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The Exact Limits and Improved Decay Estimates for All Order Derivatives of the Global Weak Solutions to a Two-Dimensional Incompressible Dissipative Quasi-Geostrophic Equation

Linghai Zhang[1,†]

Abstract We will accomplish the exact limits for all order derivatives of the global weak solutions to a two-dimensional incompressible dissipative quasi-geostrophic equation. We will also establish the improved decay estimates with sharp rates for all order derivatives. We will consider two cases for the initial function and the external force and prove the optimal results for both cases. We will couple together existing ideas (including the Fourier transformation and its properties, Parseval's identity, iteration technique, Lebesgue's dominated convergence theorem, Gagliardo-Nirenberg-Sobolev interpolation inequality, squeeze theorem, Cauchy-Schwartz's inequality, etc) existing results (the existence of global weak solutions, the existence of local smooth solution on (T, ∞) and the elementary decay estimate with a sharp rate) and a few novel ideas to obtain the main results.

Keywords Incompressible dissipative quasi-geostrophic equation, All order derivatives of global weak solution, Primary decay estimates, exact limits, Improved decay estimates with sharp rates.

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[†]The corresponding author.
Email: liz5@lehigh.edu
[1]Department of Mathematics, Lehigh University, 17 Memorial Drive East, Bethlehem, PA USA 18015

Outline of the Paper

1. Introduction

1.1. The mathematical model equations and known related results

Consider the Cauchy problem for the two-dimensional incompressible dissipative quasi-geostrophic equation

$$\frac{\partial}{\partial t}u + \alpha(-\triangle)^{\rho}u + J(u, (-\triangle)^{-1/2}u) = f(\mathbf{x}, t), \tag{1.1}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}). \tag{1.2}$$

In this problem, $\alpha > 0$ and $\rho > 0$ are positive constants, $\mathbf{x} = (x, y) \in \mathbb{R}^2$, $u = u(\mathbf{x}, t)$ represents the temperature of the fluid, $(-\triangle)^{-1/2}u$ is called the stream function. The Jacobian determinant is defined by

$$J(u, (-\triangle)^{-1/2}u) = \frac{\partial}{\partial x}u\frac{\partial}{\partial y}(-\triangle)^{-1/2}u - \frac{\partial}{\partial y}u\frac{\partial}{\partial x}(-\triangle)^{-1/2}u.$$

Unlike nonlinear functions in other nonlinear evolution equations, the Jacobian determinant is not only a nonlinear function, but also a nonlocal term. The Fourier transformation of the Jacobian determinant is a totally nontrivial problem. The model equation is called subcritical, critical or supercritical, if $\rho > 1/2$, $\rho = 1/2$ or $\rho < 1/2$, respectively.

The linear operators

$$\frac{\partial}{\partial x}(-\triangle)^{-1/2} \qquad \text{and} \qquad \frac{\partial}{\partial y}(-\triangle)^{-1/2}$$

represent the standard Riesz transformations in $\mathbb{R}^2$. The vector field

$$\mathbf{F}(\mathbf{x}, t) \stackrel{\text{def}}{=} \begin{pmatrix} -\frac{\partial}{\partial y}(-\triangle)^{-1/2}u \\ +\frac{\partial}{\partial x}(-\triangle)^{-1/2}u \end{pmatrix}$$

represents the velocity of the fluid. The fluid is incompressible because $\nabla \cdot \mathbf{F} = 0$. The model is the dimensionally correct analogue of the three-dimensional incompressible Navier-Stokes equations, if $\rho = 1/2$. It is derived from a general quasi-geostrophic equation in the special case of constant potential vorticity and buoyancy frequency. It is a model in geophysical fluid dynamics because it arises in meteorology and oceanography. Therefore, it is of great interest in applied mathematics. In particular, the critical dissipative quasi-geostrophic equation is a very important model for the investigation of the existence of the global smooth solution of the three-dimensional incompressible Navier-Stokes equations.

There have been many contributions to the existence of global smooth solution, global weak solution, elementary decay estimates of the Cauchy problem, for the case $f = 0$. While it is impossible to list all related results, let us mention some previous results closely related to this paper. For the existence of the global smooth solution, particularly the case $1/2 \leq \rho \leq 1$, and the existence of the global weak solutions, particularly the case $0 < \rho < 1/2$, see Chen, Miao and Zhang [1], Constantin, Cordoba and Wu [2], Constantin and Wu [3], Cordoba and Cordoba [4], Dong [6], Dong and Du [7], Dong and Pavlovic [8], Ning Ju [11]- [12], Kiselev, Nazarov and Volberg [13], and Miao and Xue [14]. For the existence of global weak solution, which is also a local smooth solution on some unbounded interval (T, ∞), where $T \gg 1$ is a sufficiently large positive constant, see Dabkowski [5]. For the decay estimates of the global weak solutions, see Dong and Du [7], Ferreira, Niche and Planas [9], Pu and Guo [16], Maria E. Schonbek and Tomas P. Schonbek [17]. For the asymptotic behaviors of other problems, see [10].

Note that, if $f = 0$, then there exists a unique global smooth solution for each $1/2 \leq \rho \leq 1$, there exists a global weak solution for each $0 < \rho < 1/2$. Moreover, there holds the following uniform energy estimate

$$\begin{aligned}&\left\{\iint_{\mathbb{R}^2} |u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_0^t \iint_{\mathbb{R}^2} |(-\triangle)^{\rho/2} u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}^{1/2}\\ &\leq \left\{\iint_{\mathbb{R}^2} |u_0(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right\}^{1/2} + \int_0^\infty \left\{\iint_{\mathbb{R}^2} |f(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right\}^{1/2} \mathrm{d}t.\end{aligned}$$

Moreover, if the initial function and the external force are sufficiently small, then the global weak solution coincides with a global smooth solution. If the initial function or the external force is large, then after a long time T, the global weak solution becomes small enough and sufficiently smooth on (T, ∞). For our main purposes, we need the existence of the global weak solution and the existence of the local smooth solution on (T, ∞), but we do not need any estimate on the bounded interval $(0, T)$.

Also consider the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t} v + \alpha(-\triangle)^\rho v = f(\mathbf{x}, t), \tag{1.3}$$

$$v(\mathbf{x}, t) = u_0(\mathbf{x}). \tag{1.4}$$

There exists a unique global smooth solution to the Cauchy problem for the linear equation, under appropriate conditions on the initial function $u_0 = u_0(\mathbf{x})$ and the external force $f = f(\mathbf{x}, t)$. For example, if the initial function and the external force satisfy the following conditions

$$u_0 \in L^1(\mathbb{R}^2), \qquad f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+),$$

or if

$$u_0 \in L^2(\mathbb{R}^2), \qquad f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+),$$

then

$$v \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+).$$

Note that the initial functions for the nonlinear problem and the linear problem are the same, the external forces for both problems are the same as well. We may use the global smooth solution of the linear problem to approximate the global weak solutions of the nonlinear problem.

In this paper, we will study the exact limits and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions to the two-dimensional incompressible dissipative quasi-geostrophic equation. See Subsection

1.4 for the statements of the main results. See Section 2 for the proofs of the exact limits and the improved decay estimates with sharp rates.

There exist special structures in the dissipative quasi-geostrophic equation, especially in the Fourier transformation of the Jacobian determinant $J(u,(-\triangle)^{-1/2}u)$. We will reveal the hidden special structures in this paper and we will make complete use of them to accomplish the exact limits and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions. Without using the special structures, it is impossible to obtain the exact limits, the improved decay estimates with sharp rates and other results.

Notations: We will use the following Banach spaces and Hilbert spaces:

$$
\begin{aligned}
&L^1(\mathbb{R}^2)=\left\{\phi=\phi(\mathbf{x}):\iint_{\mathbb{R}^2}|\phi(\mathbf{x})|\mathrm{d}\mathbf{x}<\infty\right\},\\
&L^2(\mathbb{R}^2)=\left\{\phi=\phi(\mathbf{x}):\iint_{\mathbb{R}^2}|\phi(\mathbf{x})|^2\mathrm{d}\mathbf{x}<\infty\right\},\\
&L^\infty(\mathbb{R}^2)=\left\{\phi=\phi(\mathbf{x}):\sup_{\mathbb{R}^2}|\phi(\mathbf{x})|<\infty\right\},\\
&L^1(\mathbb{R}^+,L^2(\mathbb{R}^2))=\left\{\psi=\psi(\mathbf{x},t):\int_0^\infty\left[\iint_{\mathbb{R}^2}|\psi(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]^{1/2}\mathrm{d}t<\infty\right\},\\
&L^2(\mathbb{R}^+,L^2(\mathbb{R}^2))=\left\{\psi=\psi(\mathbf{x},t):\int_0^\infty\iint_{\mathbb{R}^2}|\psi(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t<\infty\right\},\\
&L^\infty(\mathbb{R}^+,H^{2m}(\mathbb{R}^2))=\left\{\psi=\psi(\mathbf{x},t):\sup_{t>0}\iint_{\mathbb{R}^2}(1+|\xi|^2)^m|\widehat{\psi}(\xi,t)|^2\mathrm{d}\xi<\infty\right\},\\
&L^2(\mathbb{R}^+,H^{2m}(\mathbb{R}^2))=\left\{\psi=\psi(\mathbf{x},t):\int_0^\infty\iint_{\mathbb{R}^2}(1+|\xi|^2)^m|\widehat{\psi}(\xi,t)|^2\mathrm{d}\xi\mathrm{d}t<\infty\right\}.
\end{aligned}
$$

Definition 1.1. Let the function $\phi\in L^1(\mathbb{R}^2)$. Define its Fourier transformation by

$$\widehat{\phi}(\xi)=\iint_{\mathbb{R}^2}\exp(-\mathrm{i}\mathbf{x}\cdot\xi)\phi(\mathbf{x})\mathrm{d}\mathbf{x},$$

where

$$\mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}\in\mathbb{R}^2,\qquad \xi=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\in\mathbb{R}^2,\qquad \mathbf{x}\cdot\xi=x\xi_1+y\xi_2.$$

Definition 1.2. Let r be a real constant, typically, $r=\rho$ or $r=-1/2$. Define the fractional order derivative $(-\triangle)^r\phi$ of ϕ by using the Fourier transformation

$$\widehat{(-\triangle)^r}\phi(\xi)=|\xi|^{2r}\widehat{\phi}(\xi),$$

for all $\xi\in\mathbb{R}^2$.

1.2. The main motivations - the main purposes - the main difficulties - the main strategies - the main advances

The main motivations: Here are many very important and interesting questions about the two-dimensional dissipative quasi-geostrophic equation.

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What are the influences on the global weak solutions $u = u(\mathbf{x}, t)$ of the physical mechanisms, which are represented by the diffusion coefficient α, the order ρ in the dissipation $(-\triangle)^{\rho}$, the initial function u_0, the external force f, the order m of the derivative, and the special structures of the Jacobian determinant $J(u, (-\triangle)^{-1/2}u)$?

What mechanisms have large influences and what mechanisms have minor influences to the global weak solutions?

Can we greatly improve previous results on the decay estimates of the global weak solutions?

Can we accomplish the exact limits for all order derivatives of the global weak solutions in terms of some known information?

As the diffusion coefficient, the order of the dissipation, and the order of the derivatives increase, how fast will the value of each exact limit change? As the initial function and the external force increase, how fast will the value of the exact limit increase?

Can we use the global smooth solution of the corresponding linear equation to approximate the global weak solutions of the nonlinear equation?

If yes, will the solution of the nonlinear equation become closer and closer to the solution of the corresponding linear equation as time goes to infinity?

Are there any error estimates to the above approximations? Will the error become smaller and smaller when time approaches infinity?

Are the global weak solutions of the two-dimensional dissipative quasi-geostrophic equation stable with respect to perturbations of the initial function and the external force?

If we drop the Jacobian determinant in the nonlinear equation, will the exact limits reduce to the exact limits of the global smooth solution of the corresponding linear equation?

For the nonlinear equation and the linear equation, for each fixed order of the derivatives, will the ratios of the exact limits be the same? If yes, are there any indications about the existence of the global smooth solution to the nonlinear equation with large initial function and large external force, for the case $0 < \rho < 1/2$?

Can we provide a more accurate weather forecast ahead of time? The mathematical study of the Cauchy problem for the two-dimensional dissipative quasi-geostrophic equation may help a lot.

Another motivation: The elementary decay estimate with a sharp rate

$$C \stackrel{\text{def}}{=} \sup_{t>0} \left\{ t^{1/\rho} \iint_{\mathbb{R}^2} |u(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\}, \qquad 0 < C < \infty,$$

has been established very well. However, it is not explicit at all how the constant C depends on the initial function and the external force. Long time accurate numerical simulations are of particular values in industry, engineering, national defence and applied mathematics. The constant C in the elementary decay estimate with a sharp rate may be used to greatly improve the accuracy and stability of numerical schemes of the two-dimensional dissipative quasi-geostrophic equation. The following exact limit (which is to be accomplished)

$$\begin{aligned} &\lim_{t\to\infty} \left\{ t^{(2m+1)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\} \\ &= \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \exp(-2\alpha|\eta|^{2\rho}) \mathrm{d}\eta \end{aligned}$$

$$\cdot \left\{ \iint_{\mathbb{R}^2} u_0(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_0^\infty \iint_{\mathbb{R}^2} f(\mathbf{x},t) \mathrm{d}\mathbf{x} \mathrm{d}t \right\}^2,$$

for all constants $m \geq 0$, will help to determine the precise value of the constant C.

Another motivation of this paper is to study the influences of the physical mechanisms (represented by the diffusion coefficient α, the order ρ in the dissipation $(-\triangle)^\rho$, the initial function u_0, the external force f, the Jacobian determinant $J(u, (-\triangle)^{-1/2}u)$ and the order m of the derivative) on the exact limits for all order derivatives of the global weak solutions, so that these results have positive impacts on extremely long time numerical simulations.

The positive solutions to these problems definitely help us to better understand the properties of the global weak solutions and hopefully to discover new special structures. We will answer most of these important and interesting questions by using rigorous mathematical analysis. The study of the exact limits and the improved decay estimates of the global weak solutions of the two-dimensional equation will have positive influences on long time safe flight of airplanes and spacecrafts as well as long time voyages under the oceans.

The main purposes: Let $u = u(\mathbf{x}, t)$ represent the global weak solutions of the Cauchy problem for the two-dimensional dissipative quasi-geostrophic equation. Let $v = v(\mathbf{x}, t)$ represent the global smooth solution of the Cauchy problem for the corresponding linear equation. Note that the initial function and the external force $(u_0, f) = (u_0(\mathbf{x}), f(\mathbf{x}, t))$ in both problems are the same.

We will consider two main cases for the initial function and the external force.

The main purpose for Case 1 is to accomplish the following exact limits

$$\lim_{t\to\infty} \left\{ t^{(2m+1)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\},$$
$$\lim_{t\to\infty} \left\{ t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t) - v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \right\},$$

in terms of the constants α, ρ, m and some known integrals.

The main purpose for Case 2 is to accomplish the exact limits

$$\lim_{t\to\infty} \left\{ t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\},$$
$$\lim_{t\to\infty} \left\{ t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t) - v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \right\},$$

in terms of the constants α, ρ, m and some other known integrals.

Note that the rates of decay in the two cases are different.

The second purposes are to make use of the exact limits to accomplish the improved decay estimates with sharp rates for all order derivatives of the global weak solutions to the two-dimensional dissipative quasi-geostrophic equation. For Case 1, we will prove the following improved decay estimates with sharp rates

$$t^{(2m+1)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \leq \mathcal{A}_1(m) + \mathcal{B}_1(m) t^{-(2-2\rho)/\rho},$$
$$t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t) - v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \leq \mathcal{C}_1(m) + \mathcal{D}_1(m) t^{-(2-2\rho)/\rho},$$

for all constants $m \geq 0$ and for all sufficiently large $t \gg 1$, where

$$\mathcal{A}_1(m), \qquad \mathcal{B}_1(m), \qquad \mathcal{C}_1(m), \qquad \mathcal{D}_1(m)$$

are positive constants, independent of t.

By improved decay estimates with sharp rates, we mean that not only the constants

$$\mathcal{A}_1(m), \qquad \mathcal{B}_1(m), \qquad \mathcal{C}_1(m), \qquad \mathcal{D}_1(m)$$

are independent of t, but also these constants are independent of

(1) the integrals of any order derivatives of the initial function u_0,

(2) the integrals of any order derivatives of the external force f,

(3) the integrals of any order derivatives of the nonlinear function u^2.

Can we find the best explicit representations for the constants in terms of some known information?

For Case 2, we will prove the following improved decay estimates with sharp rates

$$\begin{aligned}
&t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \leq \mathcal{A}_2(m) + \mathcal{B}_2(m) t^{-(4-2\rho)/\rho}, \\
&t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t) - v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \leq \mathcal{C}_2(m) + \mathcal{D}_2(m) t^{-(4-2\rho)/\rho},
\end{aligned}$$

for all constants $m \geq 0$ and for all sufficiently large $t \gg 1$, where

$$\mathcal{A}_2(m), \qquad \mathcal{B}_2(m), \qquad \mathcal{C}_2(m), \qquad \mathcal{D}_2(m)$$

are positive constants, independent of t.

By improved decay estimates with sharp rates, we mean that not only the constants

$$\mathcal{A}_2(m), \qquad \mathcal{B}_2(m), \qquad \mathcal{C}_2(m), \qquad \mathcal{D}_2(m)$$

are independent t, but also these constants are independent of

(1) the integrals of any order derivatives of the functions ϕ_1 and ϕ_2,

(2) the integrals of any order derivatives of the functions ψ_1 and ψ_2,

(3) the integrals of any order derivatives of the nonlinear function u^2.

Can we find the best explicit representations for the constants in terms of some known information?

The constants in the improved decay estimates with sharp rates for all order derivatives of the global weak solutions to the two-dimensional dissipative quasi-geostrophic equation are explicit enough so that the results have significant influence on numerical simulations.

The main difficulties: Let the parameter $0 < \rho < 1/2$. When the initial function and the external force are large, the uniform energy estimates of any order derivatives of the global weak solutions have been open. Therefore, the existence and uniqueness of the global smooth solution $u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+)$ have been open.

For the main purposes of this paper, we need the existence of the global weak solutions, which are also local smooth solutions on the unbounded interval (T,∞). See Dabkowski [5]. Additionally, we will establish some new minor results: the primary decay estimates with sharp rates for all order derivatives of the global weak solutions on that interval.

The main technical difficulty for $0<\rho<1$ is the mathematical analysis of the following integrals

$$\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] \left[\mathrm{i}\sum_{k=1}^{2} \eta_k \widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right] \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

$$\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

where $\mathcal{N}(u) = J(u,(-\triangle)^{-1/2}u)$. We will apply a few novel ideas to establish optimal estimates for these integrals.

The main strategies: To accomplish the exact limits for all order derivatives of the global weak solutions of the Cauchy problem for the two-dimensional dissipative quasi-geostrophic equation, we will use the following strategies. We will couple together the Parseval's identity, a few simple properties of the Fourier transformation, the representation of the Fourier transformation of the global weak solutions, special structures of the nonlinear function $J(u,(-\triangle)^{-1/2}u)$, the interval decomposition $[0,t]=[0,(1-\varepsilon)t]\cup[(1-\varepsilon)t,t]$, Lebesgue's dominated convergence theorem, a series of new estimates about the integrals

$$\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] \left[\mathrm{i}\sum_{k=1}^{2} \eta_k \widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right] \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

$$\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

and the primary decay estimates with sharp rates for all order derivatives of the global weak solutions on (T,∞).

Note that the Jacobian determinant $J(u,(-\triangle)^{-1/2}u)$ is a nonlocal term. There exist special structures in $J(u,(-\triangle)^{-1/2}u)$. The special structures may be revealed clearly by using the Fourier transformation. We will make complete use of the special structures to accomplish the main results.

We will make use of the primary decay results to establish optimal estimates for these integrals. Then we will use the exact limits for all order derivatives of the global weak solutions to obtain the best possible upper bounds, which depend only on integrals of functions related to the initial function, integrals of functions related to the external force and integrals of functions related to the nonlinear function u^2. The upper bound does not depend on the integrals of any order derivatives of these functions.

It is worth mentioning that the exact limits play a very important role in establishing the improved decay estimates with sharp rates to make sure that the constants are independent of the integrals of the derivatives of the initial function and the derivatives of the global weak solutions.

The main advances: One of the main technical advances is that we are able to use the singular integral

$$\iint_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta$$

to control the integrals

$$\iint_{\mathbb{R}^2} |\eta|^{4m} \left|\int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\right|^2 \mathrm{d}\eta,$$
$$\iint_{\mathbb{R}^2} |\eta|^{4m} \left|\int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta,$$

for all constants $0<\delta<2\rho$, $0<\varepsilon<1$ and $m\geq 0$ and for all $t \gg 1$.

The main advance is that we are able to represent the exact limits for all order derivatives of the global weak solutions explicitly, in terms of the integrals of functions related to the initial function, the integrals of functions related to the external force and the integrals of the nonlinear function u^2, rather than the integrals of the derivatives of these functions.

1.3. The mathematical assumptions

We make the following mathematical assumptions for the two-dimensional incompressible dissipative quasi-geostrophic equation. Let $\alpha>0$, $0<\delta<2\rho$, $0<\varepsilon<1$ and $0<\rho<1$ be positive constants. Let $m\geq 0$ be any real constant.

(A1) Suppose that the initial function and the external force satisfy the following assumptions

$$\begin{aligned}&u_0 \in C^1(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2),\\ &f\in C^\infty(\mathbb{R}^2\times\mathbb{R}^+)\cap L^1(\mathbb{R}^2\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}^2)),\end{aligned}$$

such that

$$\iint_{\mathbb{R}^2} u_0(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \iint_{\mathbb{R}^2} f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t \neq 0.$$

Suppose that there exists the following limit*

$$\lim_{t\to\infty}\left\{t^{(m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m f(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right\}^2,$$

for all constants $m\geq 0$.

(A2) Suppose that the initial function and the external force satisfy the conditions

$$u_0 \in C^1(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2),$$

*Here is a slightly weaker condition

$$\sup_{t>0}\left\{t^{(m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m f(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right\}^2<\infty,$$

for all constants $m\geq 0$.

$$f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)).$$

Suppose that there exist real scalar smooth functions

$$\begin{aligned}
&\phi_1 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2),\ \ \psi_1 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+),\\
&\phi_2 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2),\ \ \psi_2 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+),
\end{aligned}$$

such that

$$u_0(\mathbf{x}) = \frac{\partial}{\partial x}\phi_1(\mathbf{x}) + \frac{\partial}{\partial y}\phi_2(\mathbf{x}), \qquad f(\mathbf{x},t) = \frac{\partial}{\partial x}\psi_1(\mathbf{x},t) + \frac{\partial}{\partial y}\psi_2(\mathbf{x},t),$$

for all $(\mathbf{x},t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

Suppose that there exist the following limits†

$$\lim_{t\to\infty}\left\{\sum_{k=1}^{2}\left[t^{(m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m\psi_k(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\},$$

for all constants $m \geq 0$.

(A3) Suppose that there exists a unique global smooth solution

$$\begin{aligned}
&u \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)), \qquad (-\triangle)^{\rho/2}u \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\\
&u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+),
\end{aligned}$$

if $1/2 \leq \rho \leq 1$, and if the initial function and the external force satisfy

$$\begin{aligned}
&u_0 \in C^1(\mathbb{R}^2) \cap H^{2m}(\mathbb{R}^2),\\
&f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)).
\end{aligned}$$

Suppose that there exists a global weak solution

$$u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)), \qquad (-\triangle)^{\rho/2}u \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^2)),$$

if $0 < \rho < 1/2$, and if the initial function and the external force satisfy

$$u_0 \in L^2(\mathbb{R}^2), \qquad f \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)).$$

(A4) Suppose that the global weak solutions become small enough and sufficiently smooth after a long time. That is, there exists a sufficiently large positive constant $T \gg 1$, such that the global weak solutions of the Cauchy problem for the two-dimensional dissipative quasi-geostrophic equation satisfy

$$\begin{aligned}
&\sup_{t>T}\left\{\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} < \infty,\\
&\int_T^\infty\iint_{\mathbb{R}^2}|(-\triangle)^{m+\rho/2}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t < \infty,
\end{aligned}$$

†Here are slightly weaker conditions

$$\sup_{t>0}\left\{\sum_{k=1}^{2}\left[t^{(m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m\psi_k(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\} < \infty,$$

for all constants $m \geq 0$.

for all constants $m \geq 0$.

(A5) Suppose that there holds the following representation for the Fourier transformation of the global weak solutions

$$\begin{aligned} &\widehat{u}(\xi,t) = \exp(-\alpha|\xi|^{2\rho}t)\widehat{u}_0(\xi) \\ &+ \int_0^t \exp[-\alpha|\xi|^{2\rho}(t-\tau)]\widehat{f}(\xi,\tau)\mathrm{d}\tau \\ &- \int_0^t \exp[-\alpha|\xi|^{2\rho}(t-\tau)]\widehat{\mathcal{N}(u)}(\xi,\tau)\mathrm{d}\tau, \end{aligned}$$

for all $(\xi,t) \in \mathbb{R}^2 \times \mathbb{R}^+$, where $\mathcal{N}(u) = J(u,(-\triangle)^{-1/2}u)$.

(A6) Suppose that there holds the following elementary decay estimate with a sharp rate

$$\sup_{t>0}\left\{t^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} < \infty.$$

Remark 1.1. Suppose that either assumption (A1) or assumption (A2), but not both, holds.

Remark 1.2. The above decay rate is sharp, only if

$$\iint_{\mathbb{R}^2} u_0(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \iint_{\mathbb{R}^2} f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t \neq 0.$$

The decay rate may be improved under additional conditions on the initial function and the external force.

For the existence of the global smooth solution or the global weak solution and the elementary decay results of the global weak solutions, see Constantin and Wu [3], Dabkowski [5], Dong [6], Ju [11]- [12], Kiselev, Nazarov and Volberg [13], Maria E. Schonbek and Tomas P. Schonbek [17]. The assumptions (A1), (A2), (A3), (A4), (A5), (A6) are made based on these results.

1.4. The main results

There are three parts in the results.

Part A: The exact limits for all order derivatives of the global weak solutions of the two-dimensional incompressible dissipative quasi-geostrophic equation.

In each case, we will state the exact limits for all order derivatives of the global weak solutions. We may use the global smooth solution of the corresponding linear equation to approximate the global weak solutions of the two-dimensional dissipative quasi-geostrophic equation.

To make the statements of the exact limits simple and clear, let us define the following notations

$$\begin{aligned} \mathcal{I}(m) &\stackrel{\text{def}}{=} \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta, \\ \mathcal{J} &\stackrel{\text{def}}{=} \left\{\iint_{\mathbb{R}^2} u_0(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \iint_{\mathbb{R}^2} f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2, \end{aligned}$$

$$
\begin{aligned}
\mathcal{K}_0 &\stackrel{\text{def}}{=} \sum_{k=1}^{2}\left\{\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
\mathcal{K} &\stackrel{\text{def}}{=} \sum_{k=1}^{2}\left\{\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&\quad+\sum_{k=1}^{2}\left\{\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2,\\
\mathcal{L} &\stackrel{\text{def}}{=} \sum_{k=1}^{2}\left\{\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2,
\end{aligned}
$$

where

$$
\mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}\in\mathbb{R}^2,\qquad \eta=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\in\mathbb{R}^2,\qquad \lambda=\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}\in\mathbb{R}^2.
$$

Theorem 1.1. *For Case 1, there hold the following exact limits for all order derivatives of the global weak solutions of the two-dimensional dissipative quasi-geostrophic equation*

$$
\lim_{t\to\infty}\left\{t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\mathcal{I}(m-1/2)\mathcal{J},
$$

for all constants $m\geq 0$.

Theorem 1.2. *For Case 2, there hold the following exact limits for all order derivatives of the global weak solutions*

$$
\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{2}\mathcal{I}(m)\mathcal{K},
$$

for all constants $m\geq 0$.

Theorem 1.3. *For both cases, there hold the following exact limits for all order derivatives of the global weak solutions*

$$
\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{2}\mathcal{I}(m)\mathcal{L},
$$

for all constants $m\geq 0$.

Theorem 1.4. *The ratio of the exact limits of the global weak solutions of the two-dimensional dissipative quasi-geostrophic equation is the same as the ratio of the exact limits of the global smooth solution of the corresponding linear equation, for each fixed constant m. For Case 2*[‡]*, there hold*

$$
\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+\rho}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}
\end{aligned}
$$

[‡]For simplicity, we will focus on Case 2 and skip Case 1.

$$
\begin{aligned}
&= \left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\ /\ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&= \frac{(2m+2)(2m+\rho+2)}{(2\alpha\rho)^2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{(2m+4\rho+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+2\rho}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\ /\ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&= \left\{\lim_{t\to\infty}\left[t^{(2m+4\rho+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+2\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\ /\ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&= \frac{(2m+2)(2m+\rho+2)(2m+2\rho+2)(2m+3\rho+2)}{(2\alpha\rho)^4},
\end{aligned}
$$

for all constants $m \geq 0$. Moreover

$$
\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+\rho}[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\ /\ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m}[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&= \left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\ /\ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&= \frac{(2m+2)(2m+\rho+2)}{(2\alpha\rho)^2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{(2m+4\rho+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+2\rho}[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\ /\ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m}[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&= \left\{\lim_{t\to\infty}\left[t^{(2m+4\rho+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+2\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\ /\ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&= \frac{(2m+2)(2m+\rho+2)(2m+2\rho+2)(2m+3\rho+2)}{(2\alpha\rho)^4},
\end{aligned}
$$

for all constants $m \geq 0$.

Part B: The primary decay estimates and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions.

Here are the primary decay estimates with sharp rates for all constants $m \geq 0$ and for all sufficiently large $t \gg 1$. Note that the rates of decay in the two cases are different.

Theorem 1.5. *For Case 1, there hold the following primary decay estimates with sharp rates for all order derivatives of the global weak solutions*

$$\sup_{t>T}\left\{t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

for all constants $m \geq 0$, where $T \gg 1$ is a sufficiently large positive constant, it is the same constant as in (A4).

Theorem 1.6. *For Case 2, there hold the following primary decay estimates with sharp rates for all order derivatives of the global weak solutions*

$$\sup_{t>T}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

for all constants $m \geq 0$.

Here are the improved decay estimates with sharp rates for all constants $m \geq 0$ and for all sufficiently large $t \gg 1$.

To make the statements of the improved decay estimates with sharp rates of the two-dimensional dissipative quasi-geostrophic equation simple and clear, first of all, let us define the following notations

$$\begin{aligned}
&\mathrm{E}_1(u_0)=\left\{\iint_{\mathbb{R}^2}|u_0(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2,\\
&\mathrm{E}_2(f)=\left\{\int_0^\infty\iint_{\mathbb{R}^2}|f(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
&\mathrm{E}_3(u)=\left\{\int_0^\infty\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
&\mathrm{E}_4(m)=\lim_{t\to\infty}\left\{\left[t^{(m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m f(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\},\\
&\mathcal{I}(m)=\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta,\\
&\mathcal{J}=\left\{\iint_{\mathbb{R}^2}u_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
&\mathcal{S}=\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta.
\end{aligned}$$

In this paper, we will use $C_0 = C_0(m) > 0$ to represent any positive constant, which is independent of (α,ρ), (δ,ε), (u_0,f), u and $(\mathbf{x},t)$. Then let us define the following quantities

$$\mathcal{A}_1(m)=\mathcal{A}_1(\alpha,\delta,\varepsilon,\rho,m)$$

$$
\begin{aligned}
&\stackrel{\text{def}}{=} 5\mathcal{I}(m-1/2)\left\{\mathrm{E}_1(u_0)+\frac{\mathrm{E}_2(f)+\mathrm{E}_3(u)}{\varepsilon^{(2m+1)/\rho}}\right\},\\
&\quad \mathcal{B}_1(m)=\mathcal{B}_1(\alpha,\delta,\varepsilon,\rho,m)\\
&\stackrel{\text{def}}{=} 10C_0(m)\mathcal{S}\left\{\mathrm{E}_4(m-\rho+(1+\delta)/2)+\mathcal{I}(-1/2)\mathcal{I}(m-\rho+(1+\delta)/2)\mathcal{J}^2\right\},\\
&\quad \mathcal{C}_1(m)=\mathcal{C}_1(\alpha,\delta,\varepsilon,\rho,m)\\
&\stackrel{\text{def}}{=} 2\mathcal{I}(m-1/2)\left\{\frac{\mathrm{E}_3(u)}{\varepsilon^{(2m+1)/\rho}}\right\},\\
&\quad \mathcal{D}_1(m)=\mathcal{D}_1(\alpha,\delta,\varepsilon,\rho,m)\\
&\stackrel{\text{def}}{=} 2C_0(m)\mathcal{S}\left\{\mathcal{I}(-1/2)\mathcal{I}(m-\rho+(1+\delta)/2)\mathcal{J}^2\right\}.
\end{aligned}
$$

Theorem 1.7. *For Case 1, there hold the following improved decay estimates with sharp rates for all order derivatives of the global weak solutions*

$$
\begin{aligned}
&t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le \mathcal{A}_1(m)+\mathcal{B}_1(m)t^{-(2-2\rho)/\rho},\\
&t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\le \mathcal{C}_1(m)+\mathcal{D}_1(m)t^{-(2-2\rho)/\rho},
\end{aligned}
$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all constants $m\ge 0$ and for all sufficiently large $t\gg 1$.

Let us define the following notations

$$
\begin{aligned}
&\mathrm{F}_1(u_0)=\sum_{k=1}^{2}\left\{\iint_{\mathbb{R}^2}|\phi_k(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2,\\
&\mathrm{F}_2(f)=\sum_{k=1}^{2}\left\{\int_0^\infty\iint_{\mathbb{R}^2}|\psi_k(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
&\mathrm{F}_3(u)=\left\{\int_0^\infty\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
&\mathrm{F}_4(m)=\lim_{t\to\infty}\left\{\sum_{k=1}^{2}\left[t^{(m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m\psi_k(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\},\\
&\mathcal{I}(m)=\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta,\\
&\mathcal{K}=\sum_{k=1}^{2}\left\{\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\sum_{k=1}^{2}\left\{\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,\\
&\mathcal{S}=\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta.
\end{aligned}
$$

Then let us define the following quantities

$$
\begin{aligned}
&\quad \mathcal{A}_2(m)=\mathcal{A}_2(\alpha,\delta,\varepsilon,\rho,m)\\
&\stackrel{\text{def}}{=} 5\mathcal{I}(m)\left\{\mathrm{F}_1(u_0)+\frac{\mathrm{F}_2(f)+\mathrm{F}_3(u)}{\varepsilon^{(2m+2)/\rho}}\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{B}_2(m)=\mathcal{B}_2(\alpha,\delta,\varepsilon,\rho,m)\\
\stackrel{\text{def}}{=}\ &10C_0(m)\mathcal{S}\left\{\mathrm{F}_4(m+1-\rho+\delta/2)+\mathcal{I}(0)\mathcal{I}(m+1-\rho+\delta/2)\mathcal{K}^2\right\},\\
&\mathcal{C}_2(m)=\mathcal{C}_2(\alpha,\delta,\varepsilon,\rho,m)\\
\stackrel{\text{def}}{=}\ &2\mathcal{I}(m)\left\{\frac{\mathrm{F}_3(u)}{\varepsilon^{(2m+2)/\rho}}\right\},\\
&\mathcal{D}_2(m)=\mathcal{D}_2(\alpha,\delta,\varepsilon,\rho,m)\\
\stackrel{\text{def}}{=}\ &2C_0(m)\mathcal{S}\left\{\mathcal{I}(0)\mathcal{I}(m+1-\rho+\delta/2)\mathcal{K}^2\right\}.
\end{aligned}
$$

Theorem 1.8. *For Case 2, there hold the following improved decay estimates with sharp rates for all order derivatives of the global weak solutions*

$$
\begin{aligned}
&t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le\mathcal{A}_2(m)+\mathcal{B}_2(m)t^{-(4-2\rho)/\rho},\\
&t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\le\mathcal{C}_2(m)+\mathcal{D}_2(m)t^{-(4-2\rho)/\rho},
\end{aligned}
$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all constants $m\ge 0$ and for all sufficiently large $t\gg 1$.

Part C: The linear results - The exact limits for all order derivatives of the global smooth solution of the linear equation.

Theorem 1.9. *For Case 1, there hold the following exact limits for all order derivatives of the global smooth solutions*

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\mathcal{I}(m-1/2)\mathcal{J},\\
&\left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/\ &\left\{\lim_{t\to\infty}\left[t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=\ &\frac{2(2m+1)(2m+\rho+1)}{(4\alpha\rho)^2},\\
&\left\{\lim_{t\to\infty}\left[t^{(2m+4\rho+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+2\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/\ &\left\{\lim_{t\to\infty}\left[t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=\ &\frac{2(2m+1)(2m+\rho+1)(2m+2\rho+1)(2m+3\rho+1)}{(4\alpha\rho)^4}.
\end{aligned}
$$

Theorem 1.10. *For Case 2, there hold the following exact limits for all order derivatives of the global smooth solutions*

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{2}\mathcal{I}(m)\mathcal{K}_0,\\
&\left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
& / \left\{ \lim_{t\to\infty} \left[t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m v(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&= \frac{2(2m+2)(2m+\rho+2)}{(4\alpha\rho)^2}, \\
& \left\{ \lim_{t\to\infty} \left[t^{(2m+4\rho+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^{m+2\rho} v(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
& / \left\{ \lim_{t\to\infty} \left[t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m v(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&= \frac{2(2m+2)(2m+\rho+2)(2m+2\rho+2)(2m+3\rho+2)}{(4\alpha\rho)^4}.
\end{aligned}
$$

The proofs of the results are given in Section 2. There are many important and interesting remarks on the main results. They are listed in Subsection 3.2 (Remarks).

2. The Mathematical Analysis and the Proofs of the Main Results

The main purposes of this section are to accomplish the exact limits and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions of the two-dimensional incompressible dissipative quasi-geostrophic equation. The main difficulties are that the existence and uniqueness of the global smooth solution are unknown, if $0 < \rho < 1/2$ and if the initial function and the external force are large. There exists no available uniform energy estimates for any order derivatives. To overcome the main difficulties, we will make use of the special structure and the semi-explicit representations of the Fourier transformations of the global weak solutions, to establish the primary decay estimates with sharp rates for all order derivatives on the interval (T, ∞), where $T \gg 1$ is a sufficiently large positive constant. Then we will couple together existing ideas, existing results and a few novel ideas to accomplish the exact limits and the improved decay estimates with sharp rates.

If there exists a unique global smooth solution to the Cauchy problem for the two-dimensional incompressible dissipative quasi-geostropic equation, then the energy $\iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}$ is finite, for all $m > 0$ and for all $t > 0$. If there exists a global weak solution, then the energy $\iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}$ is finite, for all $m > 0$ for all sufficiently large $t > T$. It is unknown if it is equal to infinity at some finite time $0 < t_0 < T$.

Let $\alpha > 0$, $0 < \delta < 2\rho$, $0 < \varepsilon < 1$ and $0 < \rho < 1$ be positive constants. Let $m \geq 0$ and $\kappa \geq 0$ be real constants.

As mentioned in Section 1, we will consider two main cases. Let us quickly review the two cases before we make the rigorous mathematical analysis.

Case 1: Suppose that the initial function and the external force satisfy

$$
\begin{aligned}
& u_0 \in C^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), \\
& f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)),
\end{aligned}
$$

such that

$$\iint_{\mathbb{R}^2} u_0(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \iint_{\mathbb{R}^2} f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t \neq 0.$$

Case 2: Suppose that the initial function and the external force satisfy

$$\begin{aligned}&u_0 \in C^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2),\\ &f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)).\end{aligned}$$

Suppose that there exist real scalar smooth functions

$$\begin{aligned}&\phi_1 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2),\ \ \psi_1 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+),\\ &\phi_2 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2),\ \ \psi_2 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+),\end{aligned}$$

such that

$$u_0(\mathbf{x}) = \frac{\partial}{\partial x}\phi_1(\mathbf{x}) + \frac{\partial}{\partial y}\phi_2(\mathbf{x}), \qquad f(\mathbf{x},t) = \frac{\partial}{\partial x}\psi_1(\mathbf{x},t) + \frac{\partial}{\partial y}\psi_2(\mathbf{x},t),$$

for all $(\mathbf{x},t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

Performing the Fourier transformation to these functions leads to

$$\widehat{u}_0(\xi) = \mathrm{i}\left\{\sum_{k=1}^{2} \xi_k \widehat{\phi}_k(\xi)\right\}, \qquad \widehat{f}(\xi,t) = \mathrm{i}\left\{\sum_{k=1}^{2} \xi_k \widehat{\psi}_k(\xi,t)\right\}.$$

Applying the change of variables $\eta = t^{1/(2\rho)}\xi$, we have

$$\begin{aligned}t^{1/(2\rho)}\widehat{u}_0(t^{-1/(2\rho)}\eta) &= \mathrm{i}\left\{\sum_{k=1}^{2} \eta_k \widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right\},\\ t^{1/(2\rho)}\widehat{f}(t^{-1/(2\rho)}\eta,t) &= \mathrm{i}\left\{\sum_{k=1}^{2} \eta_k \widehat{\psi}_k(t^{-1/(2\rho)}\eta,t)\right\},\end{aligned}$$

for all $(\eta,t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

Let the real vectors

$$\lambda = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \in \mathbb{R}^2, \qquad \eta = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \in \mathbb{R}^2.$$

As usual, we define

$$|\lambda| = (|\lambda_1|^2 + |\lambda_2|^2)^{1/2}, \qquad |\eta| = (|\eta_1|^2 + |\eta_2|^2)^{1/2}.$$

Let us define the notation $\lambda \odot \eta$ by

$$\lambda \odot \eta = \det\begin{pmatrix}\lambda_1 & \eta_1\\ \lambda_2 & \eta_2\end{pmatrix} = \lambda_1\eta_2 - \lambda_2\eta_1.$$

Let us do some basic preparations before the rigorous analysis. To further simplify the notations in the mathematical analysis, let us define

$$\mathrm{R}_1 u = \frac{\partial}{\partial x}(-\triangle)^{-1/2}u, \qquad \mathrm{R}_2 u = \frac{\partial}{\partial y}(-\triangle)^{-1/2}u.$$

We will write the interval $[0,t]$ as $[0,(1-\varepsilon)t]\cup[(1-\varepsilon)t,t]$, where $0<\varepsilon<1$. The treatments of the Fourier transformation of $J(u,(-\triangle)^{-1/2}u)$ on different subintervals will be very different. Note that

$$\begin{aligned}J(u,(-\triangle)^{-1/2}u)&=\frac{\partial}{\partial x}u\frac{\partial}{\partial y}(-\triangle)^{-1/2}u-\frac{\partial}{\partial y}u\frac{\partial}{\partial x}(-\triangle)^{-1/2}u\\&=\frac{\partial}{\partial x}\left\{u\frac{\partial}{\partial y}(-\triangle)^{-1/2}u\right\}-\frac{\partial}{\partial y}\left\{u\frac{\partial}{\partial x}(-\triangle)^{-1/2}u\right\}.\end{aligned}$$

That is

$$\mathcal{N}(u)=\frac{\partial}{\partial x}(u\mathrm{R}_2u)-\frac{\partial}{\partial y}(u\mathrm{R}_1u).$$

Now the Fourier transformation

$$\widehat{\mathcal{N}(u)}(\xi,t)=\mathrm{i}\xi_1\widehat{u\mathrm{R}_2u}(\xi,t)-\mathrm{i}\xi_2\widehat{u\mathrm{R}_1u}(\xi,t).$$

By the definition of the Fourier transformation and the Parseval's identity, we have

$$\begin{aligned}\widehat{u\mathrm{R}_1u}(\xi,t)&=-\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\frac{\mathrm{i}\lambda_1}{|\lambda|}\widehat{u}(\lambda+\xi,t)\overline{\widehat{u}}(\lambda,t)\mathrm{d}\lambda,\\\widehat{u\mathrm{R}_2u}(\xi,t)&=-\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\frac{\mathrm{i}\lambda_2}{|\lambda|}\widehat{u}(\lambda+\xi,t)\overline{\widehat{u}}(\lambda,t)\mathrm{d}\lambda.\end{aligned}$$

Now the Fourier transformation

$$\widehat{\mathcal{N}(u)}(\xi,t)=-\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\frac{\lambda_1\xi_2-\lambda_2\xi_1}{|\lambda|}\widehat{u}(\lambda+\xi,t)\overline{\widehat{u}}(\lambda,t)\mathrm{d}\lambda,$$

for all $(\xi,t)\in\mathbb{R}^2\times\mathbb{R}^+$. Moreover, there holds the following elementary estimate

$$\left|\widehat{\mathcal{N}(u)}(\xi,t)\right|\le|\xi|\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x},$$

for all $(\xi,t)\in\mathbb{R}^2\times\mathbb{R}^+$. Making the change of variables $\eta=t^{1/(2\rho)}\xi$, we have

$$\begin{aligned}&t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\\&=-\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\frac{\lambda\odot\eta}{|\lambda|}\widehat{u}(\lambda+t^{-1/(2\rho)}\eta,\tau)\overline{\widehat{u}}(\lambda,\tau)\mathrm{d}\lambda,\end{aligned}$$

for all $(\eta,t)\in\mathbb{R}^2\times\mathbb{R}^+$, where $\tau\in[(1-\varepsilon)t,t]$ and $0<\varepsilon<1$.

2.1. The elementary estimates: Case 1

Let the positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, let the constant $m\ge0$. First of all, we will establish a series of elementary estimates for Case 1.

Lemma 2.1. *There hold the following elementary estimates:*
(1)

$$\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\widehat{u}_0(t^{-1/(2\rho)}\eta)\right|^2\mathrm{d}\eta$$

$$\leq \iint_{\mathbb{R}^2} |\eta|^{4m} \exp(-2\alpha|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \iint_{\mathbb{R}^2} |u_0(\mathbf{x})| \mathrm{d}\mathbf{x} \right\}^2.$$

(2)

$$\begin{aligned}
&\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t}) \right] \widehat{f}(t^{-1/(2\rho)}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
\leq &\iint_{\mathbb{R}^2} |\eta|^{4m} \exp(-2\alpha\varepsilon|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \int_0^\infty \iint_{\mathbb{R}^2} |f(\mathbf{x}, t)| \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2.
\end{aligned}$$

(3)

$$\begin{aligned}
&\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t}) \right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
\leq &\iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \int_0^\infty \iint_{\mathbb{R}^2} |u(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2.
\end{aligned}$$

(4)

$$\begin{aligned}
&\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t}) \right] \widehat{f}(t^{-1/(2\rho)}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
\leq &C_0(m) t^{-(2-2\rho)/\rho} \left\{ \iint_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \right\} \\
&\cdot \sup_{(1-\varepsilon)t\leq\tau\leq t} \left\{ \tau^{(2m+3+\delta-2\rho)/(2\rho)} \iint_{\mathbb{R}^2} |(-\triangle)^{m-\rho+(1+\delta)/2} f(\mathbf{x}, \tau)| \mathrm{d}\mathbf{x} \right\}^2.
\end{aligned}$$

(5)

$$\begin{aligned}
&\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t}) \right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
\leq &C_0(m) t^{-(2-2\rho)/\rho} \left\{ \iint_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \right\} \\
&\cdot \sup_{(1-\varepsilon)t\leq\tau\leq t} \left\{ \tau^{1/\rho} \iint_{\mathbb{R}^2} |u(\mathbf{x}, \tau)|^2 \mathrm{d}\mathbf{x} \right\} \\
&\cdot \sup_{(1-\varepsilon)t\leq\tau\leq t} \left\{ \tau^{(2m+3+\delta-2\rho)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2} u(\mathbf{x}, \tau)|^2 \mathrm{d}\mathbf{x} \right\}.
\end{aligned}$$

These estimates are true for all constants $m \geq 0$ and for all $t > 0$.

Proof. The proof of the lemma is skipped. □

Lemma 2.2. *Define the following complex auxiliary functions*

$$\begin{aligned}
\Lambda_1(\eta, t) &= \exp(-\alpha|\eta|^{2\rho}) \widehat{u}_0(t^{-1/(2\rho)}\eta) \\
&+ \int_0^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t}) \right] \widehat{f}(t^{-1/(2\rho)}\eta, \tau) \mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau,\\
&\Gamma_1(\eta,t)=\exp(-\alpha|\eta|^{2\rho})\widehat{u}_0(t^{-1/(2\rho)}\eta)\\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\\
&-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau,
\end{aligned}
$$

for all $(\eta,t)\in\mathbb{R}^2\times\mathbb{R}^+$. *Then there holds the following estimate*

$$
\begin{aligned}
&\left|\iint_{\mathbb{R}^2}|\eta|^{4m}|\Lambda_1(\eta,t)|^2\mathrm{d}\eta-\iint_{\mathbb{R}^2}|\eta|^{4m}|\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right|\\
&\le\iint_{\mathbb{R}^2}|\eta|^{4m}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta\\
&+2\left\{\iint_{\mathbb{R}^2}|\eta|^{4m}|\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\left\{\iint_{\mathbb{R}^2}|\eta|^{4m}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2},
\end{aligned}
$$

for all positive constants $0<\delta<2\rho$ *and* $0<\varepsilon<1$, *for all* $m\ge 0$ *and for all* $t>0$.

Proof. Note that

$$
\begin{aligned}
\Lambda_1(\eta,t)-\Gamma_1(\eta,t)&=\int_{(1-\varepsilon)t}^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\\
&-\int_{(1-\varepsilon)t}^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
&|\Lambda_1(\eta,t)|^2-|\Gamma_1(\eta,t)|^2\\
&=|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2+2\ \mathrm{Re}\ [(\Lambda_1(\eta,t)-\Gamma_1(\eta,t))\overline{\Gamma_1(\eta,t)}].
\end{aligned}
$$

Now we have

$$
\begin{aligned}
&\iint_{\mathbb{R}^2}|\eta|^{4m}|\Lambda_1(\eta,t)|^2\mathrm{d}\eta-\iint_{\mathbb{R}^2}|\eta|^{4m}|\Gamma_1(\eta,t)|^2\mathrm{d}\eta\\
&=\iint_{\mathbb{R}^2}|\eta|^{4m}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta\\
&+2\ \mathrm{Re}\ \iint_{\mathbb{R}^2}|\eta|^{4m}[\Lambda_1(\eta,t)-\Gamma_1(\eta,t)]\overline{\Gamma_1(\eta,t)}\mathrm{d}\eta.
\end{aligned}
$$

By using the Cauchy-Schwartz's inequality, we have the estimate

$$
\begin{aligned}
&\left|2\ \mathrm{Re}\ \iint_{\mathbb{R}^2}|\eta|^{4m}[\Lambda_1(\eta,t)-\Gamma_1(\eta,t)]\overline{\Gamma_1(\eta,t)}\mathrm{d}\eta\right|\\
&\le 2\left\{\iint_{\mathbb{R}^2}|\eta|^{4m}|\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\left\{\iint_{\mathbb{R}^2}|\eta|^{4m}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}.
\end{aligned}
$$

Overall, we have finished the proof of the estimate

$$
\left|\iint_{\mathbb{R}^2}|\eta|^{4m}|\Lambda_1(\eta,t)|^2\mathrm{d}\eta-\iint_{\mathbb{R}^2}|\eta|^{4m}|\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right|
$$

$$\leq \iint_{\mathbb{R}^2} |\eta|^{4m} |\Lambda_1(\eta,t) - \Gamma_1(\eta,t)|^2 \mathrm{d}\eta$$
$$+ 2\left\{\iint_{\mathbb{R}^2} |\eta|^{4m}|\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2} \left\{\iint_{\mathbb{R}^2} |\eta|^{4m}|\Lambda_1(\eta,t) - \Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}.$$

The proof of the lemma is completed. □

2.2. The elementary estimates: Case 2

Let the positive constants $0 < \delta < 2\rho$ and $0 < \varepsilon < 1$, let the constant $m \geq 0$. First of all, we will establish a series of elementary estimates for Case 2.

Lemma 2.3. *There hold the following elementary estimates:*
(1)

$$\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \exp(-\alpha|\eta|^{2\rho}) \left[\mathrm{i}\sum_{k=1}^{2} \eta_k \widehat{\phi}_k(t^{-1/(2\rho)}\eta) \right] \right|^2 \mathrm{d}\eta$$
$$\leq \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta \left\{ \sum_{k=1}^{2} \left[\iint_{\mathbb{R}^2} |\phi_k(\mathbf{x})|\mathrm{d}\mathbf{x} \right]^2 \right\}.$$

(2)

$$\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] \left[\mathrm{i}\sum_{k=1}^{2} \eta_k \widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau) \right] \mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\leq \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta \left\{ \sum_{k=1}^{2} \left[\int_0^\infty \iint_{\mathbb{R}^2} \psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t \right]^2 \right\}.$$

(3)

$$\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\leq \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta \left\{ \int_0^\infty \iint_{\mathbb{R}^2} |u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2.$$

(4)

$$\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] \left[\mathrm{i}\sum_{k=1}^{2} \eta_k \widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau) \right] \mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\leq C_0(m)t^{-(4-2\rho)/\rho} \left\{ \iint_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sum_{k=1}^{2} \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{(2m+6+\delta-2\rho)/(2\rho)} \iint_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2}\psi_k(\mathbf{x},\tau)|\mathrm{d}\mathbf{x} \right]^2 \right\}.$$

(5)

$$\iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$

$$
\begin{aligned}
&\le C_0(m)t^{-(4-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\},\\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{2/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{(2m+4+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}.
\end{aligned}
$$

These estimates are true for all constants $m\ge 0$ and for all $t>0$.

Proof. The proof of the lemma is skipped. □

Lemma 2.4. *Define the following complex auxiliary functions*

$$
\begin{aligned}
\Lambda_2(\eta,t)&=\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right]\\
&+\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\\
&-\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_2(\eta,t)&=\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right]\\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\\
&-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau,
\end{aligned}
$$

for all $(\eta,t)\in\mathbb{R}^2\times\mathbb{R}^+$. Then there holds the following estimate

$$
\begin{aligned}
&\left|\iint_{\mathbb{R}^2}|\eta|^{4m}|\Lambda_2(\eta,t)|^2\mathrm{d}\eta-\iint_{\mathbb{R}^2}|\eta|^{4m}|\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right|\\
&\le\iint_{\mathbb{R}^2}|\eta|^{4m}|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)|^2\mathrm{d}\eta\\
&+2\left\{\iint_{\mathbb{R}^2}|\eta|^{4m}|\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\left\{\iint_{\mathbb{R}^2}|\eta|^{4m}|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2},
\end{aligned}
$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\ge 0$ and for all $t>0$.

Proof. The proof is very similar to that of Lemma 2.2. The details are skipped. □

2.3. The comprehensive analysis: Case 1

The main purposes of this subsection are to make use of the elementary estimates for Case 1 and the representation of the Fourier transformation of the global weak solutions to establish optimal estimates for $t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}$ and $t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}$. These estimates will play very important roles when we accomplish the primary decay estimates and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions of the two-dimensional dissipative quasi-geostrophic equation.

Lemma 2.5. *For Case 1, there holds the following optimal estimate*

$$
\begin{aligned}
&t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
\leq{}& \frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\left\{\iint_{\mathbb{R}^2}|u_0(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2\\
&+\frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_0^\infty\iint_{\mathbb{R}^2}|f(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{5t^{-1/\rho}}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_0^\infty\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{5C_0(m)}{(2\pi)^2}t^{-(2-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
&\cdot\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m-\rho+(1+\delta)/2}f(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2\\
&+\frac{5C_0(m)}{(2\pi)^2}t^{-(3-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
&\cdot\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\\
&\cdot\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\},
\end{aligned}
$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\geq 0$ and for all (sufficiently large) $t>0$.

Proof. Let $\mathcal{N}(u)=J(u,(-\triangle)^{-1/2}u)$. First of all, recall that there holds the following representation for the Fourier transformation of the global weak solutions:

$$
\begin{aligned}
&\widehat{u}(t^{-1/(2\rho)}\eta,t)=\exp(-\alpha|\eta|^{2\rho})\widehat{u}_0(t^{-1/(2\rho)}\eta)\\
&+\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\\
&-\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau,
\end{aligned}
$$

for all $(\eta,t)\in\mathbb{R}^2\times\mathbb{R}^+$.

By coupling together the Parseval's identity, a simple property of the Fourier transformation, the change of variables $\eta=t^{1/(2\rho)}\xi$, so that $|\eta|^{2\rho}=|\xi|^{2\rho}t$ and $\mathrm{d}\eta=t^{1/\rho}\mathrm{d}\xi$, the representation of the Fourier transformation $\widehat{u}(t^{-1/(2\rho)}\eta,t)$ of the global weak solutions, the elementary estimates in Subsection 2.1, we have the following computations and estimates

$$
t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}
$$

$$
\begin{aligned}
&= \frac{t^{(2m+1)/\rho}}{(2\pi)^2} \iint_{\mathbb{R}^2} |\xi|^{4m} |\widehat{u}(\xi,t)|^2 \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} |\widehat{u}(t^{-1/(2\rho)}\eta,t)|^2 \mathrm{d}\eta \\
&= \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \Big| \exp(-\alpha|\eta|^{2\rho}) \widehat{u}_0(t^{-1/(2\rho)}\eta) \\
&+ \int_0^t \exp\Big[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\Big] \widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \\
&- \int_0^t \exp\Big[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\Big] \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \Big|^2 \mathrm{d}\eta \\
&= \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \Big| \exp(-\alpha|\eta|^{2\rho}) \widehat{u}_0(t^{-1/(2\rho)}\eta) \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\Big] \widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \\
&- \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\Big] \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \\
&+ \int_{(1-\varepsilon)t}^t \exp\Big[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\Big] \widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \\
&- \int_{(1-\varepsilon)t}^t \exp\Big[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\Big] \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \Big|^2 \mathrm{d}\eta \\
&\le \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \Big| \exp(-\alpha|\eta|^{2\rho}) \widehat{u}_0(t^{-1/(2\rho)}\eta) \Big|^2 \mathrm{d}\eta \\
&+ \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \Big| \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\Big] \widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \Big|^2 \mathrm{d}\eta \\
&+ \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \Big| \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\Big] \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \Big|^2 \mathrm{d}\eta \\
&+ \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \Big| \int_{(1-\varepsilon)t}^t \exp\Big[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\Big] \widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \Big|^2 \mathrm{d}\eta \\
&+ \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \Big| \int_{(1-\varepsilon)t}^t \exp\Big[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\Big] \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \Big|^2 \mathrm{d}\eta \\
&\le \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta \Big\{ \iint_{\mathbb{R}^2} |u_0(\mathbf{x})|\mathrm{d}\mathbf{x} \Big\}^2 \\
&+ \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta \Big\{ \int_0^\infty \iint_{\mathbb{R}^2} |f(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t \Big\}^2 \\
&+ \frac{5t^{-1/\rho}}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta \Big\{ \int_0^\infty \iint_{\mathbb{R}^2} |u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t \Big\}^2 \\
&+ \frac{5C_0(m)}{(2\pi)^2} t^{-(2-2\rho)/\rho} \Big\{ \iint_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m-\rho+(1+\delta)/2}f(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2\\
&+\frac{5C_0(m)}{(2\pi)^2}t^{-(3-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
&\cdot \sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\\
&\cdot \sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\},
\end{aligned}
$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\geq 0$ and $t>0$. The proof of the lemma is finished now. $\square$

Lemma 2.6. *There holds the following optimal estimate*

$$
\begin{aligned}
&t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\\
\leq&\frac{2}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_0^\infty\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{2C_0(m)}{(2\pi)^2}t^{-(2-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
&\cdot \sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\\
&\cdot \sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\},
\end{aligned}
$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\geq 0$ and $t>0$, where $C_0=C_0(m)>0$ is a positive constant, independent of (α,ρ), (δ,ε), (u_0,f), u and $(\mathbf{x},t)$.

Proof. Recall that there holds the following representation for the Fourier transformation of $u-v$:

$$
\widehat{u}(t^{-1/(2\rho)}\eta,t)-\widehat{v}(t^{-1/(2\rho)}\eta,t)=-\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau,
$$

for all $(\eta,t)\in\mathbb{R}^2\times\mathbb{R}^+$. The main ideas and the main steps of the proof are the same. For the completeness of this paper, we will give all the details. By coupling together the Parseval's identity, a simple property of the Fourier transformation, the change of variables $\eta=t^{1/(2\rho)}\xi$, the representation of the Fourier transformation $\widehat{u}(t^{-1/(2\rho)}\eta,t)-\widehat{v}(t^{-1/(2\rho)}\eta,t)$ and the elementary estimates in Subsection 2.1, we have the following computations and estimates

$$
\begin{aligned}
&t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\\
=&\frac{t^{(2m+2)/\rho}}{(2\pi)^2}\iint_{\mathbb{R}^2}|\xi|^{4m}|\widehat{u}(\xi,t)-\widehat{v}(\xi,t)|^2\mathrm{d}\xi\\
=&\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}|t^{1/(2\rho)}[\widehat{u}(t^{-1/(2\rho)}\eta,t)-\widehat{v}(t^{-1/(2\rho)}\eta,t)]|^2\mathrm{d}\eta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_0^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&= \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \right. \\
&+ \left. \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&\le \frac{2}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&+ \frac{2}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&\le \frac{2}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta \left\{ \int_0^\infty \iint_{\mathbb{R}^2} |u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 \\
&+ \frac{2C_0(m)}{(2\pi)^2} t^{-(2-2\rho)/\rho} \left\{ \iint_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \right\} \\
&\cdot \sup_{(1-\varepsilon)t\le\tau\le t} \left\{ \tau^{1/\rho} \iint_{\mathbb{R}^2} |u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right\} \\
&\cdot \sup_{(1-\varepsilon)t\le\tau\le t} \left\{ \tau^{(2m+3+\delta-2\rho)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2} u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right\},
\end{aligned}
$$

for all positive constants $0 < \delta < 2\rho$ and $0 < \varepsilon < 1$, for all $m \ge 0$ and $t > 0$. The proof of the lemma is finished now. $\square$

2.4. The comprehensive analysis: Case 2

The main purposes of this subsection are to make use of the elementary estimates for Case 2 and the representation of the Fourier transformation of the global weak solutions to establish optimal estimates for $t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}$ and $t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t) - v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x}$. These estimates will play very important roles when we accomplish the primary decay estimates and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions of the two-dimensional dissipative quasi-geostrophic equation.

Lemma 2.7. *For Case 2, there holds the following optimal estimate*

$$
\begin{aligned}
& t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \\
&\le \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta \left\{ \sum_{k=1}^{2} \left[\iint_{\mathbb{R}^2} |\phi_k(\mathbf{x})| \mathrm{d}\mathbf{x} \right]^2 \right\} \\
&+ \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta \left\{ \sum_{k=1}^{2} \left[\int_0^\infty \iint_{\mathbb{R}^2} |\psi_k(\mathbf{x},t)| \mathrm{d}\mathbf{x}\mathrm{d}t \right]^2 \right\} \\
&+ \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta \left\{ \int_0^\infty \iint_{\mathbb{R}^2} |u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{5C_0(m)}{(2\pi)^2}t^{-(4-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\sum_{k=1}^{2}\left[\tau^{(2m+6+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}\psi_k(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}\\
&+\frac{5C_0(m)}{(2\pi)^2}t^{-(4-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{2/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{(2m+4+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\},
\end{aligned}
$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\ge 0$ and $t>0$.

Proof. As before, let $\mathcal{N}(u)=J(u,(-\triangle)^{-1/2}u)$. For Case 2, there holds the following representation for the Fourier transformation of the global weak solutions:

$$
\begin{aligned}
t^{1/(2\rho)}\widehat{u}(t^{-1/(2\rho)}\eta,t)&=\mathrm{i}\exp(-\alpha|\eta|^{2\rho})\left\{\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right\}\\
&+\mathrm{i}\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left\{\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right\}\mathrm{d}\tau\\
&-\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau,
\end{aligned}
$$

for all $(\eta,t)\in\mathbb{R}^2\times\mathbb{R}^+$.

We have the following computations and estimates

$$
\begin{aligned}
&t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
=&\frac{t^{(2m+2)/\rho}}{(2\pi)^2}\iint_{\mathbb{R}^2}|\xi|^{4m}|\widehat{u}(\xi,t)|^2\mathrm{d}\xi\\
=&\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}|t^{1/(2\rho)}\widehat{u}(t^{-1/(2\rho)}\eta,t)|^2\mathrm{d}\eta\\
=&\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right]\right.\\
+&\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\\
-&\left.\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
=&\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right]\right.\\
+&\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \\
&+ \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] \left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau \\
&- \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\Bigg|^2 \mathrm{d}\eta \\
&\le \frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right]\right|^2\mathrm{d}\eta \\
&+ \frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\right|^2\mathrm{d}\eta \\
&+ \frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta \\
&+ \frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\right|^2\mathrm{d}\eta \\
&+ \frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta \\
&\le \frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\left\{\sum_{k=1}^{2}\left[\iint_{\mathbb{R}^2}|\phi_k(\mathbf{x})|\mathrm{d}\mathbf{x}\right]^2\right\} \\
&+ \frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\sum_{k=1}^{2}\left[\int_0^{\infty}\iint_{\mathbb{R}^2}|\psi_k(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right\} \\
&+ \frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_0^{\infty}\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2 \\
&+ \frac{5C_0(m)}{(2\pi)^2}t^{-(4-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\} \\
&\cdot\sum_{k=1}^{2}\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{(2m+6+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}\psi_k(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\} \\
&+ \frac{5C_0(m)}{(2\pi)^2}t^{-(4-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\} \\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{2/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\} \\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{(2m+4+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\},
\end{aligned}
$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\ge 0$ and $t>0$. The proof of the lemma is finished now. $\square$

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Lemma 2.8. *There holds the following optimal estimate*

$$
\begin{aligned}
& t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t) - v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \\
\leq \ & \frac{2}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \int_0^\infty \iint_{\mathbb{R}^2} |u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 \\
+ \ & \frac{2C_0(m)}{(2\pi)^2} t^{-(4-2\rho)/\rho} \left\{ \iint_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2 |\eta|^{2+2\delta}} \mathrm{d}\eta \right\} \\
\cdot \ & \sup_{(1-\varepsilon)t \leq \tau \leq t} \left\{ \tau^{2/\rho} \iint_{\mathbb{R}^2} |u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right\} \\
\cdot \ & \sup_{(1-\varepsilon)t \leq \tau \leq t} \left\{ \tau^{(2m+4+\delta-2\rho)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2} u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right\},
\end{aligned}
$$

for all positive constants $0 < \delta < 2\rho$ and $0 < \varepsilon < 1$, for all $m \geq 0$ and $t > 0$, where $C_0 = C_0(m) > 0$ is positive constant, independent of (α, ρ), (δ, ε), (u_0, f), u and $(\mathbf{x}, t)$.

Proof. Recall that there holds the following representation for the Fourier transformation of $u - v$:

$$
\widehat{u}(t^{-1/(2\rho)}\eta, t) - \widehat{v}(t^{-1/(2\rho)}\eta, t) = -\int_0^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta, \tau)\mathrm{d}\tau,
$$

for all $(\eta, t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

By coupling together the Parseval's identity, a simple property of the Fourier transformation, the change of variables $\eta = t^{1/(2\rho)}\xi$, so that $|\eta|^{2\rho} = |\xi|^{2\rho} t$ and $\mathrm{d}\eta = t^{1/\rho}\mathrm{d}\xi$, the representation of the Fourier transformation $t^{1/(2\rho)}[\widehat{u}(t^{-1/(2\rho)}\eta, t) - \widehat{v}(t^{-1/(2\rho)}\eta, t)]$ of the global weak solutions, the elementary estimates in Subsection 2.2, we have the following computations and estimates

$$
\begin{aligned}
& t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t) - v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \\
= \ & \frac{t^{(2m+2)/\rho}}{(2\pi)^2} \iint_{\mathbb{R}^2} |\xi|^{4m} |\widehat{u}(\xi,t) - \widehat{v}(\xi,t)|^2 \mathrm{d}\xi \\
= \ & \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} |t^{1/(2\rho)} [\widehat{u}(t^{-1/(2\rho)}\eta, t) - \widehat{v}(t^{-1/(2\rho)}\eta, t)]|^2 \mathrm{d}\eta \\
= \ & \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_0^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
= \ & \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta, \tau)\mathrm{d}\tau \right. \\
+ \ & \left. \int_{(1-\varepsilon)t}^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
\leq \ & \frac{2}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
+ \ & \frac{2}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{2}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_0^\infty\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{2C_0(m)}{(2\pi)^2}t^{-(4-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{2/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{(2m+4+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\},
\end{aligned}
$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\ge 0$ and $t>0$. The proof of the lemma is finished now. □

2.5. The primary decay estimates with sharp rates

The main purposes of this subsection are to establish the primary decay estimates with sharp rates for both cases. The main idea is to apply the estimates obtained in the comprehensive analysis and the iteration technique. These primary decay estimates will be used to accomplish the exact limits.

Let the positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, let the real constants $m\ge 0$ and $\kappa\ge 0$.

Recall that there holds the elementary decay estimate with a sharp rate

$$\sup_{t>0}\left\{t^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty.$$

Also recall that there hold the following uniform estimates

$$\sup_{t>T}\left\{\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

for all constants $m\ge 0$.

The Proof of Theorem 1.5: Multiplying the inequality in Lemma 2.5 by $t^{-\kappa/\rho}$, we have

$$
\begin{aligned}
&t^{(2m+1-\kappa)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
&\le\frac{5t^{-\kappa/\rho}}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\left\{\iint_{\mathbb{R}^2}|u_0(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2\\
&+\frac{5t^{-\kappa/\rho}}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_0^\infty\iint_{\mathbb{R}^2}|f(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{5t^{-(1+\kappa)/\rho}}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_0^\infty\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{5C_0(m)}{(2\pi)^2}t^{-(2+\kappa-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{(2m+3+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m-\rho+(1+\delta)/2}f(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2\\
&+\frac{5C_0(m)}{(2\pi)^2}t^{-(3+\kappa-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}
\end{aligned}
$$

$$\cdot \sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}$$
$$\cdot \sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\},$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\ge 0$ and $t>0$. The positive constant $C_0=C_0(m)>0$ is independent of (α,ρ), (δ,ε), (u_0,f), $(\mathbf{x},t)$, u and κ.

Let us iterate the above procedure for a finite number of times, so that each time the decay rate increases $(1-\delta)/\rho$. Without loss of generality, we may make δ slightly smaller by letting $0<\delta<\min\{1,2\rho\}$. This way we may establish the primary decay estimates with sharp rates for each fixed positive constant $m>0$. Precisely, let r_i be the rate of decay we obtain if we let $\kappa=\kappa_i$, where $i=1,2,3,\cdots,M$, and M is a finite positive integer. We have the following iteration process

$$\begin{aligned}
&\kappa=\kappa_1=2m+\delta, \qquad r_1=\frac{1-\delta}{\rho},\\
&\kappa=\kappa_2=(2m+\delta)-(1-\delta), \qquad r_2=\frac{2(1-\delta)}{\rho},\\
&\kappa=\kappa_3=(2m+\delta)-2(1-\delta), \qquad r_3=\frac{3(1-\delta)}{\rho},\\
&\kappa=\kappa_4=(2m+\delta)-3(1-\delta), \qquad r_4=\frac{4(1-\delta)}{\rho},\\
&\cdots\cdots\cdots\cdots\\
&\kappa=\kappa_M=(2m+\delta)-(M-1)(1-\delta), \qquad r_M=\frac{M(1-\delta)}{\rho}.
\end{aligned}$$

If there exists a positive integer $M>1$, such that

$$M\frac{1-\delta}{\rho}=\frac{2m+1}{\rho},$$

then we have finished the proof of Theorem 1.5.

If there exists a positive integer $M>1$, such that

$$M\frac{1-\delta}{\rho}<\frac{2m+1}{\rho}<(M+1)\frac{1-\delta}{\rho},$$

then letting $\kappa_{M+1}=0$, we get $r_{M+1}=(2m+1)/\rho$.

The primary decay estimates with sharp rates for all order derivatives of the global weak solutions have been finished now, for Case 1.

Remark 2.1. By Theorem 1.5, Case 1, there holds the estimate

$$\sup_{t>T}\left\{t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

for all constants $m\ge 0$. Therefore, we have

$$\sup_{t>T}\left\{t^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

$$\sup_{t>T}\left\{t^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

for all constants $0<\delta<2\rho$ and $m\geq 0$. Now

$$\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$
$$\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

for all constants $0<\delta<2\rho$, $0<\varepsilon<1$ and $m\geq 0$, and for all $(1-\varepsilon)t>T$. Therefore, we have

$$\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\right|^2\mathrm{d}\eta\leq C_0(m)t^{-(2-2\rho)/\rho},$$

for all constants $0<\delta<2\rho$, $0<\varepsilon<1$ and $m\geq 0$, and for all $(1-\varepsilon)t>T$.

The Proof of Theorem 1.6: We have the following computations and estimates

$$
\begin{aligned}
&t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
=&\frac{t^{(2m+2)/\rho}}{(2\pi)^2}\iint_{\mathbb{R}^2}|\xi|^{4m}|\widehat{u}(\xi,t)|^2\mathrm{d}\xi\\
=&\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}|t^{1/(2\rho)}\widehat{u}(t^{-1/(2\rho)}\eta,t)|^2\mathrm{d}\eta\\
=&\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right]\right.\\
&+\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\\
&\left.-\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
=&\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right]\right.\\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\\
&-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\\
&+\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\\
&\left.-\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\right|^2\mathrm{d}\eta\\
\leq&\frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right]\right|^2\mathrm{d}\eta
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&+\frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&+\frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&+\frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&\le\frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\left\{\sum_{k=1}^{2}\left[\iint_{\mathbb{R}^2}|\phi_k(\mathbf{x})|\mathrm{d}\mathbf{x}\right]^2\right\}\\
&+\frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\sum_{k=1}^{2}\left[\int_0^\infty\iint_{\mathbb{R}^2}|\psi_k(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right\}\\
&+\frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_0^\infty\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{5C_0(m)}{(2\pi)^2}t^{-(4-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sum_{k=1}^{2}\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{(2m+6+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}\psi_k(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}\\
&+\frac{5C_0(m)}{(2\pi)^2}t^{-(2-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\\
&\cdot\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\\
&\le C,
\end{aligned}
$$

for all constants $0<\delta<2\rho$, $0<\varepsilon<1$, $m\ge 0$ and for all sufficiently large $t>0$. The proof of Theorem 1.6 is finished now. □

Remark 2.2. By Theorem 1.6, there holds the estimate

$$
\sup_{t>T}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,
$$

for all constants $m\ge 0$. Therefore, we have

$$
\begin{aligned}
&\sup_{t>T}\left\{t^{2/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,\\
&\sup_{t>T}\left\{t^{(2m+4+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,
\end{aligned}
$$

for all constants $0 < \delta < 2\rho$ and for all $m \geq 0$. Now

$$\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{2/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$
$$\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+4+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

for all constants $0 < \delta < 2\rho$, $0 < \varepsilon < 1$ and $m \geq 0$, and for all $(1-\varepsilon)t > T$. Therefore, we have

$$\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\right|^2\mathrm{d}\eta\leq C_0(m)t^{-(4-2\rho)/\rho},$$

for all constants $0 < \delta < 2\rho$, $0 < \varepsilon < 1$ and $m \geq 0$, and for all $(1-\varepsilon)t > T$.

2.6. The fundamental limits

Now we are ready to make use of the elementary estimates and the primary decay estimates to establish several fundamental limits for the global weak solutions of the two-dimensional dissipative quasi-geostropic equation. Let the positive constants $0 < \delta < 2\rho$ and $0 < \varepsilon < 1$. Let the constant $m \geq 0$ and let $\eta \in \mathbb{R}^2$.

Observation. Note that

$$\begin{aligned}&\iint_{\mathbb{R}^2}\eta_1^2|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\\ =&\iint_{\mathbb{R}^2}\eta_2^2|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\\ =&\frac{1}{2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta,\end{aligned}$$

and

$$\iint_{\mathbb{R}^2}\eta_1\eta_2|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta=0.$$

Therefore

$$\begin{aligned}&\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\sum_{k=1}^{2}a_k\eta_k+\mathrm{i}\sum_{k=1}^{2}b_k\eta_k\right|^2\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\\ =&\frac{1}{2}\sum_{k=1}^{2}(a_k^2+b_k^2)\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta,\end{aligned}$$

for all constants $\alpha > 0$, $0 < \rho < 1$, $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$ and $m \geq 0$. First of all, let us compute a few limits closely related to the Fourier transformation of the Jacobian determinant $J(u, (-\triangle)^{-1/2}u)$, as $t \to \infty$.

Lemma 2.9. *There holds the following limit*

$$\lim_{t\to\infty}\left\{t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\right\}=-\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\frac{\lambda\odot\eta}{|\lambda|}|\widehat{u}(\lambda,\tau)|^2\mathrm{d}\lambda,$$

for all $(\eta, t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

Proof. Recall that the Jacobian determinant is

$$\begin{aligned}J(u,(-\triangle)^{-1/2}u) &= \frac{\partial}{\partial x}u\frac{\partial}{\partial y}(-\triangle)^{-1/2}u - \frac{\partial}{\partial y}u\frac{\partial}{\partial x}(-\triangle)^{-1/2}u \\ &= \frac{\partial}{\partial x}\left\{u\frac{\partial}{\partial y}(-\triangle)^{-1/2}u\right\} - \frac{\partial}{\partial y}\left\{u\frac{\partial}{\partial x}(-\triangle)^{-1/2}u\right\}.\end{aligned}$$

Hence the Fourier transformation

$$\begin{aligned}\widehat{\mathcal{N}(u)}(\xi,t) &= \mathrm{i}\xi_1\widehat{u\mathrm{R}_2u}(\xi,t) - \mathrm{i}\xi_2\widehat{u\mathrm{R}_1u}(\xi,t) \\ &= -\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\frac{\lambda_1\xi_2-\lambda_2\xi_1}{|\lambda|}\widehat{u}(\lambda+\xi,t)\overline{\widehat{u}}(\lambda,t)\mathrm{d}\lambda,\end{aligned}$$

for all $(\xi,t)\in\mathbb{R}^2\times\mathbb{R}^+$, where

$$\begin{aligned}\widehat{u\mathrm{R}_1u}(\xi,t) &= -\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\frac{\mathrm{i}\lambda_1}{|\lambda|}\widehat{u}(\lambda+\xi,t)\overline{\widehat{u}}(\lambda,t)\mathrm{d}\lambda, \\ \widehat{u\mathrm{R}_2u}(\xi,t) &= -\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\frac{\mathrm{i}\lambda_2}{|\lambda|}\widehat{u}(\lambda+\xi,t)\overline{\widehat{u}}(\lambda,t)\mathrm{d}\lambda.\end{aligned}$$

Making the change of variables $\eta = t^{1/(2\rho)}\xi$, we have

$$t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau) = -\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\frac{\lambda\odot\eta}{|\lambda|}\widehat{u}(\lambda+t^{-1/(2\rho)}\mu,\tau)\overline{\widehat{u}}(\lambda,\tau)\mathrm{d}\lambda,$$

for all $(\eta,\tau)\in\mathbb{R}^2\times\mathbb{R}^+$.

The limit follows from the Lebesgue's dominated convergence theorem. The proof of the lemma is finished now. □

Lemma 2.10. *There holds the following limit*

$$\begin{aligned}&\lim_{t\to\infty}\left\{-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right\} \\ &= \exp(-\alpha|\eta|^{2\rho})\left\{\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda\odot\eta}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}.\end{aligned}$$

Proof. The limit follows from Lebesgue's dominated convergence theorem and the result of Lemma 2.9. □

Lemma 2.11. *For Case 1, there holds the following limit*

$$\begin{aligned}&\lim_{t\to\infty}\Big\{\exp(-\alpha|\eta|^{2\rho})\widehat{u}_0(t^{-1/(2\rho)}\eta) \\ &+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \\ &-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\Big\} \\ &=\exp(-\alpha|\eta|^{2\rho})\left\{\iint_{\mathbb{R}^2}u_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}.\end{aligned}$$

For Case 2, there holds the following limit

$$\begin{aligned}
&\lim_{t\to\infty}\Bigg\{\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right] \\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau \\
&-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\Bigg\} \\
&=\exp(-\alpha|\eta|^{2\rho})\Bigg\{\mathrm{i}\sum_{k=1}^{2}\eta_k\left[\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right] \\
&+\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda\odot\eta}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\Bigg\}.
\end{aligned}$$

Proof: The proof follows from Lebesgue's dominated convergence theorem, the continuity of the Fourier transformations and the estimates in Lemma 2.1. The details are omitted. □

Lemma 2.12. *For Case 1, there holds the following limit*

$$\begin{aligned}
&\lim_{t\to\infty}\Bigg\{\iint_{\mathbb{R}^2}|\eta|^{4m}\Bigg|\exp(-\alpha|\eta|^{2\rho})\widehat{u}_0(t^{-1/(2\rho)}\eta) \\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \\
&-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\Bigg\} \\
&=\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\left\{\iint_{\mathbb{R}^2}u_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2.
\end{aligned}$$

For Case 2, there holds the following limit

$$\begin{aligned}
&\lim_{t\to\infty}\Bigg\{\iint_{\mathbb{R}^2}|\eta|^{4m}\Bigg|\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right] \\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau \\
&-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\Bigg\} \\
&=\frac{1}{2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta \\
&\cdot\Bigg\{\sum_{k=1}^{2}\left[\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2 \\
&+\sum_{k=1}^{2}\left[\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right]^2\Bigg\}.
\end{aligned}$$

Proof: The proof follows from Lebesgue's dominated convergence theorem, the estimates in Lemma 2.1 and Lemma 2.3, and the limits in Lemma 2.11. $\square$

Lemma 2.13. *For Case 1, there holds the following limit*

$$\lim_{t\to\infty}\left\{\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right.\right.$$
$$\left.\left.-\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}=0.$$

For Case 2, there holds the following limit

$$\lim_{t\to\infty}\left\{\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau\right.\right.$$
$$\left.\left.-\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}=0.$$

Proof: The proof follows from the estimates in Remark 2.1 and Remark 2.2. The details are omitted. $\square$

Lemma 2.14. *For Case 1, there hold the following limits*

$$\begin{aligned}
&\lim_{t\to\infty}\left\{\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\widehat{u}_0(t^{-1/(2\rho)}\eta)\right.\right.\\
&+\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\\
&\left.\left.-\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}\\
&=\lim_{t\to\infty}\left\{\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\widehat{u}_0(t^{-1/(2\rho)}\eta)\right.\right.\\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\\
&\left.\left.-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}\\
&=\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\left\{\iint_{\mathbb{R}^2}u_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2.
\end{aligned}$$

For Case 2, there hold the following limits

$$\lim_{t\to\infty}\left\{\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right]\right.\right.$$
$$+\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau$$

$$
\begin{aligned}
&- \int_0^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\Big|^2 \mathrm{d}\eta\Bigg\} \\
&= \lim_{t\to\infty}\Bigg\{\iint_{\mathbb{R}^2}|\eta|^{4m}\Bigg|\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right] \\
&+ \int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau \\
&- \int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\Bigg\} \\
&= \frac{1}{2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta \\
&\cdot\Bigg\{\sum_{k=1}^{2}\left[\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2 \\
&+ \sum_{k=1}^{2}\left[\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right]^2\Bigg\}.
\end{aligned}
$$

Proof: The proof follows from the squeeze theorem, the limits in Lemma 2.12 and Lemma 2.13, and the estimates in Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4. The details are omitted. □

The Proof of Theorem 1.9: By using the explicit representation of the Fourier transformation

$$
\begin{aligned}
&\widehat{v}(t^{-1/(2\rho)}\eta,t) = \exp(-\alpha|\eta|^{2\rho})\widehat{u}_0(t^{-1/(2\rho)}\eta) \\
&+ \int_0^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau,
\end{aligned}
$$

and the ideas in this subsection, we can finish the computations of the exact limits easily. There hold the following identities

$$
\begin{aligned}
&\left\{\iint_{\mathbb{R}^2}|\eta|^{4m+4\rho+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\right\} \\
/&\left\{\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\right\} \\
=&\frac{(2m+2)(2m+\rho+2)}{(2\alpha\rho)^2}, \\
&\left\{\iint_{\mathbb{R}^2}|\eta|^{4m+8\rho+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\right\} \\
/&\left\{\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\right\} \\
=&\frac{(2m+2)(2m+\rho+2)(2m+2\rho+2)(2m+3\rho+2)}{(2\alpha\rho)^4},
\end{aligned}
$$

for all positive constants $\alpha > 0$ and $\rho > 0$, for all $m \geq 0$. By using these identities, the ratios of the exact limits are not difficult to compute. The proof is finished. □

The Proof of Theorem 1.10: There holds the following representation for the Fourier transformation of the global smooth solution of (1.3)-(1.4):

$$\begin{aligned} & t^{1/(2\rho)}\widehat{u}(t^{-1/(2\rho)}\eta,t)=\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right] \\ & +\int_0^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau, \end{aligned}$$

for all $(\eta,t)\in\mathbb{R}^2\times\mathbb{R}^+$. The details are very similar to those of Theorem 1.9 and they are omitted. □

2.7. The exact limits

The main purposes of this subsection are to make complete use of the fundamental limits to accomplish the exact limits for all order derivatives of the global weak solutions of the two-dimensional dissipative quasi-geostrophic equation. For Case 1, we will prove the exact limit

$$\begin{aligned} & \lim_{t\to\infty}\left\{t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} \\ =& \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta \\ & \cdot\left\{\iint_{\mathbb{R}^2}u_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2. \end{aligned}$$

For Case 2, we will prove the following exact limit

$$\begin{aligned} & \lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} \\ =& \frac{1}{2(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta \\ & \cdot\left\{\sum_{k=1}^{2}\left[\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right. \\ & \left.+\sum_{k=1}^{2}\left[\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right]^2\right\}. \end{aligned}$$

For both cases, we will prove the limit

$$\begin{aligned} & \lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\} \\ =& \frac{1}{2(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta \\ & \cdot\left\{\sum_{k=1}^{2}\left[\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right]^2\right\}. \end{aligned}$$

Recall that $u=u(\mathbf{x},t)$ represents the global weak solutions of the two-dimensional dissipative quasi-geostrophic equation, $v=v(\mathbf{x},t)$ represents the global smooth solution of the corresponding linear equation. Note that the initial functions for the two problems are the same, the external forces are the same as well.

The Proof of Theorem 1.1: By coupling together the Parseval's identity, a simple property of the Fourier transformation, the simple change of variables $\eta=t^{1/(2\rho)}\xi$, so that $|\eta|^{2\rho}=|\xi|^{2\rho}t$ and $\mathrm{d}\eta=t^{1/\rho}\mathrm{d}\xi$, the representation of the Fourier transformation $\widehat{u}(t^{-1/(2\rho)}\eta,t)$, the elementary estimates in Lemma 2.1 and the fundamental limits, we have the following computations

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\lim_{t\to\infty}\left\{\frac{t^{(2m+1)/\rho}}{(2\pi)^2}\iint_{\mathbb{R}^2}|\xi|^{4m}|\widehat{u}(\xi,t)|^2\mathrm{d}\xi\right\}\\
=&\lim_{t\to\infty}\left\{\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}|\widehat{u}(t^{-1/(2\rho)}\eta,t)|^2\mathrm{d}\eta\right\}\\
=&\lim_{t\to\infty}\left\{\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\widehat{u}_0(t^{-1/(2\rho)}\eta)\right.\right.\\
&+\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\\
&-\left.\left.\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}\\
=&\lim_{t\to\infty}\left\{\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\widehat{u}_0(t^{-1/(2\rho)}\eta)\right.\right.\\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{f}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\\
&-\left.\left.\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}\\
=&\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\left\{\iint_{\mathbb{R}^2}u_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2.
\end{aligned}
$$

The Proof of Theorem 1.2: There are several subtle differences between the main steps of the proof of Theorem 1.1 and the proof of Theorem 1.2. We give the details of the proof of Theorem 1.2 for the completeness of the paper. We have the following computations

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\lim_{t\to\infty}\left\{\frac{t^{(2m+2)/\rho}}{(2\pi)^2}\iint_{\mathbb{R}^2}|\xi|^{4m}|\widehat{u}(\xi,t)|^2\mathrm{d}\xi\right\}\\
=&\lim_{t\to\infty}\left\{\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}|t^{1/(2\rho)}\widehat{u}(t^{-1/(2\rho)}\eta,t)|^2\mathrm{d}\eta\right\}\\
=&\lim_{t\to\infty}\left\{\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^2\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right]\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau \\
&- \int_0^t \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\Big|^2\mathrm{d}\eta\Bigg\} \\
&= \lim_{t\to\infty}\left\{\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\Big|\exp(-\alpha|\eta|^{2\rho})\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\phi}_k(t^{-1/(2\rho)}\eta)\right]\right. \\
&+ \int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]\left[\mathrm{i}\sum_{k=1}^{2}\eta_k\widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right]\mathrm{d}\tau \\
&- \left.\int_0^{(1-\varepsilon)t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\Big|^2\mathrm{d}\eta\right\} \\
&= \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho}) \\
&\cdot \Big|\mathrm{i}\sum_{k=1}^{2}\eta_k\left[\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right] \\
&+ \frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda\odot\eta}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\Big|^2\mathrm{d}\eta \\
&= \frac{1}{2(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta \\
&\cdot \left\{\sum_{k=1}^{2}\left[\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right. \\
&+ \left.\sum_{k=1}^{2}\left[\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right]^2\right\}.
\end{aligned}
$$

The Proof of Theorem 1.3: The main ideas and the main steps of the proof of Theorem 1.3 are the same as those of Theorem 1.1. For the completeness of this paper, we give all the details here. We have the following computations

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\} \\
&= \lim_{t\to\infty}\left\{\frac{t^{(2m+2)/\rho}}{(2\pi)^2}\iint_{\mathbb{R}^2}|\xi|^{4m}|\widehat{u}(\xi,t)-\widehat{v}(\xi,t)|^2\mathrm{d}\xi\right\} \\
&= \lim_{t\to\infty}\left\{\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}|t^{1/(2\rho)}[\widehat{u}(t^{-1/(2\rho)}\eta,t)-\widehat{v}(t^{-1/(2\rho)}\eta,t)]|^2\mathrm{d}\eta\right\} \\
&= \lim_{t\to\infty}\left\{\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_0^t\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\} \\
&= \lim_{t\to\infty}\left\{\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right]t^{1/(2\rho)}\widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\} \\
&= \frac{1}{2(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\cdot\left\{\sum_{k=1}^{2}\left[\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right]^2\right\} \\
&= \frac{1}{2}\mathcal{I}(m)\mathcal{L}.
\end{aligned}
$$

The Proof of Theorem 1.4: To prove the ratios, let us make the change of variables $\zeta = (2\alpha)^{1/(2\rho)}\eta$. Then $|\zeta|^{2\rho} = 2\alpha|\eta|^{2\rho}$ and $\mathrm{d}\zeta = (2\alpha)^{1/\rho}\mathrm{d}\eta$. Moreover, we have

$$\iint_{\mathbb{R}^2} |\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta = (2\alpha)^{-(2m+2)/\rho}\iint_{\mathbb{R}^2} |\zeta|^{4m+2}\exp(-|\zeta|^{2\rho})\mathrm{d}\zeta,$$

for all positive constants $\alpha > 0$ and $\rho > 0$, for all $m \geq 0$. Let us differentiate the above equation with respect to α, twice and four times, respectively, to get

$$\begin{aligned}
&4\iint_{\mathbb{R}^2} |\eta|^{4m+4\rho+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\\
&= 4(2\alpha)^{-2-(2m+2)/\rho}\frac{2m+2}{\rho}\left(1+\frac{2m+2}{\rho}\right)\iint_{\mathbb{R}^2} |\zeta|^{4m+2}\exp(-|\zeta|^{2\rho})\mathrm{d}\zeta\\
&= \frac{4}{(2\alpha)^2}\frac{2m+2}{\rho}\left(1+\frac{2m+2}{\rho}\right)\iint_{\mathbb{R}^2} |\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta,
\end{aligned}$$

and

$$\begin{aligned}
&16\iint_{\mathbb{R}^2} |\eta|^{4m+8\rho+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\\
&= 16(2\alpha)^{-4-(2m+2)/\rho}\frac{2m+2}{\rho}\left(1+\frac{2m+2}{\rho}\right)\left(2+\frac{2m+2}{\rho}\right)\left(3+\frac{2m+2}{\rho}\right)\\
&\cdot \iint_{\mathbb{R}^2} |\zeta|^{4m+2}\exp(-|\zeta|^{2\rho})\mathrm{d}\zeta\\
&= \frac{16}{(2\alpha)^4}\frac{2m+2}{\rho}\left(1+\frac{2m+2}{\rho}\right)\left(2+\frac{2m+2}{\rho}\right)\left(3+\frac{2m+2}{\rho}\right)\\
&\cdot \iint_{\mathbb{R}^2} |\eta|^{4m+2}\exp(-|\eta|^{2\rho})\mathrm{d}\eta.
\end{aligned}$$

Now we get the basic ratios

$$\begin{aligned}
&\left\{\iint_{\mathbb{R}^2} |\eta|^{4m+4\rho+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\right\} / \left\{\iint_{\mathbb{R}^2} |\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\right\}\\
&= \frac{(2m+2)(2m+\rho+2)}{(2\alpha\rho)^2},
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\iint_{\mathbb{R}^2} |\eta|^{4m+8\rho+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\right\} / \left\{\iint_{\mathbb{R}^2} |\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\right\}\\
&= \frac{(2m+2)(2m+\rho+2)(2m+2\rho+2)(2m+3\rho+2)}{(2\alpha\rho)^4}.
\end{aligned}$$

2.8. The improved decay estimates with sharp rates

The main purposes of this subsection are to make complete use of the comprehensive analysis and the exact limits for all order derivatives of the global weak solutions to accomplish the improved decay estimates with sharp rates for all order derivatives of the global weak solutions of the two-dimensional incompressible dissipative quasi-geostrophic equation.

The Proof of Theorem 1.7: Recall that there hold the following estimates

$$
\begin{aligned}
& t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
\leq\ & \frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\left\{\iint_{\mathbb{R}^2}|u_0(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2\\
& + \frac{5}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_0^\infty\iint_{\mathbb{R}^2}|f(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
& + \frac{5t^{-1/\rho}}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_0^\infty\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
& + \frac{5C_0(m)}{(2\pi)^2}t^{-(2-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
& \cdot\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m-\rho+(1+\delta)/2}f(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2\\
& + \frac{5C_0(m)}{(2\pi)^2}t^{-(3-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
& \cdot\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\\
& \cdot\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
& t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\\
\leq\ & \frac{2}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_0^\infty\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
& + \frac{2C_0(m)}{(2\pi)^2}t^{-(2-2\rho)/\rho}\left\{\iint_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta\right\}\\
& \cdot\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\\
& \cdot\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\},
\end{aligned}
$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\geq 0$ and $t>T$.

To improve the decay estimates, we must control the following three quantities

$$
\begin{aligned}
& \sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m-\rho+(1+\delta)/2}f(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2,\\
& \sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\},\\
& \sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}.
\end{aligned}
$$

Recall that there hold the following exact limits

$$\lim_{t\to\infty}\left\{t^{(m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m f(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right\}^2=\mathrm{E}_4(m),$$

and

$$\begin{aligned}&\lim_{t\to\infty}\left\{t^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\mathcal{I}(-1/2)\mathcal{J},\\&\lim_{t\to\infty}\left\{t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\mathcal{I}(m-1/2)\mathcal{J},\end{aligned}$$

for all constants $m\geq 0$, where

$$\begin{aligned}&\mathcal{I}(m)=\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta,\\&\mathcal{J}=\left\{\iint_{\mathbb{R}^2}u_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2.\end{aligned}$$

By using the squeeze theorem, we have the following limits

$$\begin{aligned}&\lim_{t\to\infty}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{(2m+3+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m-\rho+(1+\delta)/2}f(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}\\&=\mathrm{E}_4(m-\rho+(1+\delta)/2),\end{aligned}$$

and

$$\begin{aligned}&\lim_{t\to\infty}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}=\mathcal{I}(-1/2)\mathcal{J},\\&\lim_{t\to\infty}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\&=\mathcal{I}(m-\rho+(1+\delta)/2)\mathcal{J},\end{aligned}$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\geq 0$.

Therefore, there exists a sufficiently large positive constant $T\gg 1$, such that

$$\begin{aligned}&\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m-\rho+(1+\delta)/2}f(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2\\&\leq 2\mathrm{E}_4(m-\rho+(1+\delta)/2),\end{aligned}$$

and

$$\begin{aligned}&\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\leq 2\mathcal{I}(-1/2)\mathcal{J},\\&\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{(2m+3+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}\\&\leq 2\mathcal{I}(m-\rho+(1+\delta)/2)\mathcal{J},\end{aligned}$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\geq 0$ and $t>T$.

Now by coupling together the estimates in the comprehensive analysis and all of the above estimates, we have finished the proof of the improved decay estimates with sharp rates

$$t^{(2m+1)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \leq \mathcal{A}_1(m) + \mathcal{B}_1(m) t^{-(2-2\rho)/\rho},$$
$$t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t) - v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \leq \mathcal{C}_1(m) + \mathcal{D}_1(m) t^{-(2-2\rho)/\rho},$$

for all order derivatives of the global weak solutions of the two-dimensional incompressible dissipative quasi-geostrophic equation. The proof of Theorem 1.7 is finished now. $\square$

The Proof of Theorem 1.8: Recall that there hold the following estimates

$$\begin{aligned}
& t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \\
\leq & \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \sum_{k=1}^{2} \left[\iint_{\mathbb{R}^2} |\phi_k(\mathbf{x})| \mathrm{d}\mathbf{x} \right]^2 \right\} \\
& + \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \sum_{k=1}^{2} \left[\int_0^\infty \iint_{\mathbb{R}^2} |\psi_k(\mathbf{x},t)| \mathrm{d}\mathbf{x} \mathrm{d}t \right]^2 \right\} \\
& + \frac{5}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \int_0^\infty \iint_{\mathbb{R}^2} |u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \mathrm{d}t \right\}^2 \\
& + \frac{5C_0(m)}{(2\pi)^2} t^{-(4-2\rho)/\rho} \left\{ \iint_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \right\} \\
& \cdot \sum_{k=1}^{2} \left\{ \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{(2m+6+\delta-2\rho)/(2\rho)} \iint_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2} \psi_k(\mathbf{x},\tau)| \mathrm{d}\mathbf{x} \right]^2 \right\} \\
& + \frac{5C_0(m)}{(2\pi)^2} t^{-(4-2\rho)/\rho} \left\{ \iint_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \right\} \\
& \cdot \sup_{(1-\varepsilon)t\leq\tau\leq t} \left\{ \tau^{2/\rho} \iint_{\mathbb{R}^2} |u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right\} \\
& \cdot \sup_{(1-\varepsilon)t\leq\tau\leq t} \left\{ \tau^{(2m+4+\delta-2\rho)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2} u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right\},
\end{aligned}$$

and

$$\begin{aligned}
& t^{(2\rho+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t) - v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \\
\leq & \frac{2}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \int_0^\infty \iint_{\mathbb{R}^2} |u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \mathrm{d}t \right\}^2 \\
& + \frac{2C_0(m)}{(2\pi)^2} t^{(4-2\rho)/\rho} \left\{ \iint_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \right\} \\
& \cdot \sup_{(1-\varepsilon)t\leq\tau\leq t} \left\{ \tau^{2/\rho} \iint_{\mathbb{R}^2} |u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right\} \\
& \cdot \sup_{(1-\varepsilon)t\leq\tau\leq t} \left\{ \tau^{(2m+4+\delta-2\rho)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2} u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right\},
\end{aligned}$$

for all positive constants $0 < \delta < 2\rho$ and $0 < \varepsilon < 1$, for all $m \geq 0$ and $t > T$.

To improve these decay estimates, we will make complete use of the exact limits in Assumption (A1) and Theorem 1.1 to replace

$$\sum_{k=1}^{2}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{(2m+6+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}\psi_k(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\},$$
$$\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{2/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\},$$
$$\sup_{(1-\varepsilon)t}\left\{\tau^{(2m+4+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}.$$

Recall that there hold the following exact limits

$$\lim_{t\to\infty}\sum_{k=1}^{2}\left\{t^{(m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m\psi_k(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right\}^2 = \mathrm{F}_4(m),$$

and

$$\lim_{t\to\infty}\left\{t^{2/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} = \mathcal{I}(0)\mathcal{K},$$
$$\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} = \mathcal{I}(m)\mathcal{K},$$

for all positive constants $0 < \delta < 2\rho$ and $0 < \varepsilon < 1$, for all $m \geq 0$, where

$$\mathcal{I}(m) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta,$$
$$\mathcal{K} = \sum_{k=1}^{2}\left\{\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2$$
$$+\sum_{k=1}^{2}\left\{\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2.$$

Now there exist the limits

$$\lim_{t\to\infty}\sum_{k=1}^{2}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{(2m+6+\delta-2\rho)/(2\rho)}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}\psi_k(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}$$
$$= \mathrm{F}_4(m+1-\rho+\delta/2),$$

and

$$\lim_{t\to\infty}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\} = \mathcal{I}(0)\mathcal{K},$$
$$\lim_{t\to\infty}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{(2m+4+\delta-2\rho)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$= \mathcal{I}(m+1-\rho+\delta/2)\mathcal{K},$$

for all positive constants $0 < \delta < 2\rho$ and $0 < \varepsilon < 1$, for all $m \geq 0$.

Therefore, there exists a sufficiently large positive constant $T \gg 1$, such that

$$\sum_{k=1}^{2} \sup_{(1-\varepsilon)t\le\tau\le t} \left\{\tau^{(2m+6+\delta-2\rho)/(2\rho)} \iint_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2}\psi_k(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right\}^2 \\ \le 2\mathrm{F}_4(m+1-\rho+\delta/2),$$

and

$$\sup_{(1-\varepsilon)t\le\tau\le t} \left\{\tau^{2/\rho} \iint_{\mathbb{R}^2} |u(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\} \le 2\mathcal{I}(0)\mathcal{K},$$
$$\sup_{(1-\varepsilon)t\le\tau\le t} \left\{\tau^{(2m+4+\delta-2\rho)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2}u(\mathbf{x},\tau|^2\mathrm{d}\mathbf{x}\right\} \\ \le 2\mathcal{I}(m+1-\rho+\delta/2)\mathcal{K},$$

for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\ge 0$ and $t>0$.

Therefore, by coupling together all of the above estimates, we have finished the proof of the improved decay estimates with sharp rates

$$t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x} \le \mathcal{A}_2(m)+\mathcal{B}_2(m)t^{-(4-2\rho)/\rho},$$
$$t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x} \le \mathcal{C}_2(m)+\mathcal{D}_2(m)t^{-(4-2\rho)/\rho},$$

for all order derivatives of the global weak solutions and for all sufficiently large t. The proof of Theorem 1.8 is finished now. $\square$

Note that there hold the following relationships

$$\iint_{\mathbb{R}^2} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^{2\rho})\mathrm{d}\eta = \varepsilon^{-(2m+2)/\rho} \iint_{\mathbb{R}^2} |\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta,$$
$$\iint_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta = \frac{(\alpha\varepsilon)^{\delta/\rho}}{\alpha^2} \iint_{\mathbb{R}^2} \frac{[1-\exp(-|\eta|^{2\rho})]^2}{|\eta|^{2+2\delta}}\mathrm{d}\eta,$$

for all positive constants $\alpha>0$, $0<\delta<2\rho$ and $0<\varepsilon<1$, for all $m\ge 0$.

3. Conclusion and Remarks

3.1. Summary

Consider the Cauchy problem for the two-dimensional incompressible dissipative quasi-geostrophic equation

$$\frac{\partial}{\partial t}u+\alpha(-\triangle)^{\rho}u+J(u,(-\triangle)^{-1/2}u)=f(\mathbf{x},t),$$
$$u(\mathbf{x},0)=u_0(\mathbf{x}).$$

In the system, $u=u(\mathbf{x},t)$ represents the temperature of the fluid, the Jacobian determinant $J(u,(-\triangle)^{-1/2}u)$ is defined by

$$J(u,(-\triangle)^{-1/2}u)=\frac{\partial}{\partial x}u\frac{\partial}{\partial y}(-\triangle)^{-1/2}u-\frac{\partial}{\partial y}u\frac{\partial}{\partial x}(-\triangle)^{-1/2}u.$$

The operators

$$\frac{\partial}{\partial x}(-\triangle)^{-1/2} \qquad \text{and} \qquad \frac{\partial}{\partial y}(-\triangle)^{-1/2}$$

represent the Riesz transformations in $\mathbb{R}^2$. Moreover, the real vector valued function

$$\mathbf{F}(\mathbf{x},t) \equiv \begin{pmatrix} -\frac{\partial}{\partial y}(-\triangle)^{-1/2}u \\ +\frac{\partial}{\partial x}(-\triangle)^{-1/2}u \end{pmatrix}$$

represents the velocity of the fluid. Note that $\nabla \cdot \mathbf{F} = 0$. Therefore, the fluid is incompressible.

Also consider the Cauchy problem for the corresponding linear equation

$$\begin{aligned} &\frac{\partial}{\partial t}v + \alpha(-\triangle)^{\rho} v = f(\mathbf{x},t), \\ &v(\mathbf{x},0) = u_0(\mathbf{x}). \end{aligned}$$

We have studied the following two main cases.

Case 1: Suppose that the initial function and the external force satisfy the following conditions

$$\begin{aligned} &u_0 \in C^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), \\ &f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)), \end{aligned}$$

such that

$$\iint_{\mathbb{R}^2} u_0(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \iint_{\mathbb{R}^2} f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t \neq 0.$$

Case 2: Suppose that the initial function and the external force satisfy the following conditions

$$\begin{aligned} &u_0 \in C^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), \\ &f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)). \end{aligned}$$

Suppose that there exist real scalar smooth functions

$$\begin{aligned} &\phi_1 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2), \ \psi_1 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+), \\ &\phi_2 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2), \ \psi_2 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+), \end{aligned}$$

such that

$$u_0(\mathbf{x}) = \frac{\partial}{\partial x}\phi_1(\mathbf{x}) + \frac{\partial}{\partial y}\phi_2(\mathbf{x}), f(\mathbf{x},t) = \frac{\partial}{\partial x}\psi_1(\mathbf{x},t) + \frac{\partial}{\partial y}\psi_2(\mathbf{x},t).$$

For Case 1, we have accomplished the exact limits

$$\begin{aligned} &\lim_{t\to\infty} \left\{ t^{(2m+1)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} \\ = &\frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m} \exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta \end{aligned}$$

$$\cdot \left\{\iint_{\mathbb{R}^2} u_0(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \iint_{\mathbb{R}^2} f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

for all constants $m \geq 0$.

For Case 2, we have accomplished the exact limits

$$\begin{aligned}
&\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{2(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\\
&\cdot\left\{\sum_{k=1}^2\left[\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right.\\
&\left.+\sum_{k=1}^2\left[\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right]^2\right\},
\end{aligned}$$

for all constants $m \geq 0$.

For both cases, we have accomplished the exact limits

$$\begin{aligned}
&\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{2(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\\
&\cdot\left\{\sum_{k=1}^2\left[\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right]^2\right\},
\end{aligned}$$

for all constants $m \geq 0$.

Moreover, for Case 1, we have accomplished the following improved decay estimates with sharp rates

$$\begin{aligned}
&t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\leq\mathcal{A}_1(m)+\mathcal{B}_1(m)t^{-(2-2\rho)/\rho},\\
&t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\leq\mathcal{C}_1(m)+\mathcal{D}_1(m)t^{-(2-2\rho)/\rho},
\end{aligned}$$

for all order derivatives of the global weak solutions and for all sufficiently large $t \gg 1$.

For Case 2, we have established the following improved decay estimates with sharp rates

$$\begin{aligned}
&t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\leq\mathcal{A}_2(m)+\mathcal{B}_2(m)t^{-(4-2\rho)/\rho},\\
&t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\leq\mathcal{C}_2(m)+\mathcal{D}_2(m)t^{-(4-2\rho)/\rho},
\end{aligned}$$

for all constants $m \geq 0$ and for all sufficiently large $t \gg 1$.

3.2. Remarks

Remark 3.1. If we drop the Jacobian determinant, then the exact limits for all order derivatives of the global weak solutions reduce to the exact limits for all order derivatives of the global smooth solution of the corresponding linear equation.

Remark 3.2. In Case 1, the exact limits for the nonlinear equation and the linear equation are the same. That is,

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\lim_{t\to\infty}\left\{t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\\
&\cdot\left\{\iint_{\mathbb{R}^2}u_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,
\end{aligned}
$$

for each constant $m\geq 0$.

In Case 2, the exact limit for the nonlinear equation is larger than the exact limit for the linear equation. That is,

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{2(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\\
&\cdot\left\{\sum_{k=1}^{2}\left[\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right.\\
&\left.+\sum_{k=1}^{2}\left[\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right]^2\right\}\\
>&\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{2(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\\
&\cdot\left\{\sum_{k=1}^{2}\left[\iint_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right\},
\end{aligned}
$$

for each constant $m\geq 0$.

Remark 3.3. For Case 1, the exact limits

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{(2m+1)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\
=&\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\\
&\cdot\left\{\iint_{\mathbb{R}^2}u_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\iint_{\mathbb{R}^2}f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,
\end{aligned}
$$

and

$$
\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\iint_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}
$$

$$
\begin{aligned}
&= \frac{1}{2(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha|\eta|^{2\rho}) \mathrm{d}\eta \\
&\quad \cdot \left\{ \sum_{k=1}^{2} \left[\frac{1}{(2\pi)^2} \int_0^\infty \iint_{\mathbb{R}^2} \frac{\lambda_k}{|\lambda|} |\widehat{u}(\lambda,t)|^2 \mathrm{d}\lambda \mathrm{d}t \right]^2 \right\},
\end{aligned}
$$

are independent of

(1) the integrals of any order derivatives of the initial function u_0,

(2) the integrals of any order derivatives of the external force f,

(3) the integrals of any order derivatives of the nonlinear function u^2.

For Case 2, the exact limits

$$
\begin{aligned}
&\lim_{t\to\infty} \left\{ t^{(2m+2)/\rho} \iint_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} \\
&= \frac{1}{2(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha|\eta|^{2\rho}) \mathrm{d}\eta \\
&\quad \cdot \left\{ \sum_{k=1}^{2} \left[\iint_{\mathbb{R}^2} \phi_k(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_0^\infty \iint_{\mathbb{R}^2} \psi_k(\mathbf{x},t) \mathrm{d}\mathbf{x} \mathrm{d}t \right]^2 \right. \\
&\quad \left. + \sum_{k=1}^{2} \left[\frac{1}{(2\pi)^2} \int_0^\infty \iint_{\mathbb{R}^2} \frac{\lambda_k}{|\lambda|} |\widehat{u}(\lambda,t)|^2 \mathrm{d}\lambda \mathrm{d}t \right]^2 \right\}, \\
&\lim_{t\to\infty} \left\{ \iint_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t) - v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \right\} \\
&= \frac{1}{2(2\pi)^2} \iint_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha|\eta|^{2\rho}) \mathrm{d}\eta \\
&\quad \cdot \left\{ \sum_{k=1}^{2} \left[\frac{1}{(2\pi)^2} \int_0^\infty \iint_{\mathbb{R}^2} \frac{\lambda_k}{|\lambda|} |\widehat{u}(\lambda,t)|^2 \mathrm{d}\lambda \mathrm{d}t \right]^2 \right\},
\end{aligned}
$$

are independent of

(1) the integrals of any order derivatives of the functions ϕ_1 and ϕ_2,

(2) the integrals of any order derivatives of the functions ψ_1 and ψ_2,

(3) the integrals of any order derivatives of the function u^2.

Remark 3.4. For Case 1, the exact limits are decreasing functions of α and ρ. The exact limits are increasing functions of m, $\mathcal{J}$ and $\mathcal{L}$, where

$$
\begin{aligned}
\mathcal{J} &= \left\{ \iint_{\mathbb{R}^2} u_0(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_0^\infty \iint_{\mathbb{R}^2} f(\mathbf{x},t) \mathrm{d}\mathbf{x} \mathrm{d}t \right\}^2, \\
\mathcal{L} &= \sum_{k=1}^{2} \left\{ \frac{1}{(2\pi)^2} \int_0^\infty \iint_{\mathbb{R}^2} \frac{\lambda_k}{|\lambda|} |\widehat{u}(\lambda,t)|^2 \mathrm{d}\lambda \mathrm{d}t \right\}^2.
\end{aligned}
$$

For Case 2, the exact limits are decreasing functions of α and ρ. The exact limits are increasing functions of m, $\mathcal{K}$ and $\mathcal{L}$, where

$$
\mathcal{K} = \sum_{k=1}^{2} \left\{ \iint_{\mathbb{R}^2} \phi_k(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_0^\infty \iint_{\mathbb{R}^2} \psi_k(\mathbf{x},t) \mathrm{d}\mathbf{x} \mathrm{d}t \right\}^2
$$

$$+\sum_{k=1}^{2}\left\{\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2,$$

$$\mathcal{L}=\sum_{k=1}^{2}\left\{\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2.$$

Remark 3.5. Recall that there holds the following elementary decay estimate

$$\sup_{t>0}\left\{(1+t)^{1/\rho}\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty.$$

Also recall that $0<\rho<1$. Therefore, the following integrals

$$\int_0^\infty\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_1}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t,$$
$$\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_2}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t,$$

exist.

Remark 3.6. Even though the following integrals

$$\left\{\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_1}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2,$$
$$\left\{\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_2}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2,$$

are not explicitly represented in terms of α, ρ and the integrals of u_0 and f, we may still view them as known information because we know that these integrals exist. That is why we call the main results the exact limits because they are given explicitly in terms of known information.

Remark 3.7. For the linear equation, if both the initial function and the external force are radially symmetric, then the global smooth solution is also radially symmetric.

For the nonlinear equation, this is unknown.

Remark 3.8. The main results obtained in this paper have indirect, but great influences on numerical simulations, in particular, in the accuracy and stability of numerical schemes of the equation.

Remark 3.9. By using Cauchy-Schwartz's inequality and Parseval's identity, it is easy to establish the following estimate

$$\begin{aligned}&\left\{\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_1}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2\\&+\left\{\frac{1}{(2\pi)^2}\int_0^\infty\iint_{\mathbb{R}^2}\frac{\lambda_2}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2\\&\le\left\{\int_0^\infty\iint_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2.\end{aligned}$$

3.3. Open problems

Problem 3.1. *Consider the Korteweg-de Vries-Burgers equation*

$$\begin{aligned}&\frac{\partial}{\partial t}u+\frac{\partial}{\partial x}u+\frac{\partial^3 u}{\partial x^3}-\alpha\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial}{\partial x}(u^2)=f(x,t),\\&u(x,0)=u_0(x),\end{aligned}$$

the Benjamin-Bona-Mahony-Burgers equation

$$\begin{aligned}&\frac{\partial}{\partial t}u+\frac{\partial}{\partial x}u-\frac{\partial^3 u}{\partial x^2\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial}{\partial x}(u^2)=f(x,t),\\&u(x,0)=u_0(x),\end{aligned}$$

and the Benjamin-Ono-Burgers equation

$$\begin{aligned}&\frac{\partial}{\partial t}u+\frac{\partial}{\partial x}u+H\frac{\partial^2}{\partial x^2}u-\alpha\frac{\partial^2}{\partial x^2}u+\beta\frac{\partial}{\partial x}(u^2)=f(x,t),\\&u(x,0)=u_0(x),\end{aligned}$$

where $\alpha>0$ and $\beta>0$ are real constants. In the Benjamin-Ono-Burgers equation, H represents the Hilbert singular integral operator, defined by

$$(H\phi)(x)=\ \textit{principal value}\ \frac{1}{\pi}\int_{\mathbb{R}}\frac{\phi(y)}{x-y}\mathrm{d}y,$$

for $\phi\in L^2(\mathbb{R})$.

Given the elementary decay estimate

$$\sup_{t>0}\left\{t^{1/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\}<\infty,$$

and given the fundamental exact limit

$$\lim_{t\to\infty}\left\{t^{1/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\},$$

in terms of the constants α, β and the integrals

$$\int_{\mathbb{R}}u_0(x)\mathrm{d}x,\qquad\int_0^\infty\int_{\mathbb{R}}f(x,t)\mathrm{d}x\mathrm{d}t,$$

can we accomplish the following exact limits

$$\lim_{t\to\infty}\left\{t^{m+1/2}\int_{\mathbb{R}}\left|\frac{\partial^m}{\partial x^m}u(x,t)\right|^2\mathrm{d}x\right\},$$

and the following improved decay estimates with sharp rates

$$\begin{aligned}&t^{m+1/2}\int_{\mathbb{R}}\left|\frac{\partial^m}{\partial x^m}u(x,t)\right|^2\mathrm{d}x\le\mathcal{A}(m)+\mathcal{B}(m)t^{-1/2},\\&t^{m+1}\int_{\mathbb{R}}\left|\frac{\partial^m}{\partial x^m}[u(x,t)-v(x,t)]\right|^2\mathrm{d}x\le\mathcal{C}(m)+\mathcal{D}(m)t^{-1/2},\end{aligned}$$

for all order derivatives of the global smooth solution to the Korteweg-de Vries-Burgers equation in terms of α, β and the integrals

$$\int_{\mathbb{R}} u_0(x)\mathrm{d}x, \qquad \int_0^\infty \int_{\mathbb{R}} f(x,t)\mathrm{d}x\mathrm{d}t?$$

For the Benjamin-Bona-Mahony-Burgers equation and the Benjamin-Ono-Burgers equation, can we do the same things?

3.4. Technical lemmas

We have applied many traditional technical lemmas in this paper, such as the Cauchy-Schwartz's inequality, Lebesgue's dominated convergence theorem and the regular Gagliardo-Nirenberg's interpolation inequality, etc. However, we only list two lemmas below.

Lemma 3.1. *There holds the Parseval's identity*

$$\iint_{\mathbb{R}^2} |\phi(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\widehat{\phi}(\xi)|^2 \mathrm{d}\xi,$$

for all functions $\phi \in L^2(\mathbb{R}^2)$.

Lemma 3.2. *There exists a positive constant $C_0 = C_0(m) > 0$, such that*

$$\begin{aligned}
&\left\{ \iint_{\mathbb{R}^2} |(-\triangle)^m [\phi(\mathbf{x})\psi(\mathbf{x})]| \mathrm{d}\mathbf{x} \right\}^2 \\
&\le C_0(m) \left\{ \iint_{\mathbb{R}^2} |\phi(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\} \left\{ \iint_{\mathbb{R}^2} |(-\triangle)^m \psi(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\} \\
&+ C_0(m) \left\{ \iint_{\mathbb{R}^2} |(-\triangle)^m \phi(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\} \left\{ \iint_{\mathbb{R}^2} |\psi(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\},
\end{aligned}$$

for all functions $\phi \in H^{2m}(\mathbb{R}^2)$ and $\psi \in H^{2m}(\mathbb{R}^2)$.

In this paper, $C_0 = C_0(m) > 0$ is the smallest such positive constant.

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