Journal of Nonlinear Modeling and Analysis
Volume 5, Number 1, March 2023, 95–107
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2023.95

Stability and Bifurcation Analysis in a Nonlocal Diffusive Predator-prey Model with Hunting Cooperation*

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Abstract In this paper, we propose a diffusive predator-prey model with hunting cooperation and nonlocal competition. Under a rather general selection of the kernel function, we first study the stability of the positive equilibrium of the model. Then, we obtain the conditions which Hopf bifurcation and Turing bifurcation occur. Our results show that nonlocal competition plays an important role in determining the dynamics of the model.

Keywords Predator-prey system, Nonlocal competition, Stability, Hopf bifurcation, Turing bifurcation.

MSC(2010) 35B32, 35B35, 35K57, 92D25.

1. Introduction

Recently, the predator-prey models with hunting cooperation have been widely studied by many researchers in the literature such as [1, 5, 9, 13, 16–21, 23, 24] for their importance in the real world. For instance, to better understand the impact of cooperative hunting upon the two trophic-level interactions, Alves and Hilker [1] proposed the following model with hunting cooperation in predators

$$\begin{cases} \frac{du}{dt} = ru(1-\frac{u}{K}) - (\lambda + av)uv, \\ \frac{dv}{dt} = ev(\lambda + av)u - mv, \end{cases} \tag{1.1}$$

where $u(t)$ and $v(t)$ represent prey and predator densities at the time t respectively, r is the per capita intrinsic growth rate of prey, K is the carrying capacity of prey, e is the conversion efficiency and m is the per capita mortality rate of predators. λ is the attack rate per predator and prey, and a is a parameter describing predator cooperation in hunting. All parameters are positive. They investigated the existence and stability of the positive equilibrium, and showed that hunting cooperation is beneficial to the predator population by the increasing attack rate. Introducing Allee effect into model (1.1), Jang, Zhang and Larriva [9] investigated the existence and stability of the positive equilibrium, and presented the optimal control problem by numerical simulations. Then, not only the impact of hunting cooperation among

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[*]The authors were supported by the National Natural Science Foundation of China (Grant Nos. 11971143 and 12271088).

predators but also predator-induced fear in prey population was considered by Pal et al. [13]. Sen, Ghorai and Banerjee [17] proposed a predator-prey model with Allee effect in prey growth rate and applied Holling type II functional response mechanism to describe the hunting cooperation. It is shown that the strong Allee effect in prey growth rate is able to strengthen the stability of the coexisting steady state.

More recently, the diffusion terms d_1 and d_2 have been introduced into model (1.1) and the corresponding diffusive model

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u + u(r(1-\frac{u}{K}) - (\lambda + av)v), \\ \frac{\partial v}{\partial t} = d_2\Delta v + v(e(\lambda + av)u - m) \end{cases} \tag{1.2}$$

has been studied by several scholars. Capone et al. [5] investigated the stability of the coexistence equilibria and obtained the conditions of Turing instability. Ryu and Ko [16] also obtained the asymptotic behaviour of positive steady state solutions when the cooperation effect of the predators is strong. The stability of positive constant steady state solution, Hopf bifurcation and Turing instability were studied by Wu and Zhao [23]. It is shown that the spatial model (1.2) can reserve the stability of the positive constant steady state solution, when the predation diffusion is not smaller than the prey diffusion. The complex patterns, such as spotted pattern, stripe pattern and mixed pattern, were obtained by Singh, Dubey and Mishra [19]. The results showed the effect of hunting cooperation in pattern dynamics of the diffusive model. Most recently, Song et al. [20] introduced the cross-diffusion into (1.2), and studied the stability and cross-diffusion driven Turing instability.

In addition, Singh and Banerjee [18] incorporated diffusion and Holling type II functional response in the predator-prey model with cooperative behavior in predators and also obtained complex patterns. The properties of the model were investigated by using extensive numerical simulations. Song et al. [21] considered a diffusive predator-prey model where the functional response follows the predator cooperation in hunting and the growth of the prey obeys the Allee effect. They investigated the diffusion-driven Turing instability, and derived the amplitude equation of Turing bifurcation by employing the weakly nonlinear analysis method. Wu and Song [24] introduced self-diffusion into the predator-prey model with hunting cooperation. Their research showed that Turing instability is induced by diffusion, and the conditions for Turing bifurcation to occur have been obtained.

For simplicity, we introduce the nondimensional parameters into model (1.2)

$$\sigma = \frac{r}{m} > 0, \quad \beta = \frac{e\lambda K}{m} > 0, \quad \alpha = \frac{am}{\lambda^2} \geq 0, \quad \bar{d_1} = \frac{1}{m}d_1, \quad \bar{d_2} = \frac{1}{m}d_2$$

and the other nondimensional variables

$$\bar{u} = \frac{e\lambda}{m}u, \quad \bar{v} = \frac{\lambda}{m}v, \quad \bar{t} = mt.$$

For the simplicity of notations, dropping the over-bars, then model (1.2) becomes

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u + u(\sigma(1-\frac{u}{\beta}) - (1 + \alpha v)v), \\ \frac{\partial v}{\partial t} = d_2\Delta v + v((1 + \alpha v)u - 1). \end{cases} \tag{1.3}$$

It is realized that in the real world, prey needs to interact with other prey or predators not only in the same location, but also in different locations, or even in the whole space. For example, researchers have realized that competition is often nonlocal, and it is also necessary to add the nonlocal interaction term to the reaction dynamics. Therefore, many researchers [2–4, 6–8, 10–12, 14, 15, 22] have dedicated themselves to studying the reaction-diffusion models with nonlocal competition. However, according to what we have learned, there does not exist theoretical results of nonlocal effect for the above system (1.3). Motivated by the aforementioned works, we will introduce nonlocal competition into model (1.3). In this paper, we select the nonlocal term given by Furter and Grinfeld [8]. When Ω is a one-dimensional bounded domain $(0, l\pi)$ with $l > 0$, the first equation of model (1.3) is rewritten as

$$\frac{\partial u}{\partial t} - d_1 u_{xx} = u\left[\sigma\left(1 - \frac{1}{\beta}\int_{\Omega} K(x,y)u(y,t)dy\right) - (1+\alpha v)v\right],$$

where the kernel function $K(x, y)$ is defined by

$$K(x,y) = \frac{1}{\mid \Omega \mid} = \frac{1}{l\pi}$$

in its simplest form.

Supplemented with Neumann boundary conditions and the nonnegative initial conditions, we are proposing the following model

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 u_{xx} = u\left[\sigma\left(1 - \frac{1}{\beta l\pi}\int_0^{l\pi} u(y,t)dy\right) - (1+\alpha v)v\right], & x \in (0, l\pi), t > 0, \\ \frac{\partial v}{\partial t} - d_2 v_{xx} = v[(1+\alpha v)u - 1], & x \in (0, l\pi), t > 0, \\ u_x(0,t) = u_x(l\pi,t) = 0,\ v_x(0,t) = v_x(l\pi,t) = 0, & t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in (0, l\pi). \end{cases} \tag{1.4}$$

The aim of this paper is to study the stability of the positive constant equilibrium and perform bifurcation analysis. The rest of this work is organized as follows. In Section 2, we linearize model (1.4) at the positive equilibrium, from which we are able to study the stability through qualitative theory. We want to see how the nonlocal competition term affects the stability of the positive equilibrium. In Section 3, we take diffusion coefficient as the bifurcation parameter to obtain the conditions for the occurrence of Hopf bifurcation and Turing bifurcation. Finally, we conclude this paper with a short discussion in Section 4.

2. Stability analysis

In this section, we mainly discuss the stability of model (1.4). It is easy to know that the positive equilibria of model (1.4) are the same as model (1.3). With regard to the existence of the positive equilibrium in model (1.3), Song et al. [20] proved the following results.

Lemma 2.1. *Assume that σ, α and β are positive constants.*
(1) If either $0<\sigma\alpha\le 1$ and $0<\beta\le 1$, or $\sigma\alpha>1$ and $0<\beta<\beta_*$, then model (1.3) has no positive equilibrium;
(2) If either $\sigma\alpha>0$ and $\beta>1$, or $\sigma\alpha>1$ and $\beta=1$, then model (1.3) has a unique positive equilibrium (u_*,v_*), and $P'(v_*)>0$;
(3) If $\sigma\alpha>1$ and $\beta=\beta_*$, then model (1.3) has a unique positive equilibrium (u_*,v_*) and $P'(v_*)=0$;
(4) If $\sigma\alpha>1$ and $\beta_*<\beta<1$, then model (1.3) has two positive equilibria (u_{1*},v_{1*}) and (u_{2*},v_{2*}). Further, if $0<v_{2*}<v_{1*}$, then $P'(v_{1*})>0$ and $P'(v_{2*})>0$, where $\beta_*=\frac{27\sigma\alpha}{2+9\sigma\alpha+2(1+3\sigma\alpha)\sqrt{1+3\sigma\alpha}}>0$, $\quad P(v)=\beta\alpha^2v^3+2\beta\alpha v^2+\beta(1-\alpha\sigma)v+\sigma(1-\beta)$.

Remark 2.1. It follows from Lemma 2.1 that if $\sigma\alpha>1$ and $\beta=1$, then model (1.3) has a unique positive equilibrium $(u_*,v_*)=(\frac{1}{\sqrt{\sigma\alpha}},\frac{\sqrt{\sigma\alpha}-1}{\alpha})$, which can be calculated explicitly, and if $\sigma\alpha>1$ and $\beta=\beta_*$, then the unique positive equilibrium can be written as $(u_{1*},v_{1*})=(\frac{\sqrt{1+3\sigma\alpha}-1}{\sigma\alpha},\frac{-2+\sqrt{1+3\sigma\alpha}}{3\alpha})$. If $\sigma\alpha>0$ and $\beta>1$, or $\sigma\alpha>1$ and $\beta_*<\beta<1$, we know that the positive equilibria exist, but we cannot express them explicitly.

In this paper, for the sake of convenience, we choose the case of $(u_*,v_*)=(\frac{1}{\sqrt{\sigma\alpha}},\frac{\sqrt{\sigma\alpha}-1}{\alpha})$. The method used can be applied to some other cases as well, providing that the positive equilibrium can be expressed explicitly.

Let $E_*=(u_*,v_*)=(\frac{1}{\sqrt{\sigma\alpha}},\frac{\sqrt{\sigma\alpha}-1}{\alpha})$. And the linearized system of model (1.4) at the positive equilibrium E_* is given by

$$\begin{cases} u_t=d_1u_{xx}-\dfrac{\sigma}{l\pi\sqrt{\sigma\alpha}}\displaystyle\int_0^{l\pi}u(y,t)dy-\frac{2\sqrt{\sigma\alpha}-1}{\sqrt{\sigma\alpha}}v, \; x\in(0,l\pi),\ t>0,\\ v_t=d_2v_{xx}+\frac{\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1)}{\alpha}u+\frac{\sqrt{\sigma\alpha}-1}{\sqrt{\sigma\alpha}}v, \quad x\in(0,l\pi),\ t>0,\\ u_x(0,t)=u_x(l\pi,t)=v_x(0,t)=v_x(l\pi,t)=0, \quad t>0,\\ u(x,0)=u_0(x)\ge 0,\quad v(x,0)=v_0(x)\ge 0, \quad x\in(0,l\pi). \end{cases} \tag{2.1}$$

Then, the characteristic equation of (2.1) is

$$\lambda^2-T_n\lambda+D_n=0,\quad n\in N=\{1,2,3,\ldots\}, \tag{2.2}$$

where

$$T_n=\frac{\sqrt{\sigma\alpha}-1}{\sqrt{\sigma\alpha}}-(d_1+d_2)\frac{n^2}{l^2}$$

and

$$D_n=\frac{(\sqrt{\sigma\alpha}-1)(2\sqrt{\sigma\alpha}-1)}{\alpha}-\frac{\sqrt{\sigma\alpha}-1}{\sqrt{\sigma\alpha}}d_1\frac{n^2}{l^2}+d_1d_2\frac{n^4}{l^4}.$$

From T_n and D_n, we can obtain the following conclusions.

Lemma 2.2. *Suppose that d_1, d_2, σ and α are positive constants and $\sigma\alpha>1$.*
(1)If $l^2\le\frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$, then $T_n<0$ for any $n\in N$;

DOI https://doi.org/10.12150/jnma.2023.95 | Generated on 2025-04-20 01:37:54

(2) If $l^2 > \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$*, then* $T_n < 0$ *for* $n > N^*$*, where*

$$N^* = \begin{cases} \left[\sqrt{\frac{\sqrt{\sigma\alpha}-1}{\sqrt{\sigma\alpha}d_2}l^2}\ \right] - 1, & \sqrt{\frac{\sqrt{\sigma\alpha}-1}{\sqrt{\sigma\alpha}d_2}l^2} \text{ is a positive integer,} \\ \left[\sqrt{\frac{\sqrt{\sigma\alpha}-1}{\sqrt{\sigma\alpha}d_2}l^2}\ \right], & \sqrt{\frac{\sqrt{\sigma\alpha}-1}{\sqrt{\sigma\alpha}d_2}l^2} \text{ is not a positive integer.} \end{cases} \tag{2.3}$$

However, for $1 \le n \le N^*$*,*

$$T_n \begin{cases} < 0, \ for\ d_1 > d_{1,n}^H, \\ = 0, \ for\ d_1 = d_{1,n}^H, \\ > 0, \ for\ d_1 < d_{1,n}^H, \end{cases} \tag{2.4}$$

where

$$d_{1,n}^H = \frac{(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2n^2}{\sqrt{\sigma\alpha}n^2}. \tag{2.5}$$

Proof. T_n can be rewritten as

$$T_n = \frac{1}{\sqrt{\sigma\alpha}l^2}\left[\left((\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2n^2\right) - \sqrt{\sigma\alpha}d_1n^2\right]. \tag{2.6}$$

(1) If $l^2 \le \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$, then we have $(\sqrt{\sigma\alpha}-1)l^2 \le \sqrt{\sigma\alpha}d_2 \le \sqrt{\sigma\alpha}d_2n^2$ for any $n \in N$, which implies $(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2n^2 \le 0$. Thus, it follows from (2.6) that $T_n < 0$ for any $n \in N$.
(2) If $l^2 > \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$, then we can obtain $(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2n^2 \ge 0$ for $1 \le n \le N^*$ and $(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2n^2 \le 0$ for $n > N^*$. Here, N^* is defined by (2.3). Thereby, from (2.6), we have $T_n < 0$ for $n > N^*$. However, for $1 \le n \le N^*$, along with (2.6), we have $T_n < 0 \Leftrightarrow d_1 > d_{1,n}^H$, $T_n = 0 \Leftrightarrow d_1 = d_{1,n}^H$ and $T_n > 0 \Leftrightarrow d_1 < d_{1,n}^H$, where $d_{1,n}^H$ is defined by (2.5). Thus, we complete the proof. □

Lemma 2.3. *Suppose that* d_1*,* d_2*,* σ *and* α *are positive constants and* $\sigma\alpha > 1$*.*
(1) If $l^2 \le \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$*, then* $D_n > 0$ *for any* $n \in N$*;*
(2) If $l^2 > \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$*, then* $D_n > 0$ *for* $n > N^*$*, but for* $1 \le n \le N^*$*,*

$$D_n \begin{cases} < 0, \ for\ d_1 > d_{1,n}^T, \\ = 0, \ for\ d_1 = d_{1,n}^T, \\ > 0, \ for\ d_1 < d_{1,n}^T, \end{cases} \tag{2.7}$$

where N^* *is defined by* (2.3) *and*

$$d_{1,n}^T = \frac{\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1)(2\sqrt{\sigma\alpha}-1)l^4}{\alpha n^2[(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2n^2]}. \tag{2.8}$$

Proof. We can rewrite D_n as

$$D_n = \frac{1}{\alpha\sqrt{\sigma\alpha}l^4}\{\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1)(2\sqrt{\sigma\alpha}-1)l^4 - \alpha d_1n^2[(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2n^2]\}. \tag{2.9}$$

(1) If $l^2 \le \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$, then we have $(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2 \le 0$. As a consequence, we obtain $(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2 n^2 \le 0$ for any $n \in N$. Thanks to $\sigma\alpha > 1$, that is, $\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1)(2\sqrt{\sigma\alpha}-1)l^4 > 0$, it follows from (2.9) that $D_n > 0$ for any $n \in N$.
(2) If $l^2 > \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$, it can be proved by a method similar to (2) of Lemma 2.2. The difference is that $d^T_{1,n}$ is obtained from the following equation

$$\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1)(2\sqrt{\sigma\alpha}-1)l^4 - \alpha d_1 n^2((\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2 n^2) = 0.$$

□

Combining Lemmas 2.2 and 2.3, we have the following conclusion.

Theorem 2.1. *Let $\beta = 1$. Assume that d_1, d_2, σ, α are positive constants and $\sigma\alpha > 1$. Then, the positive equilibrium $E_* = (\frac{1}{\sqrt{\sigma\alpha}}, \frac{\sqrt{\sigma\alpha}-1}{\alpha})$ of system (1.4) is locally asymptotically stable, when $l^2 \le \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$.*

Proof. From Lemmas 2.2 and 2.3, we know that $T_n < 0$ and $D_n > 0$ always hold for any $n \in N$, when $l^2 \le \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$. Therefore, the theorem is proved. □

If $l^2 > \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$, then for $d^H_{1,n}$ and $d^T_{1,n}$, $1 \le n \le N^*$, which are given by (2.4) and (2.7) respectively, the following conclusion can be easily proved.

Lemma 2.4. *Assume that d_1, d_2, σ and α are positive constants and $\sigma\alpha > 1$. If $l^2 > \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$, then the following statements hold true:*
(1) $d^H_{1,1} > d^H_{1,2} > \cdots > d^H_{1,N^*}$;
(2) $d^T_{1,1} > d^T_{1,2} > \cdots > d^T_{1,N_*} < d^T_{1,N_*+1} < d^T_{1,N_*+2} < \cdots < d^T_{1,N^*}$, where N_* is defined by

$$N_* = \begin{cases} \left[\sqrt{\frac{\sqrt{\sigma\alpha}-1}{2\sqrt{\sigma\alpha}d_2}l^2}\,\right] - 1, & \sqrt{\frac{\sqrt{\sigma\alpha}-1}{2\sqrt{\sigma\alpha}d_2}l^2} \text{ is a positive integer}, \\ \left[\sqrt{\frac{\sqrt{\sigma\alpha}-1}{2\sqrt{\sigma\alpha}d_2}l^2}\,\right], & \sqrt{\frac{\sqrt{\sigma\alpha}-1}{2\sqrt{\sigma\alpha}d_2}l^2} \text{ is not a positive integer}. \end{cases} \tag{2.10}$$

According to Lemmas 2.2, 2.3 and 2.4, we can prove the following theorem.

Theorem 2.2. *Let $\beta = 1$. Assume that d_1, d_2, σ and α are positive constants and $\sigma\alpha > 1$. If $\frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1} < l^2 < \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1} + \frac{4\sigma\sqrt{\sigma\alpha}(2\sqrt{\sigma\alpha}-1)d_2}{(\sqrt{\sigma\alpha}-1)^2}$, then the positive equilibrium $E_* = (\frac{1}{\sqrt{\sigma\alpha}}, \frac{\sqrt{\sigma\alpha}-1}{\alpha})$ of system (1.4) is locally asymptotically stable.*

Proof. If $l^2 > \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$, then it follows from Lemma 2.2(2) and Lemma 2.3(2) that $T_n < 0$ and $D_n > 0$ for $n > N^*$. However, for $1 \le n \le N^*$, along with Lemma 2.4, we have $T_n < 0$, when $d_1 > d^H_{1,1}$ and $D_n > 0$, when $d_1 < d^T_{1,N_*}$. Here, N_*, $d^H_{1,1}$ and d^T_{1,N_*} are defined by (2.10), (2.5) and (2.8) respectively. That is, for $1 \le n \le N^*$, $T_n < 0$ and $D_n > 0$ hold, providing that $d^H_{1,1} < d^T_{1,N_*}$. Thanks to $\sigma\alpha > 1$ and $l^2 < \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1} + \frac{4\sigma\sqrt{\sigma\alpha}(2\sqrt{\sigma\alpha}-1)d_2}{(\sqrt{\sigma\alpha}-1)^2}$, we obtain

$$\frac{(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}} < \frac{4\sigma(2\sqrt{\sigma\alpha}-1)d_2}{\sqrt{\sigma\alpha}-1}.$$

Noticing that

$$d^T_{1,N_*} = \frac{\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1)(2\sqrt{\sigma\alpha}-1)l^4}{\alpha N_*^2[(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2 N_*^2]} > \frac{4\sigma(2\sqrt{\sigma\alpha}-1)d_2}{\sqrt{\sigma\alpha}-1}$$

and

$$d_{1,1}^H = \frac{(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}},$$

now, it follows that $d_{1,1}^H < d_{1,N_*}^T$. As a consequence, we obtain $T_n < 0$ and $D_n > 0$ for any $n \in N$, if $\frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1} < l^2 < \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1} + \frac{4\sigma\sqrt{\sigma\alpha}(2\sqrt{\sigma\alpha}-1)d_2}{(\sqrt{\sigma\alpha}-1)^2}$, and this completes the proof of the theorem. □

3. Bifurcation analysis

Through the discussion in Section 2, we know that the stability of the positive equilibrium of (1.4) may be destroyed under the condition $l^2 > \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$. Thus, in this section, we perform bifurcation analysis for system (1.4) and derive the conditions that (1.4) undergoes Turing bifurcation and Hopf bifurcation.

Choose d_1 as the bifurcation parameter. We know that system (1.4) undergoes Hopf bifurcation at d_1^H, if d_1^H satisfies
(***H1***) $T_n(d_1^H) = 0,\ D_n(d_1^H) > 0$ and $\frac{\mathrm{d}}{\mathrm{d}d_1}Re\lambda(d_1^H) \neq 0$, for some $n \in N$, and $T_j(d_1^H) \neq 0,\ D_j(d_1^H) \neq 0$ for $j \neq n,\ j \in N$.

Morever, system (1.4) undergoes Turing bifurcation at d_1^T, if d_1^T satisfies
(***H2***) $T_n(d_1^T) \neq 0,\ D_n(d_1^T) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}d_1}D_n(d_1^T) \neq 0$, for some $n \in N$, and $T_j(d_1^T) \neq 0,\ D_j(d_1^T) \neq 0$ for $j \neq n,\ j \in N$.

For the sake of convenience of the discussion below, let

$$f(\alpha) = \sqrt{\sigma}(2\sigma - 1)\sqrt{\alpha} + 1 - \sigma, \tag{3.1}$$

and we first prove the following lemma.

Lemma 3.1. *Assume that σ, α are positive constants and $\sigma\alpha > 1$, function f is given by* (3.1).
(1) If $0 < \sigma < \frac{1}{2}$, then

$$f(\alpha)\begin{cases} < 0,\ for\ \alpha > \alpha_0, \\ = 0,\ for\ \alpha = \alpha_0, \\ > 0,\ for\ \frac{1}{\sigma} < \alpha < \alpha_0, \end{cases} \tag{3.2}$$

where

$$\alpha_0 = \frac{(1-\sigma)^2}{\sigma(1-2\sigma)^2}; \tag{3.3}$$

(2) If $\sigma \geq \frac{1}{2}$, then $f(\alpha) > 0$ holds for $\alpha > \frac{1}{\sigma}$.

Proof. (1) If $0 < \sigma < \frac{1}{2}$, then it is obvious from function (3.1) that $f(\alpha_0) = 0$, where α_0 is given by (3.3). Meanwhile, the other two cases of (3.2) are easy to be proved.
(2) If $\sigma = \frac{1}{2}$, then we have $f(\alpha) = \frac{1}{2} > 0$ for $\alpha > \frac{1}{\sigma}$. Thanks to $\sigma\alpha > 1$, which implies that $\sigma > \frac{1}{\alpha}$. Thus, if $\sigma > \frac{1}{2}$, then we have

$$f(\alpha) = \sqrt{\sigma}(2\sigma-1)\sqrt{\alpha} + 1 - \sigma > \frac{1}{\sqrt{\alpha}}(2\sigma-1)\sqrt{\alpha} + 1 - \sigma = \sigma > \frac{1}{2} > 0.$$

Thus, Lemma 3.1 is proved. □

Theorem 3.1. *Let $\beta = 1$. Assume that d_1, d_2, σ and α are positive constants, $\sigma\alpha > 1$ and $l^2 > \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}$.*
(1) If $0 < \sigma < \frac{1}{2}$, then
(1a) for $\frac{1}{\sigma} < \alpha \le \alpha_0$, system (1.4) undergoes a Hopf bifurcation at $d_{1,n}^H$, for $1 \le n \le N^*$;
(1b) for $\alpha > \alpha_0$, system (1.4) undergoes a Hopf bifurcation at $d_{1,n}^H$, for $1 \le n \le N^*$, when $l^2 \in \left(\frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha}-1}, l_+^2\right)$, where α_0 is defined by (3.3) and

$$l_+^2 = \frac{\sqrt{\sigma\alpha}d_2}{\sqrt{\sigma\alpha} - 1 - \sqrt{\sigma(\sqrt{\sigma\alpha}-1)(2\sqrt{\sigma\alpha}-1)}}. \tag{3.4}$$

(2) If $\sigma \ge \frac{1}{2}$, system (1.4) undergoes a Hopf bifurcation at $d_{1,n}^H$ for $1 \le n \le N^*$, when $\alpha > \frac{1}{\sigma}$.

Proof. It is obvious from (2.4) that $T_n(d_{1,n}^H) = 0$ for $1 \le n \le N^*$. Substituting $d_{1,n}^H$ into D_n, we have

$$D_n(d_{1,n}^H) = \frac{1}{\sigma\alpha l^4}\left[-\sigma\alpha d_2^2 n^4 + 2\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1)d_2 l^2 n^2 + (\sqrt{\sigma\alpha}-1)f(\alpha)l^4\right], \tag{3.5}$$

where $f(\alpha)$ is defined by (3.1).

Next, we want to prove the following fact

$$-\sigma\alpha d_2^2 n^4 + 2\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1)d_2 l^2 n^2 + (\sqrt{\sigma\alpha}-1)f(\alpha)l^4 > 0, \text{ for } 1 \le n \le N^*.$$

Let

$$f_1(x) = -\sigma\alpha d_2^2 x^2 + 2\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1)d_2 l^2 x + (\sqrt{\sigma\alpha}-1)f(\alpha)l^4, \ x \ge 1.$$

(1) If $0 < \sigma < \frac{1}{2}$, then from (3.2) in Lemma 3.1, we have $f(\alpha) > 0$, when $\frac{1}{\sigma} < \alpha < \alpha_0$ and $f(\alpha) = 0$, when $\alpha = \alpha_0$. Combining with the expression of $f_1(x)$, we obtain

$$\begin{cases} f_1(x) > 0 \Leftrightarrow 0 < x < \frac{2(\sqrt{\sigma\alpha}-1)}{\sqrt{\sigma\alpha}d_2}l^2, \text{ for } f(\alpha) = 0, \\ f_1(x) > 0 \Leftrightarrow x_1 < x < x_2, \qquad\qquad \text{ for } f(\alpha) > 0, \end{cases} \tag{3.6}$$

where

$$x_1 = \frac{(\sqrt{\sigma\alpha}-1)l^2 - \sqrt{\sigma(\sqrt{\sigma\alpha}-1)(2\sqrt{\sigma\alpha}-1)}l^2}{\sqrt{\sigma\alpha}d_2} \tag{3.7}$$

and

$$x_2 = \frac{(\sqrt{\sigma\alpha}-1)l^2 + \sqrt{\sigma(\sqrt{\sigma\alpha}-1)(2\sqrt{\sigma\alpha}-1)}l^2}{\sqrt{\sigma\alpha}d_2} \tag{3.8}$$

are two roots of the equation $f_1(x) = 0$.

$(1a)$ For the case with $\frac{1}{\sigma} < \alpha \le \alpha_0$.

If $\alpha = \alpha_0$, due to $(N^*)^2 < \frac{2(\sqrt{\sigma\alpha}-1)}{\sqrt{\sigma\alpha}d_2}l^2$, then it follows from the first line in (3.6) that $f_1(n^2) > 0$ for $1 \le n \le N^*$. If $\frac{1}{\sigma} < \alpha < \alpha_0$, from Lemma 3.1, we know $f(\alpha) > 0$, which implies $x_1 < 0$. It also follows from (2.3) and (3.8) that

$(N^*)^2 < x_2$. Then, we obtain $f_1(n^2) > 0$ for $1 \le n \le N^*$ from the second line in (3.6), i.e., we prove $D_n(d_{1,n}^H) > 0$ for $1 \le n \le N^*$, when $0 < \sigma < \frac{1}{2}$ and $\frac{1}{\sigma} < \alpha \le \alpha_0$.

Furthermore, we have

$$\frac{\mathrm{d}}{\mathrm{d}d_1} Re\lambda(d_{1,n}^H) = -\frac{n^2}{2l^2} \neq 0 \text{ for } 1 \le n \le N^*,$$

and when $n > N^*$, from (2) of Lemmas 2.2 and 2.3 respectively, we obtain

$$T_n(d_{1,j}^H) < 0 \text{ and } D_n(d_{1,j}^H) > 0 \text{ for } j \in N \text{ and } j \neq 1, 2, \ldots, N^*.$$

According to (***H1***), case ($1a$) is proved.

($1b$) For the case with $\alpha > \alpha_0$.

If $\alpha > \alpha_0$, it follows from (3.2) that $f(\alpha) < 0$. Thus, $f_1(x) > 0$ is valid, if and only if $x_1 < x < x_2$, where x_1 and x_2 are two positive roots of the equation $f_1(x) = 0$ and given by (3.7) and (3.8) respectively. Noticing that $(N^*)^2 < x_2$, as long as we prove $x_1 < 1$, we can get $D_n(d_{1,n}^T) > 0$ for $1 \le n \le N^*$.

In fact, we have

$$x_1 - 1 = \frac{1}{\sqrt{\sigma\alpha} d_2} \times \frac{-(\sqrt{\sigma\alpha} - 1) f(\alpha) l^4 - 2\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha} - 1) d_2 l^2 + \sigma\alpha d_2^2}{(\sqrt{\sigma\alpha} - 1) l^2 + \sqrt{\sigma(\sqrt{\sigma\alpha} - 1)(2\sqrt{\sigma\alpha} - 1)} l^2 - \sqrt{\sigma\alpha} d_2}$$

and

$$(\sqrt{\sigma\alpha} - 1) l^2 + \sqrt{\sigma(\sqrt{\sigma\alpha} - 1)(2\sqrt{\sigma\alpha} - 1)} l^2 - \sqrt{\sigma\alpha} d_2 > 0,$$

which is obtained from $l^2 > \frac{\sqrt{\sigma\alpha} d_2}{\sqrt{\sigma\alpha} - 1} > \frac{\sqrt{\sigma\alpha} d_2}{\sqrt{\sigma\alpha} - 1 + \sqrt{\sigma(\sqrt{\sigma\alpha} - 1)(2\sqrt{\sigma\alpha} - 1)}}$.

Let

$$g(l^2) = -(\sqrt{\sigma\alpha} - 1) f(\alpha) l^4 - 2\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha} - 1) d_2 l^2 + \sigma\alpha d_2^2. \tag{3.9}$$

Then from (3.9) and $\frac{\sqrt{\sigma\alpha} d_2}{\sqrt{\sigma\alpha} - 1} < l^2 < l_+^2$, we have $g(l^2) < 0$, which implies $x_1 < 1$. Therefore, we prove that $D_n(d_{1,n}^H) > 0$ is true for $1 \le n \le N^*$. The following proving process is similar to the above case $\frac{1}{\sigma} < \alpha \le \alpha_0$. Then, according to (***H1***), case ($1b$) is proved.

(2) If $\sigma \ge \frac{1}{2}$, combining case (2) of Lemma 3.1, the proof method is completely similar to that used in Theorem 3.1($1a$).

Thus, we complete the proof of this theorem. □

Theorem 3.2. *Let $\beta = 1$. Assuming that d_1, d_2, σ and α are positive constants, $\sigma\alpha > 1$ and $l^2 > \frac{\sqrt{\sigma\alpha} d_2}{\sqrt{\sigma\alpha} - 1}$.*
(1) If $0 < \sigma < \frac{1}{2}$, then
(1a) for $\frac{1}{\sigma} < \alpha \le \alpha_0$, system (1.4) undergoes a Turing bifurcation at $d_{1,n}^T$ for $1 \le n \le N^*$;
(1b) for $\alpha > \alpha_0$, system (1.4) undergoes a Turing bifurcation at $d_{1,n}^T$ for $1 \le n \le N^*$, when $l^2 \in \left(\frac{\sqrt{\sigma\alpha} d_2}{\sqrt{\sigma\alpha} - 1}, l_+^2\right)$, where α_0 and l_+^2 are defined by (3.3) and (3.4) respectively.
(2) If $\sigma \ge \frac{1}{2}$, then system (1.4) undergoes a Turing bifurcation at $d_{1,n}^H$ for $1 \le n \le N^*$, when $\alpha > \frac{1}{\sigma}$.

Proof. Obviously, it follows from (2.7) that $D_n(d_{1,n}^T) = 0$ for $1 \le n \le N^*$. Substituting $d_{1,n}^T$ into T_n, we have

$$T_n(d_{1,n}^T) = \frac{1}{\sqrt{\sigma\alpha} l^2} \times \frac{\sigma\alpha d_2^2 n^4 - 2\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1) d_2 l^2 n^2 - (\sqrt{\sigma\alpha}-1) f(\alpha) l^4}{(\sqrt{\sigma\alpha}-1) l^2 - \sqrt{\sigma\alpha} d_2 n^2}. \tag{3.10}$$

Since $\sigma\alpha > 1$ and $l^2 > \frac{\sqrt{\sigma\alpha} d_2}{\sqrt{\sigma\alpha}-1}$, we obtain $(\sqrt{\sigma\alpha}-1) l^2 - \sqrt{\sigma\alpha} d_2 n^2 > 0$. Our purpose is to prove the following fact

$$\sigma\alpha d_2^2 n^4 - 2\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1) d_2 l^2 n^2 - (\sqrt{\sigma\alpha}-1) f(\alpha) l^4 \ne 0, \text{ for } 1 \le n \le N^*.$$

Let

$$f_2(x) = \sigma\alpha d_2^2 x^2 - 2\sqrt{\sigma\alpha}(\sqrt{\sigma\alpha}-1) d_2 l^2 x - (\sqrt{\sigma\alpha}-1) f(\alpha) l^4, \ x \ge 1.$$

(1) If $0 < \sigma < \frac{1}{2}$, by Lemma 3.1 again, we know $f(\alpha) = 0$ for $\alpha = \alpha_0$ and $f(\alpha) > 0$ for $\frac{1}{\sigma} < \alpha < \alpha_0$. Then, by the expression of $f_2(x)$, we obtain

$$\begin{cases} f_2(x) = 0 \Leftrightarrow x = 0 \text{ or } x = \frac{2(\sqrt{\sigma\alpha}-1)}{\sqrt{\sigma\alpha} d_2} l^2, \text{ for } & f(\alpha) = 0, \\ f_2(x) = 0 \Leftrightarrow x = x_1 \text{ or } x = x_2, \qquad \text{for} & f(\alpha) < 0, \end{cases} \tag{3.11}$$

where x_1 and x_2 are given by (3.7) and (3.8) respectively. At this time, $x_1 < 0$ and $x_2 > 0$.

(1a) For the case with $\frac{1}{\sigma} < \alpha \le \alpha_0$.

Notice that $(N^*)^2 < \frac{2(\sqrt{\sigma\alpha}-1)}{\sqrt{\sigma\alpha} d_2} l^2$ and $(N^*)^2 < x_2$. Then, from (3.11), we have $T_n(d_{1,n}^T) \ne 0$ for $1 \le n \le N^*$.

(1b) For the case with $\alpha > \alpha_0$.

If $\alpha > \alpha_0$, then from (3.2), we have $f(\alpha) < 0$. Thus, the equation $f_2(x) = 0$ has two positive roots x_1 and x_2, which are defined by (3.7) and (3.8). Similar to the proof of case 1(b) in Theorem 3.1, if $l^2 \in \left(\frac{\sqrt{\sigma\alpha} d_2}{\sqrt{\sigma\alpha}-1}, l_+^2\right)$, then $x_1 < 1$. Noticing that $(N^*)^2 < x_2$, from (3.11) again, we also have $T_n(d_{1,n}^T) \ne 0$ for $1 \le n \le N^*$.

In addition, by computing directly, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}d_1} D_n(d_{1,n}^T) = \frac{n^2}{\sqrt{\sigma\alpha} l^4} \left(\sqrt{\sigma\alpha} d_2 n^2 - (\sqrt{\sigma\alpha}-1) l^2\right) \ne 0 \text{ for } 1 \le n \le N^*.$$

Furthermore, for $N > N^*$, by (2) of Lemma 2.2 and Lemma 2.3, we have

$$T_n(d_{1,j}^T) < 0 \text{ and } D_n(d_{1,j}^T) > 0 \text{ for } j \in N \text{ and } j \ne 1, 2, \ldots, N^*.$$

Thus, according to (***H2***), conclusion (1) is proved.

(2) If $\sigma \ge \frac{1}{2}$, by Lemma 3.1, we have $f(\alpha) > 0$ for $\alpha > \frac{1}{\sigma}$. The rest of the proving process is completely similar to the proof of the case of (1a) in Theorem 3.2.

Thus, we finish the proof of this theorem. $\square$

4. Conclusion

In this work, nonlocal prey competition has been introduced into a predator-prey model with hunting cooperation. First, we investigate the stability of the positive equilibrium E_* of (1.4). Our results show the effect of nonlocal competition terms on the kinetics of system (1.4). When l^2 is smaller than the critical value, E_* is stable and becomes unstable, when l^2 is larger than the critical value. Then, we derive the conditions that (1.4) undergoes Turing bifurcation and Hopf bifurcation. It is interesting to note that when the region is unbounded, due to the different selection of kernel functions, the corresponding characteristic equation is transcendental. Therefore, the research methods will be different, which are left for our next work.

Acknowledgement

The authors thank the two anonymous reviewers and the editors for their careful reading and helpful comments, which have significantly contributed to improving the initial version of the manuscript.

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