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Melnikov Functions for a Class of Piecewise Hamiltonian Systems

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Abstract This paper is concerned with the number of limit cycles for a class of piecewise Hamiltonian systems with two zones separated by two semi-straight lines. By constructing a Poincaré map, we obtain explicit expressions of the first, second and third order Melnikov functions. In addition, we apply their expressions to give upper bounds of the number of limit cycles bifurcated from a period annulus of a piecewise polynomial Hamiltonian system.

Keywords Piecewise smooth system, Melnikov function, Limit cycle, Bifurcation.

MSC(2010) 34C05, 34C07, 37G15.

1. Introduction

In recent years, the research on non-smooth systems has attracted more and more attention, especially on piecewise near-Hamiltonian systems, see [7,18,21,22] and the references therein. One of important topics in bifurcation problems for piecewise smooth systems is to study the number of limit cycles or periodic solutions of them, which is an extension of the Hilbert's 16th problem. As is known, there are two main methods to investigate limit cycle bifurcations: the Melnikov function method [1, 2, 6, 7, 12, 15, 19, 24] and the averaging method [3, 4, 9, 13, 16, 17]. It was proved in [5, 14] that the above two methods are equivalent in studying the number of limit cycles of planar C^∞ near-Hamiltonian systems or piecewise C^∞ near-integrable systems in two or higher dimensional spaces.

In 2010, Liu and Han [12] considered a piecewise near-Hamiltonian system of the form

$$\begin{cases} \dot{x} = H_y^+(x,y) + \epsilon f^+(x,y), \\ \dot{y} = -H_x^+(x,y) + \epsilon g^+(x,y), \end{cases} x > 0,$$

$$\begin{cases} \dot{x} = H_y^-(x,y) + \epsilon f^-(x,y), \\ \dot{y} = -H_x^-(x,y) + \epsilon g^-(x,y), \end{cases} x \le 0,$$

where $H_x^\pm, H_y^\pm, f^\pm, g^\pm \in C^\infty$ and $\epsilon \ge 0$ is a small real parameter, and established a formula of the first order Melnikov function which was widely used in studying the number of limit cycles bifurcated from periodic orbits, see [8, 10, 23] for example. Recently, more general results have appeared for piecewise smooth systems with

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multiple zones [11, 18, 20, 25]. For instance, Tian and Han [18] studied the number of limit cycles bifurcated from a period annulus of a class of planar piecewise near-Hamiltonian systems with three different switching curves. The authors [11] investigated limit cycle bifurcations in piecewise near-Hamiltonian systems with multiple switching curves and obtained a formula of the first order Melnikov function. Yang, Yang and Yu [24] studied a planar piecewise Hamiltonian system with two zones separated by the two semi-straight lines and presented expressions of the first and second order Melnikov functions. In [2], Chen, Li and Llibre considered piecewise smooth differential systems in $\mathbb{R}^n$ separated by a hyperplane and obtained some recursion formulas of higher order Melnikov functions.

Motivated by the works mentioned above, in this paper, we consider a piecewise Hamiltonian system of the form

$$\begin{cases} \dot{x} = H_y(x, y, \epsilon), \\ \dot{y} = -H_x(x, y, \epsilon), \end{cases} \tag{1.1}$$

where

$$H(x, y, \epsilon) = \begin{cases} H^+(x, y, \epsilon), (x, y) \in \Sigma_1, \\ H^-(x, y, \epsilon), (x, y) \in \Sigma_2, \end{cases}$$

$$H^\pm(x, y, \epsilon) = H_0^\pm(x, y) + \epsilon H_1^\pm(x, y) + \epsilon^2 H_2^\pm(x, y) + \cdots , \tag{1.2}$$

with $H_i^\pm(x, y) \in C^\infty,\ i = 0, 1, 2, \ldots$, $\epsilon \geq 0$ is a small real parameter, Σ_1 and Σ_2 are the regions with a common boundary consisting of two semi-straight lines

$$l_1 : y = k_1 x,\ \ \mu_1 x > 0$$

and

$$l_2 : y = k_2 x,\ \ \mu_2 x > 0,$$

where $\mu_1, \mu_2 = \pm 1$ with $(k_1, \mu_1) \neq (k_2, \mu_2)$, see Fig.1. By constructing a Poincaré map of system (1.1), we shall derive expressions of the first, second and third order Melnikov functions.

The rest of this paper is organized as follows. In Section 2, we establish a Ponincaré map of system (1.1) and present expressions of the first, second and third order Melnikov functions. In Section 3, we give an application to illustrate our results and estimate the number of limit cycles bifurcated from a piecewise polynomial Hamiltonian system.

2. Expressions of Melnikov functions

Consider system (1.1). We make the following basic assumptions for the unperturbed system $(1.1)|_{\epsilon=0}$ as in [11]:

(***A1***) There exist an interval $J = (\alpha, \beta)$ and two points $A_0(h) = (a_0(h), k_1 a_0(h)) \in l_1$ and $A_{10}(h) = (a_{10}(h), k_2 a_{10}(h)) \in l_2$ such that for $h \in J$

$$\begin{aligned} &H_0^+(A_0(h)) = H_0^+(A_{10}(h)) = h, \\ &H_0^-(A_0(h)) = H_0^-(A_{10}(h)). \end{aligned} \tag{2.1}$$

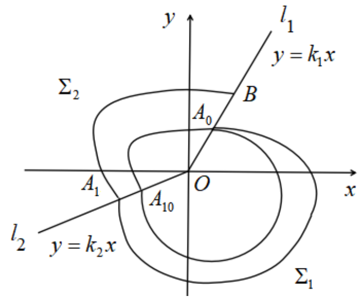

Figure 1. Phase portrait of system (1.1).

(***A2***) There is a family of closed orbits denoted by $L_h = L_h^1 \cup L_h^2, h \in J$ with clockwise orientation where L_h^1 is defined by $H_0^+(x,y) = h$, $(x,y) \in \Sigma_1$, starting from $A_0(h)$ and ending at $A_{10}(h)$, L_h^2 is defined by $H_0^-(x,y) = H^-(A_0(h))$, $(x,y) \in \Sigma_2$, starting from $A_{10}(h)$ and ending at $A_0(h)$.
(***A3***) The arcs L_h^1 and L_h^2 $(h \in J)$ are not tangent to the switching lines l_1 and l_2 at points $A_0(h)$ and $A_{10}(h)$. In other words, for each $h \in J$,

$$\begin{aligned}
&\frac{\partial H_0^\pm}{\partial x}\Big(A_{10}(h)\Big)a_{10}'(h) + k_2\frac{\partial H_0^\pm}{\partial y}\Big(A_{10}(h)\Big)a_{10}'(h) \neq 0,\\
&\frac{\partial H_0^\pm}{\partial x}\Big(A_{10}(h)\Big)a_0'(h) + k_1\frac{\partial H_0^\pm}{\partial y}\Big(A_{10}(h)\Big)a_0'(h) \neq 0.
\end{aligned} \tag{2.2}$$

Our main goal is to study the number of limit cycles bifurcated from the period annulus $\{L_h, h \in J\}$. First of all, we give a definition of bifurcation function of system (1.1). Consider the orbit of system (1.1) starting from $A_0(h) \in l_1$. For sufficiently small $|\epsilon| > 0$, it has a first intersection point with the line l_2, denoted by

$$A_1(\epsilon, h) = (a(\epsilon, h), k_2 a(\epsilon, h)). \tag{2.3}$$

For the orbit of system (1.1) starting from $A_1(\epsilon, h) \in l_2$, we denote its first intersection point with the line l_1 by

$$B(\epsilon, h) = (b(\epsilon, h), k_1 b(\epsilon, h)). \tag{2.4}$$

See Fig.1 for illustration. For smoothness of $A_0(h), A_{10}(h), A_1(\epsilon, h)$ and $B(\epsilon, h)$, we have the following lemma from [11].

Lemma 2.1. *Let assumptions (**A1**)-(**A3**) hold. Then the functions $A_0(h), A_{10}(h)$, $A_1(\epsilon, h)$ and $B(\epsilon, h)$ are C^∞ smooth with respect to (h, ϵ).*

Following [11, 12], for any integer $k \geq 1$, we can write for $h \in J$ and $|\epsilon| > 0$ sufficiently small

$$\begin{aligned}
H_0^+(B(\epsilon, h)) - H_0^+(A_0(h)) =& \epsilon F(h, \epsilon)\\
=& \sum_{j=1}^{k} \epsilon^j M_j(h) + O(\epsilon^{k+1}).
\end{aligned} \tag{2.5}$$

Here, the functions $F(h, \epsilon)$ and $M_j(h)$ in (2.5) are called a bifurcation function and the jth order Melnikov function of system (1.1), respectively. The orbit from $A_0(h)$

to $B(\epsilon, h)$ defines a Poincaré map or return map of system (1.1). From [7], we have the following bifurcation theorem.

Theorem 2.1. *[7] Under the assumptions ($\boldsymbol{A1}$) − ($\boldsymbol{A3}$), suppose that for all $1 \leq j \leq k-1$, $M_j(h) \equiv 0$ in (2.5) and that $M_k(h)$ has at most l zeros in $h \in J$, multiplicity taken into account. Then for small $|\epsilon| > 0$ system (1.1) has at most l limit cycles bifurcating from the period annulus $\{L_h, h \in J\}$, multiplicity taken into account.*

The aim of this paper is to develop formulas of the Melnikov functions up to third order. From (2.5), it is obvious that for $1 \leq j \leq k$

$$M_j(h) = \frac{1}{j!}\frac{\partial^j V_0^+}{\partial \epsilon^j}(0, h), \tag{2.6}$$

where $V_0^+(\epsilon, h) = H_0^+(B(\epsilon, h))$.

Before presenting our main results, we first give two preliminary lemmas which will be used in deducing expressions of $M_1(h)$, $M_2(h)$ and $M_3(h)$ in (2.6). For convenience, we introduce the following functions of h

$$\begin{aligned}
v_i^\pm(h) =& H_i^\pm(A_0(h)) - H_i^\pm(A_{10}(h)),\\
K_{i1}^\pm(h) =& \frac{\partial H_i^\pm}{\partial x}(A_{10}(h)) + k_2\frac{\partial H_i^\pm}{\partial y}(A_{10}(h)),\\
K_{i2}^\pm(h) =& \frac{\partial^2 H_i^\pm}{\partial x^2}(A_{10}(h)) + 2k_2\frac{\partial^2 H_i^\pm}{\partial x\partial y}(A_{10}(h)) + k_2^2\frac{\partial^2 H_i^\pm}{\partial y^2}(A_{10}(h)),\\
K_{i3}^\pm(h) =& \frac{\partial^3 H_i^\pm}{\partial x^3}(A_{10}(h)) + 3k_2\frac{\partial^3 H_i^\pm}{\partial x^2\partial y}(A_{10}(h)) + 3k_2^2\frac{\partial^3 H_i^\pm}{\partial x\partial y^2}(A_{10}(h))\\
&+ k_2^3\frac{\partial^3 H_i^\pm}{\partial y^3}(A_{10}(h)),\\
W_{i1}^\pm(h) =& \frac{\partial H_i^\pm}{\partial x}(A_0(h)) + k_1\frac{\partial H_i^\pm}{\partial y}(A_0(h)),\\
W_{i2}^\pm(h) =& \frac{\partial^2 H_i^\pm}{\partial x^2}(A_0(h)) + 2k_1\frac{\partial^2 H_i^\pm}{\partial x\partial y}(A_0(h)) + k_1^2\frac{\partial^2 H_i^\pm}{\partial y^2}(A_0(h)),\\
W_{i3}^\pm(h) =& \frac{\partial^3 H_i^\pm}{\partial x^3}(A_0(h)) + 3k_1\frac{\partial^3 H_i^\pm}{\partial x^2\partial y}(A_0(h)) + 3k_1^2\frac{\partial^3 H_i^\pm}{\partial x\partial y^2}(A_0(h))\\
&+ k_1^3\frac{\partial^3 H_i^\pm}{\partial y^3}(A_0(h)).
\end{aligned} \tag{2.7}$$

Lemma 2.2. *Under the notations in (2.7), we have for the function $a(\epsilon, h)$ in (2.3)*

$$\begin{aligned}
\frac{\partial a}{\partial \epsilon}(0,h) =& \frac{v_1^+(h)}{K_{01}^+(h)},\\
\frac{\partial^2 a}{\partial \epsilon^2}(0,h) =& \frac{1}{K_{01}^+(h)}\left(2v_2^+(h) - 2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\right),\\
\frac{\partial^3 a}{\partial \epsilon^3}(0,h) =& \frac{1}{K_{01}^+(h)}\left(6v_3^+(h) - 6K_{21}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - 3K_{12}^+(h)\left(\frac{v_1^+(h)}{K_{01}^+(h)}\right)^2\right.\\
&\left. - 3\frac{K_{11}^+(h)}{K_{01}^+(h)}\left(2v_2^+(h) - 2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\right)\right.
\end{aligned}$$

$$- 3\frac{K_{02}^+(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}\left(2v_2^+(h) - 2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - K_{02}^+(h)\left(\frac{v_1^+(h)}{K_{01}^+(h)}\right)^2\right)$$
$$- K_{03}^+(h)\left(\frac{v_1^+(h)}{K_{01}^+(h)}\right)^3\Bigg).$$

Proof. For $|\epsilon| > 0$ sufficiently small, $H_i^{\pm}(A_1(\epsilon,h))$ has the following Taylor expansion

$$H_i^{\pm}(A_1(\epsilon,h)) = H_i^{\pm}(A_{10}(h)) + \sum_{j=1}^{3}\frac{\epsilon^j}{j!}\frac{\partial^j S_i^{\pm}}{\partial \epsilon^j}(0,h) + O(\epsilon^4), \tag{2.8}$$

where $S_i^{\pm}(\epsilon,h) = H_i^{\pm}(A_1(\epsilon,h))$.

By (2.3), taking the first order partial derivative of $S_i^{\pm}(\epsilon,h)$ with respect to ϵ, we get

$$\begin{aligned}\frac{\partial S_i^{\pm}}{\partial \epsilon}(\epsilon,h) =& \frac{\partial H_i^{\pm}}{\partial x}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial \epsilon} + k_2\frac{\partial H_i^{\pm}}{\partial y}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial \epsilon}\\ =& \left(\frac{\partial H_i^{\pm}}{\partial x}(A_1(\epsilon,h)) + k_2\frac{\partial H_i^{\pm}}{\partial y}(A_1(\epsilon,h))\right)\frac{\partial a(\epsilon,h)}{\partial \epsilon}.\end{aligned} \tag{2.9}$$

It follows from (2.9) and the formula of $K_{i1}^{\pm}(h)$ in (2.7) that

$$\frac{\partial S_i^{\pm}}{\partial \epsilon}(0,h) = K_{i1}^{\pm}(h)\frac{\partial a}{\partial \epsilon}(0,h) \equiv f_{i1}^{\pm}(h). \tag{2.10}$$

Further, taking the partial derivatives with respect to ϵ on the left and right hands of (2.9), we can obtain that

$$\begin{aligned}\frac{\partial^2 S_i^{\pm}}{\partial \epsilon^2}(\epsilon,h) =& \Bigg(\frac{\partial^2 H_i^{\pm}}{\partial x^2}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial \epsilon} + k_2\frac{\partial^2 H_i^{\pm}}{\partial x\partial y}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial \epsilon}\\ &+ k_2\frac{\partial^2 H_i^{\pm}}{\partial x\partial y}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial \epsilon} + k_2^2\frac{\partial^2 H_i^{\pm}}{\partial y^2}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial \epsilon}\Bigg)\\ &\cdot\frac{\partial a(\epsilon,h)}{\partial \epsilon} + \left(\frac{\partial H_i^{\pm}}{\partial x}(A_1(\epsilon,h)) + k_2\frac{\partial H_i^{\pm}}{\partial y}(A_1(\epsilon,h))\right)\cdot\frac{\partial^2 a(\epsilon,h)}{\partial \epsilon^2}\\ =& \left(\frac{\partial^2 H_i^{\pm}}{\partial x^2}(A_1(\epsilon,h)) + 2k_2\frac{\partial^2 H_i^{\pm}}{\partial x\partial y}(A_1(\epsilon,h)) + k_2^2\frac{\partial^2 H_i^{\pm}}{\partial y^2}(A_1(\epsilon,h))\right)\\ &\cdot\left(\frac{\partial a(\epsilon,h)}{\partial \epsilon}\right)^2 + \left(\frac{\partial H_i^{\pm}}{\partial x}(A_1(\epsilon,h)) + k_2\frac{\partial H_i^{\pm}}{\partial y}(A_1(\epsilon,h))\right)\frac{\partial^2 a(\epsilon,h)}{\partial \epsilon^2}.\end{aligned} \tag{2.11}$$

By (2.11) and the formulas of $K_{i1}^{\pm}(h)$ and $K_{i2}^{\pm}(h)$ in (2.7), we have

$$\begin{aligned}\frac{\partial^2 S_i^{\pm}}{\partial \epsilon^2}(0,h) &= K_{i2}^{\pm}(h)\left(\frac{\partial a}{\partial \epsilon}(0,h)\right)^2 + K_{i1}^{\pm}(h)\frac{\partial^2 a}{\partial \epsilon^2}(0,h)\\ &\equiv f_{i2}^{\pm}(h).\end{aligned} \tag{2.12}$$

Similarly, we can obtain $\frac{\partial^3 S_i^{\pm}}{\partial \epsilon^3}(\epsilon,h)$ from (2.11) that

$$\frac{\partial^3 S_i^{\pm}}{\partial \epsilon^3}(\epsilon,h) = \Bigg(\frac{\partial^3 H_i^{\pm}}{\partial x^3}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial \epsilon} + k_2\frac{\partial^3 H_i^{\pm}}{\partial x^2\partial y}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial \epsilon} + 2k_2$$

$$
\begin{aligned}
&\cdot\frac{\partial^3 H_i^{\pm}}{\partial x^2\partial y}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial\epsilon}+2k_2^2\frac{\partial^3 H_i^{\pm}}{\partial x\partial y^2}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial\epsilon}+k_2^2\\
&\cdot\frac{\partial^3 H_i^{\pm}}{\partial x\partial y^2}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial\epsilon}+k_2^3\frac{\partial^3 H_i^{\pm}}{\partial y^3}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial\epsilon}\Bigg)\\
&\cdot\left(\frac{\partial a(\epsilon,h)}{\partial\epsilon}\right)^2+\Bigg(\frac{\partial^2 H_i^{\pm}}{\partial x^2}(A_1(\epsilon,h))+2k_2\frac{\partial^2 H_i^{\pm}}{\partial x\partial y}(A_1(\epsilon,h))+k_2^2\\
&\cdot\frac{\partial^2 H_i^{\pm}}{\partial y^2}(A_1(\epsilon,h))\Bigg)\cdot 2\cdot\frac{\partial a(\epsilon,h)}{\partial\epsilon}\frac{\partial^2 a(\epsilon,h)}{\partial\epsilon^2}+\Bigg(\frac{\partial^2 H_i^{\pm}}{\partial x^2}(A_1(\epsilon,h))\\
&\cdot\frac{\partial a(\epsilon,h)}{\partial\epsilon}+k_2\frac{\partial^2 H_i^{\pm}}{\partial x\partial y}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial\epsilon}+k_2\frac{\partial^2 H_i^{\pm}}{\partial x\partial y}(A_1(\epsilon,h))\\
&\cdot\frac{\partial a(\epsilon,h)}{\partial\epsilon}+k_2^2\frac{\partial^2 H_i^{\pm}}{\partial y^2}(A_1(\epsilon,h))\frac{\partial a(\epsilon,h)}{\partial\epsilon}\Bigg)\frac{\partial^2 a(\epsilon,h)}{\partial\epsilon^2}\\
&+\left(\frac{\partial H_i^{\pm}}{\partial x}(A_1(\epsilon,h))+k_2\frac{\partial H_i^{\pm}}{\partial y}(A_1(\epsilon,h))\right)\frac{\partial^3 a(\epsilon,h)}{\partial\epsilon^3}\\
=&\Bigg(\frac{\partial^3 H_i^{\pm}}{\partial x^3}(A_1(\epsilon,h))+3k_2\frac{\partial^3 H_i^{\pm}}{\partial x^2\partial y}(A_1(\epsilon,h))+3k_2^2\frac{\partial^3 H_i^{\pm}}{\partial x\partial y^2}(A_1(\epsilon,h))\\
&+k_2^3\frac{\partial^3 H_i^{\pm}}{\partial y^3}(A_1(\epsilon,h))\Bigg)\left(\frac{\partial a(\epsilon,h)}{\partial\epsilon}\right)^3+3\Bigg(\frac{\partial^2 H_i^{\pm}}{\partial x^2}(A_1(\epsilon,h))+2k_2\\
&\cdot\frac{\partial^2 H_i^{\pm}}{\partial x\partial y}(A_1(\epsilon,h))+k_2^2\frac{\partial^2 H_i^{\pm}}{\partial y^2}(A_1(\epsilon,h))\Bigg)\frac{\partial a(\epsilon,h)}{\partial\epsilon}\frac{\partial^2 a(\epsilon,h)}{\partial\epsilon^2}\\
&+\left(\frac{\partial H_i^{\pm}}{\partial x}(A_1(\epsilon,h))+k_2\frac{\partial H_i^{\pm}}{\partial y}(A_1(\epsilon,h))\right)\\
&\cdot\frac{\partial^3 a(\epsilon,h)}{\partial\epsilon^3}.
\end{aligned}\tag{2.13}
$$

From (2.13) and the formulas of $K_{i1}^{\pm}(h)$, $K_{i2}^{\pm}(h)$ and $K_{i3}^{\pm}(h)$ in (2.7), we obtain

$$
\begin{aligned}
\frac{\partial^3 S_i^{\pm}}{\partial\epsilon^3}(0,h)=&K_{i3}^{\pm}(h)\left(\frac{\partial a}{\partial\epsilon}(0,h)\right)^3+3K_{i2}^{\pm}(h)\frac{\partial a}{\partial\epsilon}(0,h)\cdot\frac{\partial^2 a}{\partial\epsilon^2}(0,h)\\
&+K_{i1}^{\pm}(h)\frac{\partial^3 a}{\partial\epsilon^3}(0,h)\\
\equiv&f_{i3}^{\pm}(h).
\end{aligned}\tag{2.14}
$$

From (2.10), (2.12) and (2.14), we can rewrite $H_i^{\pm}(A_1(\epsilon,h))$ in (2.8) as

$$
H_i^{\pm}(A_1(\epsilon,h))=H_i^{\pm}(A_{10}(h))+\epsilon f_{i1}^{\pm}(h)+\frac{\epsilon^2}{2!}f_{i2}^{\pm}(h)+\frac{\epsilon^3}{3!}f_{i3}^{\pm}(h)+O(\epsilon^4).\tag{2.15}
$$

We obtain from (1.2) that

$$
\begin{aligned}
H^{\pm}(A_1(\epsilon,h),\epsilon)=&H_0^{\pm}(A_1(\epsilon,h))+\epsilon H_1^{\pm}(A_1(\epsilon,h))+\epsilon^2 H_2^{\pm}(A_1(\epsilon,h))+\epsilon^3 H_3^{\pm}(A_1(\epsilon,h))\\
&+O(\epsilon^4).
\end{aligned}
$$

Combining (2.15) and the above equation, we have

$$
H^{\pm}(A_1(\epsilon,h),\epsilon)=\left(H_0^{\pm}(A_{10}(h))+\epsilon f_{01}^{\pm}(h)+\frac{\epsilon^2}{2!}f_{02}^{\pm}(h)+\frac{\epsilon^3}{3!}f_{03}^{\pm}(h)+O(\epsilon^4)\right)
$$

$$
\begin{aligned}
&+\epsilon\Big(H_1^{\pm}(A_{10}(h))+\epsilon f_{11}^{\pm}(h)+\frac{\epsilon^2}{2!}f_{12}^{\pm}(h)+\frac{\epsilon^3}{3!}f_{13}^{\pm}(h)+O(\epsilon^4)\Big)\\
&+\epsilon^2\Big(H_2^{\pm}(A_{10}(h))+\epsilon f_{21}^{\pm}(h)+\frac{\epsilon^2}{2!}f_{22}^{\pm}(h)+\frac{\epsilon^3}{3!}f_{23}^{\pm}(h)+O(\epsilon^4)\Big)\\
&+\epsilon^3\Big(H_3^{\pm}(A_{10}(h))+\epsilon f_{31}^{\pm}(h)+\frac{\epsilon^2}{2!}f_{32}^{\pm}(h)+\frac{\epsilon^3}{3!}f_{33}^{\pm}(h)+O(\epsilon^4)\Big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
H^{\pm}(A_1(\epsilon,h),\epsilon)=&H_0^{\pm}(A_{10}(h))+\epsilon\Big(f_{01}^{\pm}(h)+H_1^{\pm}(A_{10}(h))\Big)+\epsilon^2\Big(\frac{1}{2!}f_{02}^{\pm}(h)+f_{11}^{\pm}(h)\\
&+H_2^{\pm}(A_{10}(h))\Big)+\epsilon^3\Big(\frac{1}{3!}f_{03}^{\pm}(h)+\frac{1}{2!}f_{12}^{\pm}(h)+f_{21}^{\pm}(h)+H_3^{\pm}(A_{10}(h))\Big)\\
&+O(\epsilon^4).
\end{aligned}\tag{2.16}
$$

From (1.2) again

$$
H^{+}(A_0(h),\epsilon)=H_0^{+}(A_0(h))+\epsilon H_1^{+}(A_0(h))+\epsilon^2H_2^{+}(A_0(h))+\epsilon^3H_3^{+}(A_0(h))+O(\epsilon^4).\tag{2.17}
$$

Then note that system (1.1) is Hamiltonian and

$$
H^{+}(A_1(\epsilon,h),\epsilon)=H^{+}(A_0(h),\epsilon).
$$

Inserting (2.17) and the expansion of $H^{+}(A_1(\epsilon,h),\epsilon)$ in (2.16) into the above equality, and comparing the like powers of ϵ, we can obtain

$$
\begin{aligned}
&f_{01}^{+}(h)+H_1^{+}(A_{10}(h))=H_1^{+}(A_0(h)),\\
&\frac{1}{2!}f_{02}^{+}(h)+f_{11}^{+}(h)+H_2^{+}(A_{10}(h))=H_2^{+}(A_0(h)),\\
&\frac{1}{3!}f_{03}^{+}(h)+\frac{1}{2!}f_{12}^{+}(h)+f_{21}^{+}(h)+H_3^{+}(A_{10}(h))=H_3^{+}(A_0(h)).
\end{aligned}\tag{2.18}
$$

In the following we will solve $\frac{\partial a}{\partial\epsilon}(0,h)$, $\frac{\partial^2 a}{\partial\epsilon^2}(0,h)$ and $\frac{\partial^3 a}{\partial\epsilon^3}(0,h)$ from the three equations in (2.18), respectively.

Firstly, substituting (2.10) into the first equation of (2.18), we have

$$
K_{01}^{+}(h)\frac{\partial a}{\partial\epsilon}(0,h)+H_1^{+}(A_{10}(h))=H_1^{+}(A_0(h)),
$$

which implies that

$$
\frac{\partial a}{\partial\epsilon}(0,h)=\frac{H_1^{+}(A_0(h))-H_1^{+}(A_{10}(h))}{K_{01}^{+}(h)}=\frac{v_1^{+}(h)}{K_{01}^{+}(h)}.\tag{2.19}
$$

Secondly, combining (2.10), (2.12) and the second equation of (2.18), we have

$$
\begin{aligned}
&\frac{1}{2}\Big(K_{02}^{+}(h)\Big(\frac{\partial a}{\partial\epsilon}(0,h)\Big)^2+K_{01}^{+}(h)\frac{\partial^2 a}{\partial\epsilon^2}(0,h)\Big)+K_{11}^{+}(h)\frac{\partial a}{\partial\epsilon}(0,h)\\
&+H_2^{+}(A_{10}(h))=H_2^{+}(A_0(h)).
\end{aligned}
$$

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We can solve from the above equation that

$$\begin{aligned}\frac{\partial^2 a}{\partial \epsilon^2}(0,h) =&\frac{1}{K_{01}^+(h)}\Big(2H_2^+(A_0(h)) - 2H_2^+(A_{10}(h)) - 2K_{11}^+(h)\frac{\partial a}{\partial \epsilon}(0,h)\\ &- K_{02}^+(h)\Big(\frac{\partial a}{\partial \epsilon}(0,h)\Big)^2\Big).\end{aligned}$$

Substituting (2.19) into the above equation, we have

$$\frac{\partial^2 a}{\partial \epsilon^2}(0,h) =\frac{1}{K_{01}^+(h)}\Big(2v_2^+(h) - 2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big). \tag{2.20}$$

Finally, it follows from (2.10), (2.12), (2.14) and the third equation of (2.18) that

$$\begin{aligned}&\frac{1}{3!}\Big(K_{03}^+(h)\Big(\frac{\partial a}{\partial \epsilon}(0,h)\Big)^3 + 3K_{02}^+(h)\frac{\partial a}{\partial \epsilon}(0,h)\cdot\frac{\partial^2 a}{\partial \epsilon^2}(0,h) + K_{01}^+(h)\frac{\partial^3 a}{\partial \epsilon^3}(0,h)\Big)\\ &+ \frac{1}{2!}\Big(K_{12}^+(h)\Big(\frac{\partial a}{\partial \epsilon}(0,h)\Big)^2 + K_{11}^+(h)\frac{\partial^2 a}{\partial \epsilon^2}(0,h)\Big) + K_{21}^+(h)\frac{\partial a}{\partial \epsilon}(0,h)\\ &+ H_3^+(A_{10}(h)) = H_3^+(A_0(h)).\end{aligned}$$

We can solve from the above equation that

$$\begin{aligned}\frac{\partial^3 a}{\partial \epsilon^3}(0,h) =&\frac{1}{K_{01}^+(h)}\Big(6H_3^+(A_0(h)) - 6H_3^+(A_{10}(h)) - 6K_{21}^+(h)\frac{\partial a}{\partial \epsilon}(0,h)\\ &- 3\Big(K_{12}^+(h)\Big(\frac{\partial a}{\partial \epsilon}(0,h)\Big)^2 + K_{11}^+(h)\frac{\partial^2 a}{\partial \epsilon^2}(0,h)\Big)\\ &- 3K_{02}^+(h)\frac{\partial a}{\partial \epsilon}(0,h)\cdot\frac{\partial^2 a}{\partial \epsilon^2}(0,h) - K_{03}^+(h)\Big(\frac{\partial a}{\partial \epsilon}(0,h)\Big)^3\Big).\end{aligned}$$

Substituting (2.19) and (2.20) into the above equation, we obtain that

$$\begin{aligned}\frac{\partial^3 a}{\partial \epsilon^3}(0,h) =&\frac{1}{K_{01}^+(h)}\Big(6v_3^+(h) - 6K_{21}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - 3K_{12}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\\ &- 3\frac{K_{11}^+(h)}{K_{01}^+(h)}\Big(2v_2^+(h) - 2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big)\\ &- 3\frac{K_{02}^+(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}\Big(2v_2^+(h) - 2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - K_{02}^+(h)\\ &\cdot\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big) - K_{03}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3\Big).\end{aligned} \tag{2.21}$$

Combining (2.19), (2.20) and (2.21) gives the conclusion of Lemma 2.2. This ends the proof. □

Lemma 2.3. *Under the notations in (2.7), we have for the function $b(\epsilon,h)$ in (2.4)*

$$\frac{\partial b}{\partial \epsilon}(0,h) =\frac{1}{W_{01}^-(h)}\Big(-v_1^-(h) + \frac{K_{01}^-(h)}{K_{01}^+(h)}v_1^+(h)\Big),$$

$$
\begin{aligned}
\frac{\partial^2 b}{\partial \epsilon^2}(0,h) =&\frac{1}{W_{01}^-(h)}\Bigg(-2v_2^-(h)+2K_{11}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}-2\frac{W_{11}^-(h)}{W_{01}^-(h)}\Big(\frac{K_{01}^-(h)}{K_{01}^+(h)}v_1^+(h)\\
&-v_1^-(h)\Big)+\frac{K_{01}^-(h)}{K_{01}^+(h)}\Big(2v_2^+(h)-2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-K_{02}^+(h)\\
&\cdot\big(\frac{v_1^+(h)}{K_{01}^+(h)}\big)^2\Big)+K_{02}^-(h)\big(\frac{v_1^+(h)}{K_{01}^+(h)}\big)^2-\frac{W_{02}^-(h)}{(W_{01}^-(h))^2}\Big(\frac{K_{01}^-(h)}{K_{01}^+(h)}v_1^+(h)\\
&-v_1^-(h)\Big)^2\Bigg),\\
\frac{\partial^3 b}{\partial \epsilon^3}(0,h) =&\frac{1}{W_{01}^-}\Bigg\{\frac{K_{01}^-}{K_{01}^+}\Bigg(6v_3^+(h)-6K_{21}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-3\frac{K_{11}^+(h)}{K_{01}^+(h)}\Big(2v_2^+(h)-2K_{11}^+(h)\\
&\cdot\frac{v_1^+(h)}{K_{01}^+(h)}-K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big)-3K_{12}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2-3\frac{K_{02}^+(h)}{K_{01}^+(h)}\\
&\cdot\frac{v_1^+(h)}{K_{01}^+(h)}\Big(2v_2^+(h)-2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big)-K_{03}^+(h)\\
&\cdot\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3\Bigg)+3\frac{K_{02}^-(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}\Big(2v_2^+(h)-2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}\\
&-K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big)+K_{03}^-(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3+3\frac{K_{11}^-(h)}{K_{01}^+(h)}\Big(2v_2^+(h)\\
&-2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big)+3K_{12}^-(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\\
&+6K_{21}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}-6v_3^-(h)-3\frac{W_{02}^-(h)}{(W_{01}^-(h))^2}\Big(-v_1^-(h)+K_{01}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}\Big)\\
&\cdot\Bigg(-2v_2^-(h)+2K_{11}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}-2\frac{W_{11}^-(h)}{W_{01}^-(h)}\big(-v_1^-(h)+K_{01}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}\big)\\
&+\frac{K_{01}^-(h)}{K_{01}^+(h)}\big(2v_2^+(h)-2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\big)+K_{02}^-(h)\\
&\cdot\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2-\frac{W_{02}^-(h)}{W_{01}^-(h)}\big(-v_1^-(h)+K_{01}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}\big)^2\Bigg)-\frac{W_{03}^-(h)}{(W_{01}^-(h))^3}\\
&\cdot\Big(-v_1^-(h)+K_{01}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3-3\frac{W_{11}^-(h)}{W_{01}^-(h)}\Bigg(-2v_2^-(h)+2K_{11}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}\\
&-2\frac{W_{11}^-(h)}{W_{01}^-(h)}\big(-v_1^-(h)+K_{01}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}\big)+\frac{K_{01}^-(h)}{K_{01}^+(h)}\big(2v_2^+(h)-2K_{11}^+(h)\\
&\cdot\frac{v_1^+(h)}{K_{01}^+(h)}-K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\big)+K_{02}^-(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2-\frac{W_{02}^-(h)}{W_{01}^-(h)}\\
&\cdot\Big(-v_1^-(h)+K_{01}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Bigg)-3\frac{W_{12}^-(h)}{(W_{01}^-(h))^2}\cdot\Big(-v_1^-(h)+K_{01}^-(h)\\
&\cdot\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2-6\frac{W_{21}^-(h)}{W_{01}^-(h)}\Big(-v_1^-(h)+K_{01}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}\Big)\Bigg\}.
\end{aligned}
$$

Proof. For $|\epsilon|>0$ sufficiently small, $H_i^{\pm}(B(\epsilon,h))$ has the following Taylor expansion

$$H_i^{\pm}(B(\epsilon,h))=H_i^{\pm}(A_0(h))+\sum_{j=1}^{3}\frac{\epsilon^j}{j!}\frac{\partial^j V_i^{\pm}}{\partial\epsilon^j}(0,h))+O(\epsilon^4), \tag{2.22}$$

where $V_i^{\pm}(\epsilon,h)=H_i^{\pm}(B(\epsilon,h))$.

Through a similar computing process as (2.9), (2.11) and (2.13), we have

$$\frac{\partial V_i^{\pm}}{\partial\epsilon}(\epsilon,h)=\left(\frac{\partial H_i^{\pm}}{\partial x}(B(\epsilon,h))+k_1\frac{\partial H_i^{\pm}}{\partial y}(B(\epsilon,h))\right)\frac{\partial b(\epsilon,h)}{\partial\epsilon}, \tag{2.23}$$

$$\begin{aligned}\frac{\partial^2 V_i^{\pm}}{\partial\epsilon^2}(\epsilon,h)=&\left(\frac{\partial^2 H_i^{\pm}}{\partial x^2}(B(\epsilon,h))+2k_1\frac{\partial^2 H_i^{\pm}}{\partial x\partial y}(B(\epsilon,h))+k_1^2\frac{\partial^2 H_i^{\pm}}{\partial y^2}(B(\epsilon,h))\right)\\&\cdot\left(\frac{\partial b(\epsilon,h)}{\partial\epsilon}\right)^2+\left(\frac{\partial H_i^{\pm}}{\partial x}(B(\epsilon,h))+k_1\frac{\partial H_i^{\pm}}{\partial y}(B(\epsilon,h))\right)\frac{\partial^2 b(\epsilon,h)}{\partial\epsilon^2},\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\frac{\partial^3 V_i^{\pm}}{\partial\epsilon^3}(\epsilon,h)=&\left(\frac{\partial^3 H_i^{\pm}}{\partial x^3}(B(\epsilon,h))+3k_1\frac{\partial^3 H_i^{\pm}}{\partial x^2\partial y}(B(\epsilon,h))+3k_1^2\frac{\partial^3 H_i^{\pm}}{\partial x\partial y^2}(B(\epsilon,h))\right.\\&\left.+k_1^3\frac{\partial^3 H_i^{\pm}}{\partial y^3}(B(\epsilon,h))\right)\left(\frac{\partial b(\epsilon,h)}{\partial\epsilon}\right)^3+3\left(\frac{\partial^2 H_i^{\pm}}{\partial x^2}(B(\epsilon,h))+2k_1\right.\\&\left.\cdot\frac{\partial^2 H_i^{\pm}}{\partial x\partial y}(B(\epsilon,h))+k_1^2\frac{\partial^2 H_i^{\pm}}{\partial y^2}(B(\epsilon,h))\right)\frac{\partial b(\epsilon,h)}{\partial\epsilon}\frac{\partial^2 b(\epsilon,h)}{\partial\epsilon^2}\\&+\left(\frac{\partial H_i^{\pm}}{\partial x}(B(\epsilon,h))+k_1\frac{\partial H_i^{\pm}}{\partial y}(B(\epsilon,h))\right)\frac{\partial^3 b(\epsilon,h)}{\partial\epsilon^3}.\end{aligned} \tag{2.25}$$

It is direct to see from (2.23) and the formula of $W_{i1}^{\pm}(h)$ in (2.7) that

$$\frac{\partial V_i^{\pm}}{\partial\epsilon}(0,h)=W_{i1}^{\pm}(h)\frac{\partial b}{\partial\epsilon}(0,h)\equiv g_{i1}^{\pm}(h). \tag{2.26}$$

From (2.24) and the formulas of $W_{i1}^{\pm}(h)$ and $W_{i2}^{\pm}(h)$ in (2.7), we have

$$\frac{\partial^2 V_i^{\pm}}{\partial\epsilon^2}(0,h)=W_{i2}^{\pm}(h)\Big(\frac{\partial b}{\partial\epsilon}(0,h)\Big)^2+W_{i1}^{\pm}(h)\frac{\partial^2 b}{\partial\epsilon^2}(0,h)\equiv g_{i2}^{\pm}(h). \tag{2.27}$$

Based on (2.25) and the formulas of $W_{i1}^{\pm}(h)$, $W_{i2}^{\pm}(h)$ and $W_{i3}^{\pm}(h)$ in (2.7), we obtain

$$\begin{aligned}\frac{\partial^3 V_i^{\pm}}{\partial\epsilon^3}(0,h)=&W_{i3}^{\pm}(h)\Big(\frac{\partial b}{\partial\epsilon}(0,h)\Big)^3+3W_{i2}^{\pm}(h)\frac{\partial b}{\partial\epsilon}(0,h)\cdot\frac{\partial^2 b}{\partial\epsilon^2}(0,h)\\&+W_{i1}^{\pm}(h)\frac{\partial^3 b}{\partial\epsilon^3}(0,h)\\\equiv&g_{i3}^{\pm}(h).\end{aligned} \tag{2.28}$$

By (2.26)-(2.28), $H_i^{\pm}(B(\epsilon,h))$ in (2.22) can be expressed as

$$H_i^{\pm}(B(\epsilon,h)) = H_i^{\pm}(A_0(h)) + \epsilon g_{i1}^{\pm}(h) + \frac{\epsilon^2}{2!} g_{i2}^{\pm}(h) + \frac{\epsilon^3}{3!} g_{i3}^{\pm}(h) + O(\epsilon^4). \tag{2.29}$$

We obtain from (1.2) that

$$\begin{aligned} H^-(B(\epsilon,h),\epsilon) = & H_0^-(B(\epsilon,h)) + \epsilon H_1^-(B(\epsilon,h)) + \epsilon^2 H_2^-(B(\epsilon,h)) + \epsilon^3 H_3^-(B(\epsilon,h)) \\ & + O(\epsilon^4). \end{aligned}$$

Combining the expansion of $H_i^-(B(\epsilon,h))$ in (2.29) and the above equation, similar to (2.16), we have

$$\begin{aligned} H^-(B(\epsilon,h),\epsilon) = & H_0^-(A_0(h)) + \epsilon\Big(g_{01}^-(h) + H_1^-(A_0(h))\Big) + \epsilon^2\Big(\frac{1}{2!} g_{02}^-(h) + g_{11}^-(h) \\ & + H_2^-(A_0(h))\Big) + \epsilon^3\Big(\frac{1}{3!} g_{03}^-(h) + \frac{1}{2!} g_{12}^-(h) + g_{21}^-(h) + H_3^-(A_0(h))\Big) \\ & + O(\epsilon^4). \end{aligned} \tag{2.30}$$

Note that system (1.1) is Hamiltonian and

$$H^-(A_1(\epsilon,h),\epsilon) = H^-(B(\epsilon,h),\epsilon).$$

Substituting (2.30) and the expansion of $H^-(A_1(\epsilon,h),\epsilon)$ in (2.16) into the above equality, we have

$$\begin{aligned} & H_0^-(A_{10}(h)) + \epsilon\Big(f_{01}^-(h) + H_1^-(A_{10}(h))\Big) + \epsilon^2\Big(\frac{1}{2!} f_{02}^-(h) + f_{11}^-(h) + H_2^-(A_{10}(h))\Big) \\ & + \epsilon^3\Big(\frac{1}{3!} f_{03}^-(h) + \frac{1}{2!} f_{12}^-(h) + f_{21}^-(h) + H_3^-(A_{10}(h))\Big) + O(\epsilon^4) \\ & = H_0^-(A_0(h)) + \epsilon\Big(g_{01}^-(h) + H_1^-(A_0(h))\Big) + \epsilon^2\Big(\frac{1}{2!} g_{02}^-(h) + g_{11}^-(h) + H_2^-(A_0(h))\Big) \\ & + \epsilon^3\Big(\frac{1}{3!} g_{03}^-(h) + \frac{1}{2!} g_{12}^-(h) + g_{21}^-(h) + H_3^-(A_0(h))\Big) + O(\epsilon^4). \end{aligned} \tag{2.31}$$

Comparing the like powers of ϵ on the left and right sides of (2.31), we have

$$\begin{aligned} & f_{01}^-(h) + H_1^-(A_{10}(h)) = g_{01}^-(h) + H_1^-(A_0(h)), \\ & \frac{1}{2!} f_{02}^-(h) + f_{11}^-(h) + H_2^-(A_{10}(h)) = \frac{1}{2!} g_{02}^-(h) + g_{11}^-(h) + H_2^-(A_0(h)), \\ & \frac{1}{3!} f_{03}^-(h) + \frac{1}{2!} f_{12}^-(h) + f_{21}^-(h) + H_3^-(A_{10}(h)) = \frac{1}{3!} g_{03}^-(h) + \frac{1}{2!} g_{12}^-(h) + g_{21}^-(h) \\ & + H_3^-(A_0(h)). \end{aligned} \tag{2.32}$$

In the following we will solve $\frac{\partial b}{\partial \epsilon}(0,h)$, $\frac{\partial^2 b}{\partial \epsilon^2}(0,h)$ and $\frac{\partial^3 b}{\partial \epsilon^3}(0,h)$ from the three equations in (2.32), respectively.

Firstly, substituting (2.10) and (2.26) into the first equation of (2.32), we have

$$K_{01}^-(h)\frac{\partial a}{\partial \epsilon}(0,h) + H_1^-(A_{10}(h)) = W_{01}^-(h)\frac{\partial b}{\partial \epsilon}(0,h) + H_1^-(A_0(h)).$$

We can solve from the above equation that

$$\frac{\partial b}{\partial \epsilon}(0,h)=\frac{1}{W_{01}^{-}(h)}\Big(H_1^{-}(A_{10}(h))-H_1^{-}(A_0(h))+K_{01}^{-}(h)\frac{\partial a}{\partial \epsilon}(0,h)\Big).$$

Then by (2.19) in Lemma 2.2 and the formula of $v_1^{\pm}(h)$ in (2.7), we have

$$\frac{\partial b}{\partial \epsilon}(0,h)=\frac{1}{W_{01}^{-}(h)}\Big(-v_1^{-}(h)+\frac{K_{01}^{-}(h)}{K_{01}^{+}(h)}v_1^{+}(h)\Big). \tag{2.33}$$

Secondly, substituting (2.10), (2.12), (2.26) and (2.27) into the second equation of (2.32), we have

$$\begin{aligned}&\frac{1}{2}\Big(K_{02}^{-}(h)\Big(\frac{\partial a}{\partial \epsilon}(0,h)\Big)^2+K_{01}^{-}(h)\frac{\partial^2 a}{\partial \epsilon^2}(0,h)\Big)+K_{11}^{-}(h)\frac{\partial a}{\partial \epsilon}(0,h)\\&+H_2^{-}(A_{10}(h))=\frac{1}{2}\Big(W_{02}^{-}(h)\Big(\frac{\partial b}{\partial \epsilon}(0,h)\Big)^2+W_{01}^{-}(h)\frac{\partial^2 b}{\partial \epsilon^2}(0,h)\Big)+W_{11}^{-}(h)\\&\cdot\frac{\partial b}{\partial \epsilon}(0,h)+H_2^{-}(A_0(h)).\end{aligned}$$

We can solve from the above equation that

$$\begin{aligned}\frac{\partial^2 b}{\partial \epsilon^2}(0,h)=&\frac{1}{W_{01}^{-}(h)}\Big(2H_2^{-}(A_{10}(h))-2H_2^{-}(A_0(h))+2K_{11}^{-}(h)\frac{\partial a}{\partial \epsilon}(0,h)\\&-2W_{11}^{-}(h)\frac{\partial b}{\partial \epsilon}(0,h)+K_{01}^{-}(h)\frac{\partial^2 a}{\partial \epsilon^2}(0,h)+K_{02}^{-}(h)\\&\cdot\Big(\frac{\partial a}{\partial \epsilon}(0,h)\Big)^2-W_{02}^{-}(h)\Big(\frac{\partial b}{\partial \epsilon}(0,h)\Big)^2\Big).\end{aligned} \tag{2.34}$$

Then from (2.19), (2.20) and (2.33), we have

$$\begin{aligned}\frac{\partial^2 b}{\partial \epsilon^2}(0,h)=&\frac{1}{W_{01}^{-}(h)}\Big(-2v_2^{-}(h)+2K_{11}^{-}(h)\frac{v_1^{+}(h)}{K_{01}^{+}(h)}-2\frac{W_{11}^{-}(h)}{W_{01}^{-}(h)}\Big(\frac{K_{01}^{-}(h)}{K_{01}^{+}(h)}v_1^{+}(h)\\&-v_1^{-}(h)\Big)+\frac{K_{01}^{-}(h)}{K_{01}^{+}(h)}\Big(2v_2^{+}(h)-2K_{11}^{+}(h)\frac{v_1^{+}(h)}{K_{01}^{+}(h)}-K_{02}^{+}(h)\\&\cdot(\frac{v_1^{+}(h)}{K_{01}^{+}(h)})^2\Big)+K_{02}^{-}(h)(\frac{v_1^{+}(h)}{K_{01}^{+}(h)})^2-\frac{W_{02}^{-}(h)}{(W_{01}^{-})^2(h)}\Big(\frac{K_{01}^{-}(h)}{K_{01}^{+}(h)}v_1^{+}(h)\\&-v_1^{-}(h)\Big)^2\Big).\end{aligned} \tag{2.35}$$

Finally, inserting (2.10), (2.12), (2.14) and (2.26)-(2.28) into the third equation of (2.32), we have

$$\begin{aligned}&\frac{1}{3!}\bigg(K_{03}^-(h)\Big(\frac{\partial}{\partial\epsilon}(0,h)\Big)^3+3K_{02}^-(h)\frac{\partial a}{\partial\epsilon}(0,h)\cdot\frac{\partial^2 a}{\partial\epsilon^2}(0,h)+K_{01}^-(h)\frac{\partial^3 a}{\partial\epsilon^3}(0,h)\bigg)\\&\quad+\frac{1}{2!}\bigg(K_{12}^-(h)\Big(\frac{\partial a}{\partial\epsilon}(0,h)\Big)^2+K_{11}^-(h)\frac{\partial^2 a}{\partial\epsilon^2}(0,h)\bigg)+K_{21}^-(h)\frac{\partial a}{\partial\epsilon}(0,h)\\&\quad+H_3^-(A_{10}(h))=\frac{1}{3!}\bigg(W_{03}^-(h)\Big(\frac{\partial b}{\partial\epsilon}(0,h)\Big)^3+3W_{02}^-\frac{\partial b}{\partial\epsilon}(0,h)\cdot\frac{\partial^2 b}{\partial\epsilon^2}(0,h)+W_{01}^-(h)\\&\quad\cdot\frac{\partial^3 b}{\partial\epsilon^3}(0,h)\bigg)+\frac{1}{2!}\bigg(W_{12}^-(h)\Big(\frac{\partial b}{\partial\epsilon}(0,h)\Big)^2+W_{11}^-(h)\frac{\partial^2 b}{\partial\epsilon^2}(0,h)\bigg)+W_{21}^-(h)\\&\quad\cdot\frac{\partial b}{\partial\epsilon}(0,h)+H_3^-(A_0(h)),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{\partial^3 b}{\partial\epsilon^3}(0,h)=&\frac{1}{W_{01}^-(h)}\bigg(K_{01}^-(h)\frac{\partial^3 a}{\partial\epsilon^3}(0,h)+3K_{02}^-(h)\frac{\partial a}{\partial\epsilon}(0,h)\cdot\frac{\partial^2 a}{\partial\epsilon^2}(0,h)\\&+K_{03}^-(h)\cdot\Big(\frac{\partial a}{\partial\epsilon}(0,h)\Big)^3+3K_{11}^-(h)\frac{\partial^2 a}{\partial\epsilon^2}(0,h)+3K_{12}^-(h)\\&\cdot\Big(\frac{\partial a}{\partial\epsilon}(0,h)\Big)^2+6K_{21}^-(h)\frac{\partial a}{\partial\epsilon}(0,h)-6v_3^-(h)-3W_{02}^-(h)\\&\cdot\frac{\partial b}{\partial\epsilon}(0,h)\cdot\frac{\partial^2 b}{\partial\epsilon^2}(0,h)-W_{03}^-(h)\Big(\frac{\partial b}{\partial\epsilon}(0,h)\Big)^3-3W_{11}^-(h)\\&\cdot\frac{\partial^2 b}{\partial\epsilon^2}(0,h)-3W_{12}^-(h)\Big(\frac{\partial b}{\partial\epsilon}(0,h)\Big)^2-6W_{21}^-(h)\frac{\partial b}{\partial\epsilon}(0,h)\bigg).\end{aligned}\tag{2.36}$$

Substituting (2.19)-(2.21), (2.33) and (2.35) into (2.36), we have

$$\begin{aligned}\frac{\partial^3 b}{\partial\epsilon^3}(0,h)=&\frac{1}{W_{01}^-(h)}\bigg\{\frac{K_{01}^-(h)}{K_{01}^+(h)}\bigg(6v_3^+(h)-6K_{21}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-3\frac{K_{11}^+(h)}{K_{01}^+(h)}\Big(2v_2^+(h)\\&-2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big)-3K_{12}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\\&-3\frac{K_{02}^+(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}\Big(2v_2^+(h)-2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big)\\&-K_{03}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3\bigg)+3\frac{K_{02}^-(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}\Big(2v_2^+(h)-2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}\\&-K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big)+K_{03}^-(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3+3\frac{K_{11}^-(h)}{K_{01}^+(h)}\Big(2v_2^+(h)\\&-2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big)+3K_{12}^-(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\\&+6K_{21}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}-6v_3^-(h)-3\frac{W_{02}^-(h)}{(W_{01}^-(h))^2}(-v_1^-(h)+K_{01}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)})\\&\cdot\bigg(-2v_2^-(h)+2K_{11}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}-2\frac{W_{11}^-(h)}{W_{01}^-(h)}(-v_1^-(h)+K_{01}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)})\\&+\frac{K_{01}^-(h)}{K_{01}^+(h)}(2v_2^+(h)-2K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-K_{02}^+(h)\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2)\end{aligned}$$

$$
\begin{aligned}
&+K_{02}^{-}(h)\Big(\frac{v_1^{+}(h)}{K_{01}^{+}(h)}\Big)^2-\frac{W_{02}^{-}(h)}{W_{01}^{-}(h)}\Big(-v_1^{-}(h)+K_{01}^{-}(h)\frac{v_1^{+}(h)}{K_{01}^{+}(h)}\Big)^2\Bigg)\\
&-\frac{W_{03}^{-}(h)}{(W_{01}^{-}(h))^3}\Big(-v_1^{-}(h)+K_{01}^{-}(h)\frac{v_1^{+}(h)}{K_{01}^{+}(h)}\Big)^3-3\frac{W_{11}^{-}(h)}{W_{01}^{-}(h)}\Bigg(-2v_2^{-}(h)\\
&+2K_{11}^{-}(h)\frac{v_1^{+}(h)}{K_{01}^{+}(h)}-2\frac{W_{11}^{-}(h)}{W_{01}^{-}(h)}\big(-v_1^{-}(h)+K_{01}^{-}(h)\frac{v_1^{+}(h)}{K_{01}^{+}(h)}\big)+\frac{K_{01}^{-}(h)}{K_{01}^{+}(h)}\\
&\cdot\big(2v_2^{+}(h)-2K_{11}^{+}(h)\frac{v_1^{+}(h)}{K_{01}^{+}(h)}-K_{02}^{+}(h)\Big(\frac{v_1^{+}(h)}{K_{01}^{+}(h)}\Big)^2\big)+K_{02}^{-}(h)\Big(\frac{v_1^{+}(h)}{K_{01}^{+}(h)}\Big)^2\\
&-\frac{W_{02}^{-}(h)}{W_{01}^{-}(h)}\Big(-v_1^{-}(h)+K_{01}^{-}(h)\frac{v_1^{+}(h)}{K_{01}^{+}(h)}\Big)^2\Bigg)-3\frac{W_{12}^{-}(h)}{(W_{01}^{-}(h))^2}\Big(-v_1^{-}(h)\\
&+K_{01}^{-}(h)\frac{v_1^{+}(h)}{K_{01}^{+}(h)}\Big)^2-6\frac{W_{21}^{-}(h)}{W_{01}^{-}(h)}\Big(-v_1^{-}(h)+K_{01}^{-}(h)\frac{v_1^{+}(h)}{K_{01}^{+}(h)}\Big)\Bigg\}. \qquad (2.37)
\end{aligned}
$$

This ends the proof. □

Based on the previous discussion, now we present the explicit formulas of the first, second and third order Melnikov functions of piecewise Hamiltonian system (1.1).

Theorem 2.2. *Consider system (1.1) with the assumptions (**A1**)-(**A3**). The first order Melnikov function of system (1.1) is given by*

$$
M_1(h)=\frac{W_{01}^{+}(h)}{W_{01}^{-}(h)}\Bigg(-v_1^{-}(h)\Bigg)+v_1^{+}(h),
$$

where $W_{01}^{\pm}(h)$ and $v_1^{\pm}(h)$ are defined in (2.7).

Proof. From (2.6), the first order Melnikov function $M_1(h)$ of system (1.1) is

$$
M_1(h)=\frac{\partial V_0^{+}}{\partial \epsilon}(0,h). \qquad (2.38)
$$

Substituting (2.26) and (2.33) in Lemma 2.3 into (2.38), we have

$$
\begin{aligned}
M_1(h)=&W_{01}^{+}(h)\frac{\partial b}{\partial \epsilon}(0,h)\\
=&\frac{W_{01}^{+}(h)}{W_{01}^{-}(h)}\Bigg(-v_1^{-}(h)+\frac{K_{01}^{-}(h)}{K_{01}^{+}(h)}v_1^{+}(h)\Bigg), \qquad (2.39)
\end{aligned}
$$

where $v_1^{\pm}(h)$, $W_{01}^{\pm}(h)$ and $K_{01}^{\pm}(h)$ are defined in (2.7).

We claim that

$$
\frac{W_{01}^{+}(h)}{W_{01}^{-}(h)}\frac{K_{01}^{-}(h)}{K_{01}^{+}(h)}=1. \qquad (2.40)
$$

In fact, noting that

$$
A_0(h)=(a_0(h),k_1a_0(h)) \text{ and } A_{10}(h)=(a_{10}(h),k_2a_{10}(h)),
$$

and differentiating two equalities in (2.1) with respect to h, we can obtain

$$
\Big(\frac{\partial H_0^{+}}{\partial x}(A_0(h))+k_1\frac{\partial H_0^{+}}{\partial y}(A_0(h))\Big)a_0'(h)=\Big(\frac{\partial H_0^{+}}{\partial x}(A_{10}(h))+k_2\frac{\partial H_0^{+}}{\partial y}(A_{10}(h))\Big)a_{10}'(h)=1, \qquad (2.41)
$$

$$\Big(\frac{\partial H_0^-}{\partial x}(A_0(h))+k_1\frac{\partial H_0^-}{\partial y}(A_0(h))\Big)a_0'(h)=\Big(\frac{\partial H_0^-}{\partial x}(A_{10}(h))+k_2\frac{\partial H_0^-}{\partial y}(A_{10}(h))\Big)a_{10}'(h).$$

It follows from (2.41) that

$$\frac{\frac{\partial H_0^+}{\partial x}(A_0(h))+k_1\frac{\partial H_0^+}{\partial y}(A_0(h))}{\frac{\partial H_0^-}{\partial x}(A_0(h))+k_1\frac{\partial H_0^-}{\partial y}(A_0(h))}\cdot\frac{\frac{\partial H_0^-}{\partial x}(A_{10}(h))+k_2\frac{\partial H_0^-}{\partial y}(A_{10}(h))}{\frac{\partial H_0^+}{\partial x}(A_{10}(h))+k_2\frac{\partial H_0^+}{\partial y}(A_{10}(h))}=1,$$

which means that (2.40) holds.

By (2.39) and (2.40), we obtain the formula of the first order Melnikov function

$$M_1(h)=\frac{W_{01}^+(h)}{W_{01}^-(h)}\bigg(-v_1^-(h)\bigg)+v_1^+(h).$$

This finishes the proof. □

Remark 2.1. By applying the more general result of the formula of the first order Melnikov function in [11], we can give a shorter proof of Theorem 2.2. From [11], the first order Melnikov function of system (1.1) is

$$\begin{aligned}M_1(h)=&\int_{\widehat{A_0A_{10}}}-H_{1x}^+dx-H_{1y}^+dy+\frac{H_{0x}^+(A_0)+k_1H_{0y}^+(A_0)}{H_{0x}^-(A_0)+k_1H_{0y}^-(A_0)}\\&\cdot\int_{\widehat{A_{10}A_0}}-H_{1x}^-dx-H_{1y}^-dy.\end{aligned}$$

We can prove that the formula of the first order Melnikov function in Theorem 2.2 is equivalent to the above formula. In fact,

$$\begin{aligned}M_1(h)&=\int_{\widehat{A_{10}A_0}}H_{1x}^+dx+H_{1y}^+dy+\frac{H_{0x}^+(A_0)+k_1H_{0y}^+(A_0)}{H_{0x}^-(A_0)+k_1H_{0y}^-(A_0)}\int_{\widehat{A_0A_{10}}}H_{1x}^-dx+H_{1y}^-dy\\&=\int_{\widehat{A_{10}A_0}}dH_1^++\frac{H_{0x}^+(A_0)+k_1H_{0y}^+(A_0)}{H_{0x}^-(A_0)+k_1H_{0y}^-(A_0)}\int_{\widehat{A_0A_{10}}}dH_1^-\\&=H_1^+(A_0(h))-H_1^+(A_{10}(h))+\frac{W_{01}^+(h)}{W_{01}^-(h)}\Big(H_1^-(A_{10}(h))-H_1^-(A_0(h))\Big)\\&=v_1^+(h)+\frac{W_{01}^+(h)}{W_{01}^-(h)}\bigg(-v_1^-(h)\bigg).\end{aligned}$$

Theorem 2.3. *Consider system (1.1) with the assumptions (**A1**)-(**A3**). If $M_1(h)\equiv 0$, then the second order Melnikov function of system (1.1) is given by*

$$M_2(h)=\frac{W_{01}^+(h)}{W_{01}^-(h)}M_{21}(h)+M_{22}(h),$$

where

$$\begin{aligned}M_{21}(h)&=-v_2^-(h)+K_{11}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}+\frac{K_{02}^-(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2,\\M_{22}(h)&=v_2^+(h)-K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-\frac{K_{02}^+(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2,\end{aligned}$$

$W_{01}^\pm(h)$, $K_{01}^\pm(h)$, $K_{02}^\pm(h)$, $K_{11}^\pm(h)$, $v_1^\pm(h)$ and $v_2^\pm(h)$ are defined in (2.7).

Proof. From (2.2) and (2.7), we know that $W_{01}^{\pm}(h) \neq 0$. Then it follows from (2.39) that $M_1(h) \equiv 0$ if and only if

$$\frac{\partial b}{\partial \epsilon}(0,h) \equiv 0. \tag{2.42}$$

According to (2.6), the second order Melnikov function $M_2(h)$ of system (1.1) is

$$M_2(h) = \frac{1}{2!}\frac{\partial^2 V_0^+}{\partial \epsilon^2}(0,h). \tag{2.43}$$

By (2.27) in Lemma 2.3 and (2.42), the function $M_2(h)$ in (2.43) can be rewritten as

$$\begin{aligned} M_2(h) =& \frac{W_{01}^+(h)}{2!}\frac{\partial^2 b}{\partial \epsilon^2}(0,h) \\ =& \frac{W_{01}^+(h)}{2!W_{01}^-(h)}\Bigg(-2v_2^-(h) + 2K_{11}^-(h)\frac{\partial a}{\partial \epsilon}(0,h) \\ & + K_{01}^-(h)\frac{\partial^2 a}{\partial \epsilon^2}(0,h) + K_{02}^-(h)\left(\frac{\partial a}{\partial \epsilon}(0,h)\right)^2\Bigg). \end{aligned} \tag{2.44}$$

Inserting (2.42) and the expressions of $\frac{\partial a}{\partial \epsilon}(0,h)$, $\frac{\partial^2 a}{\partial \epsilon^2}(0,h)$ in Lemma 2.2 into (2.44), we obtain that

$$\begin{aligned} M_2(h) =& \frac{W_{01}^+(h)}{W_{01}^-(h)}\Bigg(-v_2^-(h) + K_{11}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)} + \frac{K_{02}^-(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Bigg) \\ & + \frac{W_{01}^+(h)}{W_{01}^-(h)}\frac{K_{01}^-(h)}{K_{01}^+(h)}\Bigg(v_2^+(h) - K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - \frac{K_{02}^+(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Bigg) \\ =& \frac{W_{01}^+(h)}{W_{01}^-(h)}M_{21}(h) + M_{22}(h), \end{aligned} \tag{2.45}$$

where where $M_{21}(h)$ and $M_{22}(h)$ are defined in Theorem 2.3. This completes the proof. □

Theorem 2.4. *Consider system (1.1) with the assumptions (**A1**)-(**A3**). If $M_1(h) = M_2(h) \equiv 0$, then the third order Melnikov function of system (1.1) is given by*

$$M_3(h) = \frac{W_{01}^+(h)}{W_{01}^-(h)}M_{31}(h) + M_{32}(h),$$

where

$$\begin{aligned} M_{31}(h) =& -v_3^-(h) + K_{21}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)} + \frac{K_{12}^-(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2 + \frac{K_{03}^-(h)}{6}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3 \\ & + \frac{K_{11}^-(h)}{K_{01}^+(h)}M_{22}(h) + \frac{K_{02}^-(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}M_{22}(h), \\ M_{32}(h) =& v_3^+(h) - K_{21}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - \frac{K_{12}^+(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2 - \frac{K_{03}^+(h)}{6}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3 \\ & - \frac{K_{11}^+(h)}{K_{01}^+(h)}M_{22}(h) - \frac{K_{02}^+(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}M_{22}(h), \end{aligned}$$

$W_{01}^{\pm}(h)$, $K_{01}^{\pm}(h)$, $K_{02}^{\pm}(h)$, $K_{03}^{\pm}(h)$, $K_{11}^{\pm}(h)$, $K_{12}^{\pm}(h)$, $K_{21}^{\pm}(h)$, $v_1^{\pm}(h)$ and $v_3^{\pm}(h)$ are defined in (2.7).

Proof. Note that $W_{01}^{\pm}(h) \neq 0$. From (2.39) and (2.44), it is easy to see that $M_1(h) = M_2(h) \equiv 0$ if and only if

$$\frac{\partial b}{\partial \epsilon}(0,h) = \frac{\partial^2 b}{\partial \epsilon^2}(0,h) \equiv 0. \tag{2.46}$$

According to (2.6), the third order Melnikov function of system (1.1) is

$$M_3(h) = \frac{1}{3!}\frac{\partial^3 V_0^+}{\partial \epsilon^3}(0,h). \tag{2.47}$$

It follows from(2.28), (2.36), (2.46) and (2.47) that

$$\begin{aligned}
M_3(h) =& \frac{W_{01}^+(h)}{3!}\frac{\partial^3 b}{\partial \epsilon^3}(0,h)\\
=& \frac{W_{01}^+(h)}{3!W_{01}^-(h)}\bigg(K_{01}^-(h)\frac{\partial^3 a}{\partial \epsilon^3}(0,h) + 3K_{02}^-(h)\frac{\partial a}{\partial \epsilon}(0,h)\cdot\frac{\partial^2 a}{\partial \epsilon^2}(0,h) + K_{03}^-(h)\\
&\cdot\Big(\frac{\partial a}{\partial \epsilon}(0,h)\Big)^3 + 3K_{11}^-(h)\frac{\partial^2 a}{\partial \epsilon^2}(0,h) + 3K_{12}^-(h)\Big(\frac{\partial a}{\partial \epsilon}(0,h)\Big)^2\\
&+ 6K_{21}^-(h)\frac{\partial a}{\partial \epsilon}(0,h) - 6v_3^-(h)\bigg).
\end{aligned} \tag{2.48}$$

Based on (2.40) and Lemma 2.2, the third order Melnikov function $M_3(h)$ in (2.48) can be rewritten as

$$\begin{aligned}
M_3(h) =& \frac{W_{01}^+(h)}{W_{01}^-(h)}\bigg[-v_3^- + K_{21}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)} + \frac{K_{12}^-(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2 + \frac{K_{03}^-(h)}{6}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3\\
&+ \frac{K_{11}^-(h)}{K_{01}^+(h)}\Big(v_2^+(h) - K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - \frac{K_{02}^+(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big) + \frac{K_{02}^-(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}\\
&\cdot\Big(v_2^+(h) - K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - \frac{K_{02}^+(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big)\bigg] + \bigg[v_3^+(h) - K_{21}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}\\
&- \frac{K_{12}^+(h)}{2}\Big(\frac{v_1^+}{K_{01}^+}\Big)^2 - \frac{K_{03}^+(h)}{6}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3 - \frac{K_{11}^+(h)}{K_{01}^+(h)}\Big(v_2^+(h) - K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}\\
&- \frac{K_{02}^+(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big) - \frac{K_{02}^+(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}\Big(v_2^+(h) - K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - \frac{K_{02}^+(h)}{2}\\
&\cdot\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2\Big)\bigg]\\
=& \frac{W_{01}^+(h)}{W_{01}^-(h)}M_{31}(h) + M_{32}(h),
\end{aligned}$$

where $M_{31}(h)$ and $M_{32}(h)$ are defined in Theorem 2.4. The proof is completed. □

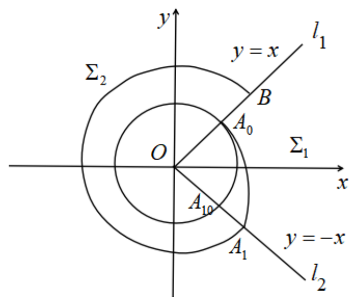

Figure 2. Phase portrait of system (3.1).

3. Applications

In this section, as an application of the main results, we consider a Hamiltonian system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y \\ -x \end{pmatrix} + \epsilon \begin{pmatrix} H_1^{+\prime}(y) \\ 0 \end{pmatrix} + \epsilon^2 \begin{pmatrix} H_2^{+\prime}(y) \\ 0 \end{pmatrix}, & (x,y) \in \Sigma_1, \\ \begin{pmatrix} y \\ -x \end{pmatrix} + \epsilon \begin{pmatrix} H_1^{-\prime}(y) \\ 0 \end{pmatrix} + \epsilon^2 \begin{pmatrix} H_2^{-\prime}(y) \\ 0 \end{pmatrix}, & (x,y) \in \Sigma_2, \end{cases} \tag{3.1}$$

where $0 < \epsilon \ll 1$,

$$H_1^{\pm}(y) = \sum_{i=0}^{m+1} a_i^{\pm} y^i, \; H_2^{\pm}(y) = \sum_{i=0}^{m+1} b_i^{\pm} y^i, \tag{3.2}$$

$m \geq 1$ is an integer, Σ_1 and Σ_2 are the regions bounded by the two semi-straight lines $l_1 : y = x, \; x > 0$ and $l_2 : y = -x, \; x > 0$. See Fig.2 for illustration.

Obviously, the unperturbed system $(3.1)|_{\epsilon=0}$ is Hamiltonian with the following Hamiltonian function given by

$$H(x,y) = \frac{1}{2}(x^2 + y^2),$$

and has a family of periodic orbits with clockwise orientation denoted by

$$L_h = \{(x,y) | \frac{1}{2}(x^2 + y^2) = h\},$$

with $h > 0$. In this case, we have $A_0(h) = (\sqrt{h}, \sqrt{h})$ and $A_{10}(h) = (\sqrt{h}, -\sqrt{h})$. For any integer $m \geq 1$, we have the following result for piecewise polynomial Hamiltonian system (3.1).

Theorem 3.1. *If the first order Melnikov function $M_1(h)$ is not zero identically, then for sufficiently small $|\epsilon| > 0$, system (3.1) has at most $[\frac{m}{2}]$ limit cycles bifurcated from the period annulus L_h, multiplicity taken into account, where $[\cdot]$ denotes the integer part function. Moreover, this upper bound can be reached.*

Proof. According to Theorem 2.2, the first order Melnikov function of system (3.1) is

$$M_1(h) = \frac{W_{01}^+(h)}{W_{01}^-(h)}(-v_1^-(h)) + v_1^+(h). \tag{3.3}$$

Next, we compute the coefficients $W_{01}^{\pm}(h)$ and $v_1^{\pm}(h)$. For convenience of calculation, we discuss the case m odd and the case m even, respectively.

For m is even, we compute from (2.7) that

$$W_{01}^+(h) = 2\sqrt{h},\ W_{01}^-(h) = 2\sqrt{h}, \tag{3.4}$$

$$v_1^+(h) = 2\sum_{j=0}^{\frac{m}{2}} a_{2j+1}^+(\sqrt{h})^{2j+1},\ v_1^-(h) = 2\sum_{j=0}^{\frac{m}{2}} a_{2j+1}^-(\sqrt{h})^{2j+1}. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we have

$$M_1(h) = 2\sqrt{h}\sum_{j=0}^{\frac{m}{2}}(a_{2j+1}^+ - a_{2j+1}^-)h^j. \tag{3.6}$$

For m is odd, by direct calculation, we have

$$v_1^+(h) = 2\sum_{j=0}^{\frac{m-1}{2}} a_{2j+1}^+(\sqrt{h})^{2j+1},\ v_1^-(h) = 2\sum_{j=0}^{\frac{m-1}{2}} a_{2j+1}^-(\sqrt{h})^{2j+1}. \tag{3.7}$$

Substituting (3.4) and (3.7) into (3.3), we can obtain

$$M_1(h) = 2\sqrt{h}\sum_{j=0}^{\frac{m-1}{2}}(a_{2j+1}^+ - a_{2j+1}^-)h^j. \tag{3.8}$$

From (3.6) and (3.8), it is easy to observe that $M_1(h)$ has at most $[\frac{m}{2}]$ zeros in h on $(0,+\infty)$, multiplicity taken into account. Hence, by Theorem 2.1, system (3.1) has at most $[\frac{m}{2}]$ limit cycles bifurcated from the period annulus L_h, multiplicity taken into account. Moreover, noting that $a_{2j+1}^{\pm}$ are independent with each other, there exists a choice of parameters $a_{2j+1}^{\pm}$ such that $M_1(h)$ has exactly $[\frac{m}{2}]$ zeros in h on $(0,+\infty)$, which means that this upper bound can be reached. $\square$

Theorem 3.2. *Suppose that $M_1(h) \equiv 0$ and $M_2(h) \not\equiv 0$. For sufficiently small $|\epsilon| > 0$, system (3.1) has at most $m-1$ limit cycles bifurcated from the period annulus L_h, multiplicity taken into account.*

Proof. If $M_1(h) \equiv 0$, we need to use the second order Melnikov function $M_2(h)$ to estimate the number of limit cycles of system (3.1). According to Theorem 2.3 and (3.4), we have

$$\begin{aligned} M_2(h) =& \Big(v_2^+(h) - v_2^-(h)\Big) - \frac{1}{K_{01}^+(h)}\Big(K_{11}^+(h) - K_{11}^-(h)\Big)v_1^+(h) - \frac{K_{02}^+(h) - K_{02}^-(h)}{2} \\ & \cdot \Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2. \end{aligned} \tag{3.9}$$

Similarly to Theorem 3.1, for m is even, we can calculate from (2.7) that

$$v_2^+(h) = 2\sum_{j=0}^{\frac{m}{2}} b_{2j+1}^+(\sqrt{h})^{2j+1},\ v_2^-(h) = 2\sum_{j=0}^{\frac{m}{2}} b_{2j+1}^-(\sqrt{h})^{2j+1},$$

$$K_{02}^+(h) = K_{02}^-(h) = 2,\ v_2^+(h) - v_2^-(h) = 2\sum_{j=0}^{\frac{m}{2}}(b_{2j+1}^+ - b_{2j+1}^-)(\sqrt{h})^{2j+1}, \tag{3.10}$$

$$K_{11}^+(h) = -\sum_{j=1}^{m+1} ja_j^+(-\sqrt{h})^{j-1},\ K_{11}^-(h) = -\sum_{j=1}^{m+1} ja_j^-(-\sqrt{h})^{j-1}.$$

It follows from (3.6) that $M_1(h) \equiv 0$ if and only if $a_{2j+1}^+ = a_{2j+1}^-$ for $j = 0, 1, \cdots, \frac{m}{2}$. Hence, $K_{11}^+(h) - K_{11}^-(h)$ can be reduced to

$$\begin{aligned} K_{11}^+(h) - K_{11}^-(h) &= -\sum_{j=1}^{\frac{m}{2}} 2j(a_{2j}^+ - a_{2j}^-)(-\sqrt{h})^{2j-1} \\ &= \sum_{j=1}^{\frac{m}{2}} 2j(a_{2j}^+ - a_{2j}^-)(\sqrt{h})^{2j-1}. \end{aligned} \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we can obtain

$$\begin{aligned} M_2(h) =& 2\sum_{j=0}^{\frac{m}{2}}(b_{2j+1}^+ - b_{2j+1}^-)(\sqrt{h})^{2j+1} - \sqrt{h}\sum_{j=1}^{\frac{m}{2}} 2j(a_{2j}^+ - a_{2j}^-)(\sqrt{h})^{2j-2} \\ & \cdot \sum_{j=0}^{\frac{m}{2}} a_{2j+1}^+(\sqrt{h})^{2j} \\ =& 2\sqrt{h}G_1(h), \end{aligned} \tag{3.12}$$

where

$$G_1(h) = \sum_{j=0}^{\frac{m}{2}}(b_{2j+1}^+ - b_{2j+1}^-)h^j - \sum_{j=1}^{\frac{m}{2}} 2j(a_{2j}^+ - a_{2j}^-)h^{j-1} \cdot \sum_{j=0}^{\frac{m}{2}} a_{2j+1}^+ h^j,$$

and $\deg(G_1) = \frac{m}{2} - 1 + \frac{m}{2} = m - 1$, which implies that $M_2(h)$ has at most $m-1$ zeros in h on $(0, +\infty)$, multiplicity taken into account, if it is not zero identically.

For m is odd, we can calculate the following coefficients from (2.7)

$$v_2^+(h) = 2\sum_{j=0}^{\frac{m-1}{2}} b_{2j+1}^+(\sqrt{h})^{2j+1},\ v_2^-(h) = 2\sum_{j=0}^{\frac{m-1}{2}} b_{2j+1}^-(\sqrt{h})^{2j+1},$$

$$K_{02}^+(h) = K_{02}^-(h) = 2,\ v_2^+(h) - v_2^-(h) = 2\sum_{j=0}^{\frac{m-1}{2}}(b_{2j+1}^+ - b_{2j+1}^-)(\sqrt{h})^{2j+1}, \tag{3.13}$$

$$K_{11}^+(h) = -\sum_{j=1}^{m+1} ja_j^+(-\sqrt{h})^{j-1},\ K_{11}^-(h) = -\sum_{j=1}^{m+1} ja_j^-(-\sqrt{h})^{j-1}.$$

As a matter of fact from (3.8), $M_1(h) \equiv 0$ if and only if $a^+_{2j+1} = a^-_{2j+1}$ for $j = 0, 1, \cdots, \frac{m-1}{2}$. Then $K^+_{11}(h) - K^-_{11}(h)$ can be reduced to

$$\begin{aligned} K^+_{11}(h) - K^-_{11}(h) &= -\sum_{j=1}^{\frac{m+1}{2}} 2j(a^+_{2j} - a^-_{2j})(-\sqrt{h})^{2j-1} \\ &= \sum_{j=1}^{\frac{m+1}{2}} 2j(a^+_{2j} - a^-_{2j})(\sqrt{h})^{2j-1}. \end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.9), we have

$$\begin{aligned} M_2(h) =& 2\sum_{j=0}^{\frac{m-1}{2}} (b^+_{2j+1} - b^-_{2j+1})(\sqrt{h})^{2j+1} - \sqrt{h}\sum_{j=1}^{\frac{m+1}{2}} 2j(a^+_{2j} - a^-_{2j})(\sqrt{h})^{2j-2} \\ & \cdot \sum_{j=0}^{\frac{m-1}{2}} a_{2j+1}(\sqrt{h})^{2j} \\ =& 2\sqrt{h}G_2(h), \end{aligned} \tag{3.15}$$

where

$$G_2(h) = \sum_{j=0}^{\frac{m-1}{2}} (b^+_{2j+1} - b^-_{2j+1})h^j - \sum_{j=1}^{\frac{m+1}{2}} 2j(a^+_{2j} - a^-_{2j})h^{j-1} \cdot \sum_{j=0}^{\frac{m-1}{2}} a_{2j+1}h^j,$$

and $\deg(G_2) = \frac{m+1}{2} - 1 + \frac{m-1}{2} = m - 1$. It is easy to see that $M_2(h)$ has at most $m-1$ zeros in h on $(0, +\infty)$, multiplicity taken into account, if it is not zero identically.

Thus, according to Theorem 2.1 together with (3.12) and (3.15), system (3.1) has at most $m-1$ limit cycles bifurcated from the period annulus L_h by the second order Melnikov function, multiplicity taken into account. $\square$

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