{X_1}x_n - J_{X_1}x_{n-1}\rangle.
\end{aligned} \tag{3.7}$$

By (3.6) and (3.7), we get

$$\phi(x_n, w_n) \leq k_2{\alpha_n}^2\|J_{X_1}x_n - J_{X_1}x_{n-1}\|^2. \tag{3.8}$$

Similarly,

$$\phi(y, t_n) \leq \phi(y, y_n) + \phi(y_n, t_n) + \alpha_n\phi(y_n, y_{n-1}) + \alpha_n\phi(y, y_n) - \alpha_n\phi(y, y_{n-1}) \tag{3.9}$$

and

$$\phi(y_n, t_n) \leq k_2{\alpha_n}^2\|J_{X_2}y_n - J_{X_2}y_{n-1}\|^2. \tag{3.10}$$

Now,

$$\begin{aligned}
&- 2\gamma\langle Su_n - Tv_n, J_{X_3}(Sw_n - Tt_n)\rangle \\
&= -2\gamma\|Sw_n - Tt_n\|^2 - 2\gamma\langle Su_n - Tv_n, J_{X_3}(Sw_n - Tt_n)\rangle \\
&+ 2\gamma\langle Sw_n - Tt_n, J_{X_3}(Sw_n - Tt_n)\rangle \\
&= -2\gamma\|Sw_n - Tt_n\|^2 + 2\gamma\langle S(w_n - u_n), J_{X_3}(Sw_n - Tt_n)\rangle \\
&+ 2\gamma\langle T(v_n - t_n), J_{X_3}(Sw_n - Tt_n)\rangle \\
&= -2\gamma\|Sw_n - Tt_n\|^2 + 2\gamma\langle w_n - u_n, S^*J_{X_3}(Sw_n - Tt_n)\rangle \\
&+ 2\gamma\langle v_n - t_n, T^*J_{X_3}(Sw_n + Tt_n)\rangle \\
&= -2\gamma\|Sw_n - Tt_n\|^2 \\
&+ 2\gamma\langle w_n - J_{X_1}^{-1}(J_{X_1}w_n - \gamma S^*J_{X_3}(Sw_n - Tt_n)), S^*J_{X_3}(Sw_n - Tt_n)\rangle \\
&+ 2\gamma\langle J_{X_2}^{-1}(J_{X_2}t_n + \gamma T^*J_{X_3}(Sw_n + Tt_n)) - t_n, T^*J_{X_3}(Sw_n + Tt_n)\rangle \\
&= -2\gamma\|Sw_n - Tt_n\|^2 \\
&+ 2\gamma\langle J_{X_1}^{-1}J_{X_1}w_n - J_{X_1}^{-1}(J_{X_1}w_n - \gamma S^*J_{X_3}(Sw_n - Tt_n)), S^*J_{X_3}(Sw_n - Tt_n)\rangle \\
&+ 2\gamma\langle J_{X_2}^{-1}(J_{X_2}t_n + \gamma T^*J_{X_3}(Sw_n + Tt_n)) - J_{X_2}^{-1}J_{X_2}t_n, T^*J_{X_3}(Sw_n + Tt_n)\rangle \\
&\leq -2\gamma\|Sw_n - Tt_n\|^2 + \frac{2\gamma^2\|S\|^2}{c}\|(Sw_n - Tt_n))\|^2 + \frac{2\gamma^2\|T\|^2}{c}\|(Sw_n - Tt_n)\|^2 \\
&= -\Big(2\gamma - \frac{2\gamma^2(\|S\|^2 + \|T\|^2)}{c}\Big)\|(Sw_n - Tt_n)\|^2.
\end{aligned} \tag{3.11}$$

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

Putting (3.6), (3.9) and (3.11) in (3.5) and $0 < \gamma < \frac{c}{\|S\|^2+\|T\|^2}$, we have

$$\begin{aligned}
&\phi(x, x_{n+1}) + \phi(y, y_{n+1}) \\
&\leq \phi(x, x_n) + \phi(y, y_n) \\
&+ \alpha_n[(\phi(x, x_n) + \phi(y, y_n)) - (\phi(x, x_{n-1}) + \phi(y, y_{n-1}))] \\
&- (2\gamma - \frac{2\gamma^2(\|S\|^2 + \|T\|^2)}{c})\|(Sw_n - Tt_n)\|^2 \\
&+ k_2{\alpha_n}^2(\|J_{X_1}x_n - J_{X_1}x_{n-1}\|^2 + \|J_{X_2}y_n - J_{X_2}y_{n-1}\|^2) \\
&+ \alpha_n(\phi(x_n, x_{n-1}) + \phi(y_n, y_{n-1})) \\
&\leq \phi(x, x_n) + \phi(y, y_n) \\
&+ \alpha_n[(\phi(x, x_n) + \phi(y, y_n)) - (\phi(x, x_{n-1}) + \phi(y, y_{n-1}))] \\
&+ k_2\alpha_n(\|J_{X_1}x_n - J_{X_1}x_{n-1}\|^2 + \|J_{X_2}y_n - J_{X_2}y_{n-1}\|^2) \\
&+ \alpha_n(\phi(x_n, x_{n-1}) + \phi(y_n, y_{n-1})).
\end{aligned} \tag{3.12}$$

From Lemma 2.5, we have that $\Theta_n(x, y)$ is convergent, where $\Theta_n(u, v) := \phi(u, x_n) + \phi(v, y_n)$. Therefore, the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently, we have that $\{w_n\}$, $\{t_n\}$, $\{u_n\}$ and $\{v_n\}$ are also bounded. By (3.12), we have

$$\lim_{n\to\infty} \|Sw_n - Tt_n\| = 0, \tag{3.13}$$

$$\begin{aligned}
\|u_n - w_n\| &= \|J_{X_1}^{-1}(J_{X_1}w_n - \gamma S^* J_{X_3}(Sw_n - Tt_n)) - J_{X_1}^{-1}(J_{X_1}w_n\| \\
&\leq \frac{\gamma\|S\|}{c}\|Sw_n - Tt_n\|.
\end{aligned} \tag{3.14}$$

By (3.13) and (3.14), we obtain

$$\lim_{n\to\infty} \|u_n - w_n\| = 0.$$

Similarly,

$$\lim_{n\to\infty} \|v_n - t_n\| = 0.$$

Since $J_{X_1}^{-1}$ and $J_{X_2}^{-1}$ are uniformly continuous on bounded sets, then from Algorithm 2, (3.8) and (3.10), we have

$$\lim_{n\to\infty} \|x_n - w_n\| = 0$$

and

$$\lim_{n\to\infty} \|y_n - t_n\| = 0.$$

Then,

$$\|v_n - y_n\| \leq \|v_n - t_n\| + \|t_n - y_n\| \to 0 \ as \ n \to \infty.$$

From the definition of ϕ, the quasi-ϕ-nonexpansiveness of B and Lemma 2.6, we get

$$\begin{aligned}
\phi(x, x_{n+1}) &= \phi(x, J_{X_1}^{-1}(a_n J_{X_1}u_n + (1 - a_n)J_{X_1}Bu_n)) \\
&= \|x\|^2 - 2\langle x, a_n J_{X_1}u_n + (1 - a_n)J_{X_1}Bu_n\rangle \\
&+ \|a_n J_{X_1}u_n + (1 - a_n)J_{X_1}Bu_n\|^2 \\
&\leq |x\|^2 - 2\langle x, a_n J_{X_1}u_n + (1 - a_n)J_{X_1}Bu_n\rangle \\
&+ a_n\|u_n\|^2 + (1 - a_n)\|Bu_n\|^2 - a_n(1 - a_n)g_1(\|J_{X_1}u_n - J_{X_1}Bu_n\|) \\
&= a_n\phi(x, u_n) + (1 - a_n)\phi(x, Bu_n) - a_n(1 - a_n)g_1(\|J_{X_1}u_n - J_{X_1}Bu_n\|) \\
&\leq \phi(x, u_n) - a_n(1 - a_n)g_1(\|J_{X_1}u_n - J_{X_1}Bu_n\|).
\end{aligned}$$

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

By (3.2) and (3.6), we obtain

$$\begin{aligned}\phi(x,x_{n+1}) \le\ & \phi(x,x_n)+\alpha_n(\phi(x,x_n)-\phi(x,x_{n-1}))\\ &+\phi(x_n,w_n)+\alpha_n\phi(x_n,x_{n-1})\\ &-2\gamma\langle Su_n-Sx, J_{X_3}(Sw_n-Tt_n)\rangle\\ &-a_n(1-a_n)g_1(\|J_{X_1}u_n-J_{X_1}Bu_n\|).\end{aligned} \tag{3.15}$$

Similarly,

$$\begin{aligned}\phi(y,y_{n+1}) \le\ & \phi(y,y_n)+\alpha_n(\phi(y,y_n)-\phi(y,y_{n-1}))\\ &+\phi(y_n,t_n)+\alpha_n\phi(y_n,y_{n-1})\\ &+2\gamma\langle Tv_n-Ty, J_{X_3}(Sw_n-Tt_n)\rangle\\ &-a_n(1-a_n)g_2(\|J_{X_2}v_n-J_{X_2}Av_n\|).\end{aligned} \tag{3.16}$$

From (3.15), (3.16), $Sx=Ty$, (3.11) and the condition on γ, we obtain

$$\begin{aligned}\Theta_{n+1}(x,y) \le\ & \Theta_n(x,y)+\alpha_n(\Theta_n(x,y)-\Theta_{n-1}(x,y)\\ &+k_2(\alpha_n\|J_{X_1}x_n-J_{X_1}x_{n-1}\|^2+\alpha_n\|J_{X_2}y_n-J_{X_2}y_{n-1}\|^2)\\ &+\alpha_n(\phi(x_n,x_{n-1})+\phi(y_n,y_{n-1}))\\ &-a_n(1-a_n)[g_1(\|J_{X_1}u_n-J_{X_1}Bu_n\|)+g_2(\|J_{X_2}v_n-J_{X_2}Sv_n\|)].\end{aligned} \tag{3.17}$$

Since the limit of $\Theta_n(x,y)$ exists,

$$\lim_{n\to\infty}\alpha_n\|J_{X_1}x_n-J_{X_1}x_{n-1}\|=0=\lim_{n\to\infty}\alpha_n\|J_{X_2}y_n-J_{X_2}y_{n-1}\|$$

and

$$\lim_{n\to\infty}\alpha_n\phi(x_n,x_{n-1})=0=\lim_{n\to\infty}\alpha_n\phi(y_n,y_{n-1}),$$

and we obtain from (3.16) that

$$\lim_{n\to\infty}g_1(\|J_{X_1}u_n-J_{X_1}Bu_n\|)=0=\lim_{n\to\infty}g_2(\|J_{X_2}v_n-J_{X_2}Av_n\|).$$

Using the properties of g_1 and g_2, we get

$$\lim_{n\to\infty}\|J_{X_1}u_n-J_{X_1}Bu_n\|=0=\lim_{n\to\infty}\|J_{X_2}v_n-J_{X_2}Av_n\|.$$

Since $J_{X_1}^{-1}$ and $J_{X_2}^{-1}$ are uniformly continuous on bounded sets,

$$\lim_{n\to\infty}\|u_n-Bu_n\|=0=\lim_{n\to\infty}\|v_n-Av_n\|.$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, there exist the subsequences $\{x_{n_k}\}$ of $\{x_n\}$, $\{y_{n_k}\}$ of $\{y_n\}$ such that $x_{n_k}\rightharpoonup x^*$ and $y_{n_k}\rightharpoonup y^*$ for some $x^*\in X_1$, $y^*\in X_2$.
Since

$$\lim_{n\to\infty}\|u_n-x_n\|=0=\lim_{n\to\infty}\|v_n-y_n\|,$$

we have $u_{n_k}\rightharpoonup x^*$ and $v_{n_k}\rightharpoonup y^*$. By demiclosedness of $I-B$ and $I-A$ at 0, we have $x^*\in F(B)$ and $y^*\in F(A)$. Since $\|.\|$ is weakly lower semi continuous, we have

$$\|Sx^*-Ty^*\|\le\liminf_{k\to\infty}\|Sx_{n_k}-Ty_{n_k}\|=\lim_{k\to\infty}\|Sx_{n_k}-Ty_{n_k}\|=0,$$

which implies $Sx^*=Ty^*$. Therefore, $(x^*,y^*)\in\Omega$.

Let $\{x_{n_j}\}$ be an arbitrary subsequence of $\{x_n\}$ such that $x_{n_j}\rightharpoonup p$ as $j\to\infty$. We claim $p=x^*$. Suppose this claim is false, then $p\neq x^*$. Since X_1 satisfies the Opial condition, we get

$$\liminf_{n\to\infty}\|x_n-x^*\|<\liminf_{n\to\infty}\|x_n-p\|<\liminf_{n\to\infty}\|x_n-x^*\|,$$

and this contradiction gives $p=x^*$.

Hence, $\{x_n\}$ has a single weak cluster point. Therefore, $x_n\rightharpoonup x^*$.

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

Following the similar argument, we get that $\{y_n\}$ converges weakly to y^*. □

Corollary 3.1. *Let X_1 and X_2 be l_p spaces, $1 < p \leq 2$, and let X_3 be a smooth real Banach space. Setting $\Omega = \{(x, y) \in F(B) \times F(A) : Sx = Ty\}$ and assuming $\Omega \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 2 converges weakly to some point (x^*, y^*) in Ω.*

3.2. Strong convergence

Theorem 3.2. *Let X_1, X_2 and X_3 be real Banach spaces as in Theorem 3.1. Let S, T, A and B also be mappings as in Theorem 3.1 such that B and A are semi-compact. Set $\Omega = \{(x, y) \in F(B) \times F(A) : Sx = Ty\}$ and assume $\Omega \neq \emptyset$. Then, the sequence $\{(x_n, y_n)\}$ generated by Algorithm 2 strongly converges to some point (x^*, y^*) in Ω.*

Proof. Following the same proof as that of Theorem 3.1, we obtain

$$\lim_{n\to\infty} \|u_n - x_n\| = 0 = \lim_{n\to\infty} \|v_n - y_n\| \tag{3.18}$$

and

$$\lim_{n\to\infty} \|u_n - Bu_n\| = 0 = \lim_{n\to\infty} \|v_n - Sv_n\|. \tag{3.19}$$

Therefore, we have $u_n \rightharpoonup x^*$ and also $v_n \rightharpoonup y^*$. Since B and A are semi-compact, there exist the subsequences $\{u_{n_j}\}$ of $\{u_n\}$ and $\{v_{n_j}\}$ of $\{v_n\}$ such that $u_{n_j} \to x^*$ and $v_{n_j} \to y^*$, as $j \to \infty$. Let $\{u_{n_i}\}$ be another subsequence of $\{u_n\}$ such that $u_{n_i} \to q$, as $i \to \infty$. Let

$$m := \liminf_{n\to\infty} (\phi(q, u_n) + \phi(x^*, u_n)).$$

Then,

$$\phi(q, u_n) + \phi(x^*, u_n) = 2\langle x^* - q, J_{X_1} u_n\rangle + \|q\|^2 - \|x\|^2. \tag{3.20}$$

Using (3.20), $u_{n_i} \to q$, as $i \to \infty$ and $u_{n_j} \to x^*$ as $j \to \infty$, we have

$$m = 2\langle x^* - q, J_{X_1} x^*\rangle + \|q\|^2 - \|x\|^2,$$
$$m = 2\langle x^* - q, J_{X_1} q\rangle + \|q\|^2 - \|x\|^2.$$

Thus, $\langle x^* - q, J_{X_1} x^* - J_{X_1} q\rangle = 0$. Hence, $x^* = q$. By the strict monotonicity of J_{X_1}, $\{u_n\}$ strongly converges to x^*. From (3.18), $\{x_n\}$ converges strongly to x^*. Using the similar argument, we have that $\{y_n\}$ converges strongly to y^*.

Corollary 3.2. *Let X_1 and X_2 be l_p spaces, $1 < p \leq 2$, and let X_3 be a real Banach space. Let S, T, A and B also be mappings as in Theorem 3.1 such that B and A are semi-compact, and $(I - B)$ and $(I - A)$ are demiclosed at zero. Setting $\Omega = \{(x, y) \in F(B) \times F(A) : Sx = Ty\}$ and assuming $\Omega \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 2 strongly converges to (x^*, y^*) in Ω.*

□

4. Applications

4.1. Split equality equilibrium problem (SEEP)

Let D and R be nonempty closed and convex subsets of a real Banach space X, and let $H : D \times D \to \mathbb{R}$ be a bifunction. The following problem:

$$\text{find } t \in D \text{ such that } H(t, s) \geq 0, \ \forall\, s \in D$$

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

is the Equilibrium Problem, and the solution set of the problem is denoted by $EP(H)$. To solve the equilibrium problem, the bifunction H is usually assumed to satisfy the following conditions:

(1) $H(r,r) = 0$, for all $r \in D$.

(2) H is monotone, i.e., $H(r,s) + H(s,r) \leq 0$, $\forall r, s \in D$.

(3) For each $r, s, t \in D$ and $k \in (0,1)$, $\lim_{k \to 0} H(kt + (1-k)r, s) \leq H(r,s)$.

(4) For each $r \in D$, $s \longmapsto H(r,s)$ is lower semi continuous and convex.

Next we define the *split equality equilibrium problem (SEEP)* which is a generalization of the equilibrium problem. The SEEP is to find

$$r^* \in D,\ s^* \in R \text{ such that } g(r^*, r) \geq 0,\ f(s^*, s) \geq 0 \text{ and } Sr^* = Ts^*,$$

for all $r \in D$, $s \in R$, where $g : D \times D \to \mathbb{R}$, $f : R \times R \to \mathbb{R}$ are bifunctions that satisfy (1) to (4). We denote the set of solutions of the split equality equilibrium problem by Γ.

Lemma 4.1 ([35]). *Let X be a strictly convex, uniformly smooth and reflexive Banach space, and let D be a nonempty closed and convex subset of X. Let $H : D \times D \to \mathbb{R}$ be a bifunction which satisfies assumptions (1) to (4) above. Then, for $r \in X$ and $d > 0$, there exists a unique $t \in D$ such that*

$$H(z,y) + \frac{1}{d}\langle s - t, Jt - Jr\rangle \geq\ 0,\ \forall s\ \in D.$$

Lemma 4.2 ([35]). *Let X be a smooth strictly convex and reflexive Banach space, and let $D \neq \emptyset$ be a convex and closed subset of X. Let $H : D \times D \to \mathbb{R}$ be bifunctional which satisfies assumptions (1)-(4). Given $r \in X$ and $d > 0$, if we define a map $G_d : X \to D$ by*

$$G_d r = \{r \in D : H(t,s) + \frac{1}{d}\langle s - t, Jt - Jr\rangle \geq\ 0,\ \forall s\ \in D\},$$

the following holds

1. *G_d is single-valued.*
2. *G_d is firmly nonexpansive type, i.e.,*
$$\langle G_d r - G_d s, JG_d r - JG_d s\rangle \leq \langle G_d r - G_d s, r - s\rangle,\ \forall r, s\ \in\ X.$$
3. *$F(G_d) = EP(H)$.*
4. *$EP(H)$ is convex and closed.*
5. *$\phi(r, G_d w) + \phi(G_d w, w) \leq \phi(r, w), \forall r \in F(G_d), w \in X$.*

Algorithm 3.

Step 1: Choose the positive sequences $\{\epsilon_n\}$ and $\{a_n\}$ satisfying $\sum_{n=1}^{\infty} \epsilon_n < \infty$, $0 < a_n < 1$, $0 < \gamma < \frac{c}{\|S\|^2 + \|T\|^2}$, $c = \min\{c_1, c_2\}$, where c_1, c_2 are constants as in Lemma 2.2.

Step 2: Select the arbitrary starting points $x_0, x_1 \in X_1$, $y_0, y_1 \in X_2$, $\alpha \in (0,1)$ and choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n = \begin{cases} \min\Big\{\alpha, \epsilon_n \|J_{X_1} x_n - J_{X_1} x_{n-1}\|^{-2},\ \epsilon_n \phi(x_n, x_{n-1})^{-1}, \\ \ \epsilon_n \|J_{X_1} y_n - J_{X_1} y_{n-1}\|^{-2},\ \epsilon_n \phi(y_n, y_{n-1})^{-1}\Big\}, \quad x_n \neq x_{n-1}, y_n \neq y_{n-1}; \\ \alpha, \quad \mathit{otherwise}. \end{cases}$$

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

Step 3: Compute

$$w_n = J_{X_1}^{-1}(J_{X_1}x_n + \alpha_n(J_{X_1}x_n - J_{X_1}x_{n-1}))$$

and

$x_{n+1} = J_{X_1}^{-1}(a_n J_{X_1}u_n + (1-a_n)J_{X_1}G_d u_n), u_n = J_{X_1}^{-1}(J_{X_1}w_n - \gamma S^* J_{X_3}(Sw_n - Tt_n)$.

Step 4: Compute

$$t_n = J_{X_2}^{-1}(J_{X_2}y_n + \alpha_n(J_{X_2}y_n - J_{X_2}y_{n-1}))$$

and

$y_{n+1} = J_{X_2}^{-1}(a_n J_{X_2}v_n + (1-a_n)J_{X_2}T_d v_n), v_n = J_{X_2}^{-1}(J_{X_2}t_n + \gamma B^* J_{X_3}(Sw_n + Tt_n)$.

Step 5: Set $n = n+1$ and go to Step 2.

Theorem 4.1. *Let X_1, X_2 and X_3 be real Banach spaces as in Theorem 3.1. Let D and R be nonempty convex and closed subsets of X_1 and X_2, respectively, and let $g : D \times D \to \mathbb{R}$ and $f : R \times R \to \mathbb{R}$ be bifunctions which satisfy (1) to (4) with $EP(g) \neq \emptyset$ and $EP(f) \neq \emptyset$. Let S and T be bounded operators as in Theorem 3.1. Assuming that Ω is nonempty, we let the sequence $\{(x_n, y_n)\}$ be generated by Algorithm 3, where $G_d r = \{t \in D : g(t,s) + \frac{1}{d}\langle s-t, J_{X_1}t - J_{X_1}r\rangle \geq 0, \forall s \in D\}, r \in X_1$, $T_d v = \{x \in R : h(x,s) + \frac{1}{d}\langle s-x, J_{X_2}x - J_{X_2}v\rangle \geq 0, \forall s \in R\}, v \in X_2, d > 0$. Then, the sequence $\{(x_n, y_n)\}$ converges weakly to (x^*, y^*) in Ω.*

Proof. Letting $B = G_d$ and $A = T_d$, by Lemma 4.2, we obtain that A and B are quasi-ϕ-nonexpansive. Therefore, by Theorem 3.1 and Lemma 4.2(3), the result follows. □

4.2. Split equality variational inclusion problem (SEVIP)

Let $N : X_1 \to 2^{X_1{}^*}$ and $M : X_2 \to 2^{X_2{}^*}$ be maximal monotone operators. The split equality variational inclusion problem is to

$$\text{find} \quad x \in N^{=1}(0), y \in M^{-1}(0) \;\text{ such that }\; Sx = Ty,$$

where $N^{-1}(0) = \{z \in X_1 : 0 \in Nz\}$ and $M^{-1}(0) = \{x \in X_2 : 0 \in Mx\}$ are the sets of zeros of N and M respectively.

Algorithm 4.

Step 1: Choose the positive sequences $\{\epsilon_n\}$ and $\{a_n\}$ satisfying $\sum_{n=1}^{\infty} \epsilon_n < \infty$, $0 < a_n < 1$, $0 < \gamma < \frac{c}{\|S\|^2 + \|T\|^2}$, $c = \min\{c_1, c_2\}$, where c_1, c_2 are constants as in Lemma 2.2.

Step 2: Select the arbitrary starting points $x_0, x_1 \in X_1$, $y_0, y_1 \in X_2$, $\alpha \in (0,1)$ and choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n = \begin{cases} \min\Big\{\alpha, \epsilon_n\|J + X_1 x_n - J_{X_1}x_{n-1}\|^{-2},\ \epsilon_n\phi(x_n, x_{n-1})^{-1}, \\ \ \epsilon_n\|J_{X_2}y_n - J_{X_2}y_{n-1}\|^{-2},\ \epsilon_n\phi(y_n, y_{n-1})^{-1}\Big\}, \quad x_n \neq x_{n-1}, y_n \neq y_{n-1}; \\ \alpha, \quad otherwise. \end{cases}$$

Step 3: Compute

$$w_n = J_{X_1}^{-1}(J_{X_1}x_n + \alpha_n(J_{X_1}x_n - J_{X_1}x_{n-1}))$$

and

$x_{n+1} = J_{X_1}^{-1}(a_n J_{X_1}u_n + (1-a_n)J_{X_1}{Q_r}^N u_n), u_n = J_{X_1}^{-1}(J_{X_1}w_n - \gamma S^* J_{X_3}(Sw_n - Tt_n)$.

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

Step 4: Compute

$$t_n = J_{X_2}^{-1}(J_{X_2}y_n + \alpha_n(J_{X_2}y_n - J_{X_2}y_{n-1}))$$

and

$y_{n+1} = J_{X_2}^{-1}(a_nJ_{X_2}v_n + (1-a_n)J_{X_2}{Q_r}^M v_n), v_n = J_{X_2}^{-1}(J_{X_2}t_n + \gamma T^* J_{X_3}(Sw_n + Tt_n)$.

Step 5: Set $n = n + 1$ and go to Step 2.

Theorem 4.2. *Let X_1, X_2 and X_3 be real Banach spaces as in Theorem 3.1. Let $N : X_1 \to 2^{X_1^*}$ and $M : X_2 \to 2^{X_2^*}$ be maximal monotone mappings with $N^{-1}(0)$ and $M^{-1}(0)$ nonempty. Let $S : X_1 \to X_3$ and $T : X_2 \to X_3$ be operators as in Theorem 3.1. Setting $\Omega = \{(x, y) \in N^{-1}(0) \times M^{-1}(0) : Sx = Ty\} \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 4, where $Q_r^N = (J_{X_1} + rN)^{-1}J_{X_1}$, $Q_r^M = (J_{X_2} + rM)^{-1}J_{X_2}, Sr > 0$ converges weakly to (x^*, y^*) in Ω.*

Proof. Putting $B = Q_r^N$ and $A = Q_r^M$, by Lemma 2.4, Q_r^N and Q_r^M are quasi-ϕ-nonexpansive. Therefore, by Theorem 3.1, the result follows. □

4.3. Split equality problem

The problem of finding

$$x \in D, y \in R \text{ such that } Sx = Ty$$

is called split equality problem.

Algorithm 5.

Step 1: Choose the positive sequences $\{\epsilon_n\}$ and $\{a_n\}$ satisfying $\sum_{n=1}^{\infty} \epsilon_n < \infty$, $0 < a_n < 1$, $0 < \gamma < \frac{c}{\|S\|^2+\|T\|^2}$, $c = \min\{c_1, c_2\}$, where c_1, c_2 are constants as in Lemma 2.2.

Step 2: Select the arbitrary starting points $x_0, x_1 \in X_1$ $y_0, y_1 \in X_2$, $\alpha \in (0, 1)$ and choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n = \begin{cases} \min\Big\{\alpha, \epsilon_n\|J_{X_1}x_n - J_{X_1}x_{n-1}\|^{-2},\ \epsilon_n\phi(x_n, x_{n-1})^{-1}, \\ \epsilon_n\|J_{X_2}y_n - J_{X_2}y_{n-1}\|^{-2},\ \epsilon_n\phi(y_n, y_{n-1})^{-1}\Big\}, \quad x_n \neq x_{n-1}, y_n \neq y_{n-1}; \\ \alpha, \quad otherwise. \end{cases}$$

Step 3: Compute

$$w_n = J_{X_1}^{-1}(J_{X_1}x_n + \alpha_n(J_{X_1}x_n - J_{X_1}x_{n-1}))$$

and

$x_{n+1} = J_{X_1}^{-1}(a_nJ_{X_1}u_n + (1-a_n)J_{X_1}\Pi_D u_n), u_n = J_{X_1}^{-1}(J_{X_1}w_n - \gamma B^* J_{X_3}(Sw_n - Tt_n)$.

Step 4: Compute

$$t_n = J_{X_2}^{-1}(J_{X_2}y_n + \alpha_n(J_{X_2}y_n - J_{X_2}y_{n-1}))$$

and

$y_{n+1} = J_{X_2}^{-1}(a_nJ_{X_2}v_n + (1 - a_n)J_{X_2}\Pi_R v_n), v_n = J_{X_2}^{-1}(J_{X_2}t_n + \gamma T^* J_{X_3}(Sw_n + Tt_n)$.

Step 5: Set $n = n + 1$ and go back to Step 2.

Theorem 4.3. *Let X_1, X_2 and X_3 be real Banach spaces as in Theorem 3.1. Let S and T be operators as in Theorem 3.1.*
Assuming $\Omega \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 5 converges weakly to some (x^*, y^*) in Ω.

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

Proof. Letting $B = \Pi_D$ and $A = \Pi_R$, by Lemma 2.3, Π_D and Π_R are quasi-ϕ-nonexpansive. Therefore, by Theorem 3.1, the result follows. □

5. Numerical illustrations

We shall examine the effect of the inertial extrapolation term in accelerating the convergence of the sequence generated by our algorithm.

Example 5.1.

In Algorithms 1 and 2, set $X_1 = \mathbb{R}$, $X_2 = \mathbb{R}^2$ and $X_3 = \mathbb{R}^2$. Let $S : X_1 \to X_3$ and let $T : X_2 \to X_3$ be defined by

$$Sr = \left(\frac{r}{2}, \frac{r}{3}\right), \quad T(r,s) = \big(r + 2s, s\big)$$

respectively. We can easily verify

$$S^*(n,m) = \frac{n}{2} + \frac{m}{3} \quad \text{and} \quad T^*(n,m) = (n, 2n+m).$$

Let $B : X_1 \to X_1$ and $A : X_2 \to X_2$ be defined by

$$Bx = \frac{r}{2} \quad \text{and} \quad A(n,m) = (n,m).$$

We can see that B and A are quasi-ϕ-nonexpansive, and $(I - B)$ and $(I - A)$ are demiclosed at zero. Furthermore, since $0 \in \Omega$, then $\Omega \neq \emptyset$. In Algorithm 1, we take $\gamma = 0.35$, $a_n = \frac{1}{(n+1)^2}$, and in Algorithm 2, we take $\epsilon_n = \frac{1}{n^6}$, $\alpha_n = \bar{\alpha}_n$, $\alpha = 0.5$, $\gamma = 0.3$, $a_n = \frac{1}{(n+1)^2}$. It is clear that the parameters satisfy the hypothesis of these Algorithms. Using a tolerance 10^{-8} and setting $n = 100$, we have the following:

Table 1. Numerical results of Example 5.1

Table of values choosing $x_0 = -2$, $x_1 = 1$, $y_0 = (1,-2)^T$ and $y_1 = (0,3)^T$				
	Algorithm 1		Algorithm 2	
n	$\lvert x_{n+1} - x_n\rvert$	$\lVert y_{n+1} - y_n\rVert$	$\lvert x_{n+1} - x_n\rvert$	$\lVert y_{n+1} - y_n\rVert$
1	0.421	6.1527	0.3072	5.2737
10	0.7097	3.9251	0.0274	0.1429
20	0.4238	2.3554	7.33E-4	0.0209
30	0.2546	1.4165	1.68E-4	0.0122
40	0.1531	0.8524	9.43E-5	0.0072
50	0.0921	0.513	5.53E-5	0.0042
60	0.0554	0.3088	3.25E-5	0.0024
70	0.0333	0.1859	1.91E-5	0.0014
80	0.0201	0.1119	1.12E-5	8.64E-4
90	0.0121	0.0673	6.62E-6	5.08E-4
99	0.0076	0.0426	4.11E-6	3.15E-4

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

Table 2. Numerical results of Example 5.1

Table of values choosing $x_0 = 0.5$, $x_1 = 1.5$, $y_0 = (1,0)^T$ and $y_1 = (0,0.25)^T$				
	Algorithm 1		Algorithm 2	
n	$\lvert x_{n+1} - x_n \rvert$	$\lVert y_{n+1} - y_n \rVert$	$\lvert x_{n+1} - x_n \rvert$	$\lVert y_{n+1} - y_n \rVert$
1	0.608	1.2187	0.6015	1.0446
10	0.1665	0.9209	0.0066	0.0332
20	0.0993	0.5526	1.22E-4	0.0011
30	0.0597	0.3323	9.128E-6	5.52E-4
40	0.0359	0.2	4.27E-6	3.25E-4
50	0.0216	0.1203	2.49E-6	1.91E-4
60	0.013	0.0724	1.46E-6	1.12E-4
70	0.0078	0.0436	8.63E-7	6.62E-5
80	0.0047	0.0262	5.07E-7	3.89E-5
90	0.0028	0.0158	2.98E-7	2.29E-5
99	0.0017	0.01	1.85E-7	1.42E-5

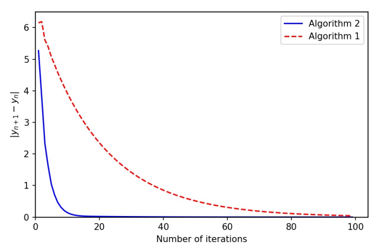

Figure 1(a). Some iterates of Algorithms 1 and 2 choosing $x_0 = -2$ and $x_1 = 1$

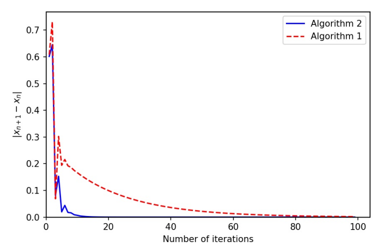

Figure 1(b). Some iterates of Algorithms 1 and 2 choosing $x_0 = 0.5$ and $x_1 = 1.5$

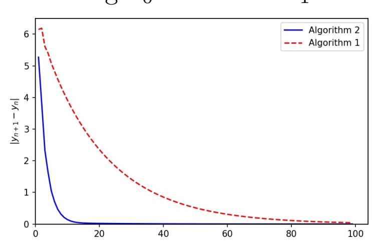

Figure 2(a). Some iterates of Algorithms 1 and 2 choosing $y_0 = (1,-2)^T$ and $y_1 = (0,3)^T$

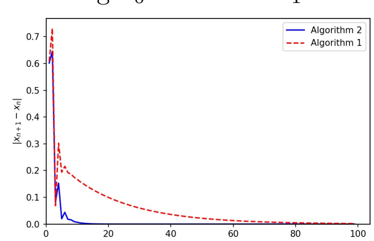

Figure 2(b). Some iterates of Algorithms 1 and 2 choosing $y_0 = (1,0)^T$ and $y_1 = (0,0.25)^T$

Example 5.2.

In Algorithms 1 and 2, set $X_1 = X_2 = X_3 = L_2([0,1])$. Let $S : X_1 \to X_3$ and

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

$T : X_2 \to X_3$ be defined by

$$(Sy)(g) = 2y(g) \quad \text{and} \quad (Ty)(g) = y(g), \quad \text{then} \quad S^* = S \quad \text{and} \quad T^* = T.$$

Let $B : X_1 \to X_1$ and $A : X_2 \to X_2$ be defined by

$$(By)(g) = \frac{y(g)}{8} \quad \text{and} \quad (Ay)(g) = \frac{y(g)}{2}.$$

Obviously, B and A are semi-compact and quasi-ϕ-nonexpansive, and $(I - B)$ and $(I-A)$ are demiclosed at zero. Furthermore, since zero belongs to Ω, Ω is nonempty. In Algorithm 1, we take $\gamma = 0.1$, $a = 0.2$, and Algorithm 2, we take $\epsilon_n = \frac{1}{(n+1)^6}$, $\alpha_n = \bar{\alpha}_n$, $\alpha = 0.8$, $\gamma = 0.01$, $a_n = \frac{1}{(n+1)^2}$. It is clear that the parameters satisfy the hypothesis of our theorems.

Using a tolerance 10^{-8} and setting $n = 7$, we have the following.

Table 3. Numerical results of Example 5.2

Table of values choosing $x_0(t) = t^2 + 1$, $x_1(t) = t$, $y_0(t) = t + 1$ and $y_1(t) = \sin t$				
	Algorithm 1		Algorithm 2	
n	$\|x_{n+1} - x_n\|$	$\|y_{n+1} - y_n\|$	$\|x_{n+1} - x_n\|$	$\|y_{n+1} - y_n\|$
1	0.5389	0.2917	0.6122	0.6483
2	0.2161	0.2041	0.1768	0.1097
3	0.0501	0.1156	0.0259	0.006
4	0.0597	0.0617	0.002	0.0097
5	0.0015	0.032	4.7E-4	0.0044
6	2.83E-4	0.1203	2.39E-4	0.0013

Table 4. Numerical results of Example 5.2

Table of values choosing $x_0(t) = 2$, $x_1(t) = e^t$, $y_0(t) = t + \cos t$ and $y_1(t) = 1 + 2\sin t$				
	Algorithm 1		Algorithm 2	
n	$\|x_{n+1} - x_n\|$	$\|y_{n+1} - y_n\|$	$\|x_{n+1} - x_n\|$	$\|y_{n+1} - y_n\|$
1	1.2562	0.8021	1.4127	1.6572
2	0.5044	0.566	0.4223	0.3769
3	0.1173	0.3222	0.068	0.0767
4	0.0217	0.1722	0.0083	0.0285
5	0.0037	0.0895	0.0014	0.0108
6	6.88E-4	0.0458	5.17E-4	0.0038

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

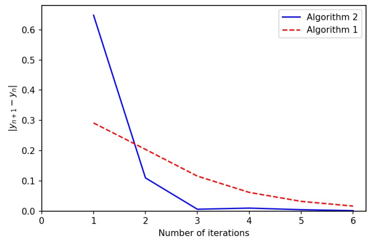

Figure 3(a). Some iterates of Algorithms 1 and 2 choosing $x_0(t) = t^2 + 1$ and $x_1(t) = t$,

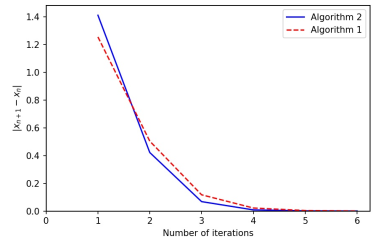

Figure 3(b). Some iterates of Algorithms 1 and 2 choosing $x_0(t) = 2$ and $x_1(t) = e^t$

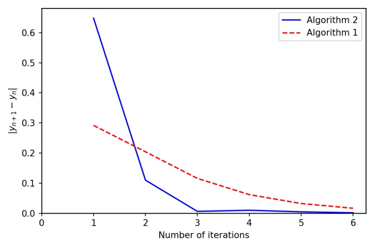

Figure 4(a). Some iterates of Algorithms 1 and 2 choosing $y_0(t) = t + 1$ and $y_1(t) = \sin t$

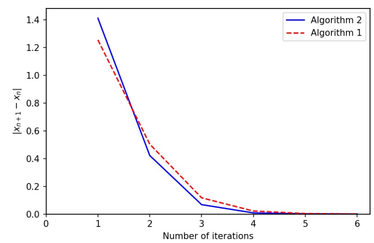

Figure 4(b). Some iterates of Algorithms 1 and 2 choosing $y_0(t) = t + \cos t$ and $y_1(t) = 1 + 2\sin t$

Acknowledgements

The authors would like to thank the reviewers and editors for their esteemed comments that have greatly improved this paper. Furthermore, they appreciate their institutions for their generous support.

References

[1] A. Adamu and A. A. Adam, *Approximation of solutions of split equality fixed point problems with applications*, Carpathian Journal of Mathematics, 2021, 37(3), 381–392.

[2] A. Adamu, D. Kitkuan, A. Padcharoen, et al., *Inertial viscosity-type iterative method for solving inclusion problems with applications*, Mathematics and Computers in Simulation, 2022, 194, 445–459.

[3] Y. Alber and I. Ryazantseva, *Nonlinear Ill Posed Problems of Monotone Type*, Springer, Dordrecht, 2006.

[4] Ya. I. Alber, *Metric and Generalized Projection Operators in Banach Spaces: Properties and Applications*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996.

[5] F. Alvarez, *Weak Convergence of a Relaxed and Inertial Hybrid Projection-Proximal Point Algorithm for Maximal Monotone Operators in Hilbert Space*, SIAM Journal on Optimization, 2004, 14(3), 773–782.

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

[6] F. Alvarez and H. Attouch, *An Inertial Proximal Method for Maximal Monotone Operators via Discretization of a Nonlinear Oscillator with Damping*, Set-Valued Analysis, 2001, 9(1–2), 3–11.

[7] A. Beck and M. Teboulle, *A Fast Iterative Shrinkage-Thresholding Algorithm for Linear Inverse Problems*, SIAM Journal on Imaging Science, 2009, 2(1), 183–202.

[8] R. I. Bot, E. R. Csetnek and C. Hendrich, *Inertial Douglas–Rachford splitting for monotone inclusion problems*, Applied Mathematics and Computation, 2015, 256, 472–487.

[9] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems, 2002, 18(2), 441–453.

[10] Y. Censor and A. Segal, *The split common fixed point problem for directed operators*, Journal of Convex Analysis, 2009, 16(2), 587–600.

[11] R. Chen, R. H. Chan, S. Ma and J. Yang, *Inertial Proximal ADMM for Linearly Constrained Separable Convex Optimization*, SIAM Journal on Imaging Sciences, 2015, 8(4), 2239–2267.

[12] C. E. Chidume and A. Adamu, *On split generalized mixed equality equilibrium and split equality fixed point problems*, Applied Set-Valued Analysis and Optimization, 2020, 2(3), 273–283.

[13] C. E. Chidume and A. Adamu, *A new iterative algorithm for split feasibility and fixed point problem*, Journal of Nonlinear and Varaiational Analysis, 2021, 5(2), 201–210.

[14] C. E. Chidume and A. Adamu, *Solving Split Equality Fixed Point Problem for Quasi-Phi-Nonexpansive Mappings*, Thai Journal of Mathematics, 2021, 19(4), 1699-1717.

[15] C. E. Chidume, A. Adamu and M. O. Nnakwe, *Strong convergence of an inertial algorithm for maximal monotone inclusions with applications*, Fixed Point Theory and Applications, 2020, 13, 22 pages.

[16] C. E. Chidume, A. Adamu and M. O. Nnakwe, *An Inertial Algorithm for Solving Hammerstein Equations*, Symmetry, 2021, 13(3), 376, 13 pages.

[17] C. E. Chidume, A. Adamu and L. C. Okereke, *A Krasnoselskii-Type Algorithm for Approximating Solutions of Variational Inequality Problems and Convex Feasibility Problems*, Journal of Nonlinear Variational Analysis, 2018, 2(2), 203–218.

[18] C. E. Chidume, S. I. Ikechukwu and A. Adamu, *Inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps*, Fixed Point Theory and Applications, 2018, 9, 9 pages.

[19] C. E. Chidume, P. Kumam and A. Adamu, *A hybrid inertial algorithm for approximating solution of convex feasibility problems with applications*, Fixed Point Theory and Applications, 2020, 12, 17 pages.

[20] C. E. Chidume, L. Okereke and A. Adamu, *A hybrid algorithm for approximating solutions of a variational inequality problem and a convex feasibility problem*, Advances in Nonlinear Variational Inequalities, 2018, 21(1), 46–64.

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

[21] C. E. Chidume, P. Ndambomve and A. U. Bello, *The split equality fixed point problem for demicontractive mappings*, Journal of Nonlinear Analysis and Optimization, 2015, 6(1), 61–69.

[22] C. E. Chidume, O. M. Romanus and U. V. Nyaba, *An iterative algorithm for solving split equality fixed point problems for a class of nonexpansive-type mappings in Banach spaces*, Numerical Algorithms, 2019, 82(3), 987–1007.

[23] Q. Dong, H. Yuan, Y. J. Cho and Th. M. Rassias, *Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings*, Optimization Letters, 2018, 12(1), 87–102.

[24] D. M. Giang, J. J. Strodiot and V. H. Nguyen, *Strong convergence of an iterative method for solving the multiple-set split equality fixed point problem in a real Hilbert space*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 2017, 111(4), 983–998.

[25] J. P. Gossez and E. Lami-Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific Journal of Mathematics, 1972, 40(3), 365–573.

[26] A. H. Ibrahim, P. Kumam, A. B. Abubakar and A. Adamu, *Accelerated derivative-free method for nonlinear monotone equations with an application*, Numerical Linear Algebra with with Applications, 2022, 29(3), e2424, 27 pages.

[27] S. Kamimura and W. Takahashi, *Strong Convergence of a Proximal-Type Algorithm in a Banach Space*, SIAM Journal on Optimization, 2002, 13(3), 938–945.

[28] D. A. Lorenz and T. Pock, *An Inertial Forward-Backward Algorithm for Monotone Inclusions*, Journal of Mathematical Imaging, 2015, 51(2), 311–325.

[29] A. Moudafi, *Alternating CQ-algorithm for convex feasibility and split fixed-point problems*, Journal of Nonlinear Convex Analysis, 2014, 15(4), 809–818.

[30] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bulletin of the American Mathematical Society, 1967, 73(4), 591–597.

[31] B. T. Polyak, *Some methods of speeding up the convergence of iteration methods*, USSR Computational Mathematics and Mathematical Physics, 1964, 4(5), 1–17.

[32] L. Wei and H. Zhou, *The new iterative scheme with errors of zero point for maximal operator in Banach space*, Mathematica Applicata, 2006, 19(1), 101–105.

[33] Y. Wu, R. Chen and L. Shi, *Split equality problem and multiple-sets split equality problem for quasi-nonexpansive multi-valued mappings*, Journal of Inequalities and Applications, 2014, 428, 8 pages.

[34] H. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Analysis: Theory, Methods & Applications, 1991, 16(12), 1127–1138.

[35] S. Zhang, *Generalized mixed equilibrium problem in Banach spaces*, Applied Mathematics and Mechanics, 2009, 30(9), 1105–1112.

[36] J. Zhao, *Solving split equality fixed-point problem of quasi-nonexpansive mappings without prior knowledge of operators norms*, Optimization, 2015, 64(12), 2619–2630.

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS

[37] J. Zhao, H. Zong, G. Liu and H. Zhang, *Solving variational inequality and split equality common fixed-point problem without prior knowledge of operator norms*, Journal of Nonlinear Science and Applications, 2016, 9(9), 5428–5440.

DOI https://doi.org/10.12150/jnma.2023.54 | Generated on 2024-12-19 05:20:11
OPEN ACCESS